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HOMOLOGY OF BASED GAUGE GROUPS ASSOCIATED WITH SPINOR GROUP

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ABSTRACT. We study the homology of based gauge groups associated with the principal $Spin(n)$ bundles over the four-sphere using the Eilenberg-Moore spectral sequence and the Serre spectral sequence with the Dyer-Lashof operations.

1. Introduction

Let G be a compact, connected simple Lie group. The fact that $\pi_3(G) = \pi_4(BG) = Z$ leads to the classification of principal G bundles P_k over S^4 by the integer k in Z. For a given P_k, the orbit spaces of connections up to based gauge equivalence is homotopy equivalent to the triple loop space of G [2]. That is, $\mathcal{C}_k = \mathcal{A}_k/\mathcal{G}_k^b(G) \simeq \Omega_k^3 G$ where $\mathcal{A}_k$ is the space of the all connections in P_k and $\mathcal{G}_k^b(G)$ is the based gauge group which consists of all base point preserving automorphisms on P_k. Since $\mathcal{A}_k$ is a linear space, it is contractible. Hence $B\mathcal{G}_k^b(G) \simeq \Omega_k^3 G$, so $\mathcal{G}_k^b(G) \simeq \Omega(\Omega_k^3 G)$ where $\simeq$ is the homotopy equivalence. Moreover $\mathcal{G}_k^b(G)$ is infinite dimensional and each $\mathcal{G}_k^b(G)$ is homotopy equivalent to $\mathcal{G}_0^b(G)$ for any component k.

Let $Spin(n)$ be the n-th spinor group which is the universal covering space of $SO(n)$ for $n \geq 3$. In this paper we study the homology of based gauge groups associated with the principal $Spin(n)$ bundles over the four-sphere.

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2. Preliminaries and basic facts

For a $(n+1)$-fold loop space, there are homology operations,

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; \mathbb{F}_p) \longrightarrow H_{pq+i(p-1)}(\Omega^{n+1}X; \mathbb{F}_p)$$

defined for $0 \leq i \leq n$ when $p = 2$ and for $0 \leq i \leq n$, $i \equiv q$ for mod 2 when p is an odd prime which is natural for a $(n+1)$-fold loop space. Let Q_i^a be the iterated operation $Q_i \dots Q_i$(a times) and β be the mod p Bockstein operation. We refer [9] for the condensed treatment of these homology operations. Throughout this paper, the subscript of an element always means the degree of an element, for example the degree of a_i is i.

Since $\pi_3(G) = Z$ for a compact, connected simple Lie group G, $\pi_0(\Omega^3 G) = Z$. Let $\Omega_0^3 G$ be the zero component of $\Omega^3 G$ and $E(x)$ be the exterior algebra on x.

THEOREM 2.1. [9, Theorem 3.1.4] *In the path-loop fibration*

$$\Omega^{n+2}X \to P\Omega^{n+1}X \to \Omega^{n+1}X,$$

we have the following.

(a) *If $x \in H_*(\Omega^{n+1}X; \mathbb{F}_p)$ is transgressive in the Serre spectral sequence, then so is $Q_i x$ and $\tau \circ Q_{i(p-1)}x = Q_{(i+1)(p-1)} \circ \tau x$ for each i, $0 \leq i \leq n$ where τ is the transgression.*

(b) *For $p > 2$ and $n \geq 1$, $d^{2q(p-1)}(x^{p-1} \otimes \tau(x)) = -\beta Q_{(p-1)}\tau(x)$ if $x \in H_{2q}(\Omega^{n+1}X; \mathbb{F}_2)$.*

(c) *For $p = 2$, $Sq_*^1 Q_i x = Q_{i-1}x$ if $x \in H_q(\Omega^{n+1}X; \mathbb{F}_2)$ and $q+i$ is even.*

Let $H_*(\Omega^4 S^n; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b Q_3^c \iota_{n-4} : a, b, c \geq 0]$, $n > 4$. Recall that $H_*(\Omega_0^4 S^4; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b Q_3^c[1] * [-2^{a+b+c}] : a, b, c \geq 0]$, where $\Omega_0^4 S^4$ is the zero component in $\Omega^4 S^4$ and a homology class [1] is the image of the generator in $\tilde{H}_0(S^0; \mathbb{F}_2)$ for the map: $S^0 \longrightarrow \Omega^4 S^4$. It is also known in [11] that

$$\begin{aligned} &H_*(\Omega_0^4 S^3; \mathbb{F}_2) = E(Q_1^a Q_2^b[1] * [2^{-a-b}] : a+b \geq 1) \\ (1)\quad &\otimes \mathbb{F}_2[(Q_1^a Q_3^b x_1)^2 : a, b \geq 0] \otimes \mathbb{F}_2[Q_1^a Q_2^b Q_4^c Q_2 x_1 : a, b, c \geq 0]\,. \end{aligned}$$

3. Mod 2 homology of based gauge groups

We have the following homotopy equivalences by the Bott periodicity:

$$\begin{aligned}
&\Omega SO \simeq SO/U,\\
&\Omega\,(O/U) \simeq U/Sp,\\
&\Omega\,(U/Sp) \simeq BSp \times Z,\\
&\Omega\,BSp \simeq Sp,\\
&\Omega Sp \simeq Sp/U.
\end{aligned}$$

Since $Spin(n)$ is the double covering space of $SO(n)$, $\Omega Spin(n) \simeq \Omega_0 SO(n)$ and $\Omega^k Spin(n) \simeq \Omega^k SO(n)$ for all $k \geq 2$. Moreover we know the homology of every space involved in above the Bott periodicity [5]. Hence we have the following Lemma from the Bott periodicity.

LEMMA 3.1. *The (co)homologies of iterated loop spaces for Spin are as follow:*

$$\begin{aligned}
\text{(a)}\quad &H^*(Spin;\mathbb{F}_2) = \mathbb{F}_2[u_{2i+3} : i \geq 0],\\
&H^*(\Omega Spin;\mathbb{F}_2) = \mathbb{F}_2[v_{4i+2} : i \geq 0],\\
&H^*(\Omega^2 Spin;\mathbb{F}_2) = E(w_{4i+1} : i \geq 0),\\
&H^*(\Omega_0^3 Spin;\mathbb{F}_2) = \mathbb{F}_2[x_{4i} : i \geq 1],\\
&H^*(\Omega_0^4 Spin;\mathbb{F}_2) = E(y_{4i-1} : i \geq 1),\\
&H^*(\Omega_0^5 Spin;\mathbb{F}_2) = E(u_{2i} : i \geq 1).
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad &H_*(\Omega Spin;\mathbb{F}_2) = E(a_{2i} : i \geq 1),\\
&H_*(\Omega^2 Spin;\mathbb{F}_2) = E(b_{4i+1} : i \geq 0),\\
&H_*(\Omega_0^3 Spin;\mathbb{F}_2) = \mathbb{F}_2[c_{4i} : i \geq 1],\\
&H_*(\Omega_0^4 Spin;\mathbb{F}_2) = E(d_{4i-1} : i \geq 1),\\
&H_*(\Omega_0^5 Spin;\mathbb{F}_2) = \mathbb{F}_2[u_{4i-2} : i \geq 1].
\end{aligned}$$

Now we study the behavior of the Serre spectral sequence for the following fibration:

$$\Omega_0^4 Spin(n) \longrightarrow \Omega_0^4 Spin(n+1) \longrightarrow \Omega_0^4 S^n\,.$$

THEOREM 3.2. *Let $\{E^r_{*,*}, d^r\}$ be the Serre spectral sequence converging to $H_*(\Omega_0^4 Spin(n+1); \mathbb{F}_2)$ for the following fibration:*

$$\Omega_0^4 Spin(n) \to \Omega_0^4 Spin(n+1) \to \Omega_0^4 S^n \, , n \geq 3.$$

Then $E^2 = E^\infty$ if and only if $n = 4k+3$ for some $k \geq 0$.

Proof. Note that $Spin(4) \simeq Spin(3) \times Spin(3)$ and $Spin(5) \simeq Sp(2)$. Using the same method in the proof of Theorem 2.1 in [7], we can get

$$H_*(\Omega^4 Sp(2); \mathbb{F}_2) \cong H_*(\Omega^4 S^3; \mathbb{F}_2) \otimes H_*(\Omega^4 S^7; \mathbb{F}_2) \, . \tag{2}$$

From this we get the conclusion for $n = 3, 4$. Now we assume that $n > 5$. Since the Serre spectral sequence for the above fibration is the spectral sequence of Hopf algebras, by the naturality of the Dyer– Lashof operation it is enough to check whether the transgression $\tau(\iota_{n-4})$ from $E_{n-4,0}$ to $E_{0,n-5}$ is trivial or not where $H_*(\Omega^4 S^n; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b Q_3^c \iota_{n-4} : a, b, c \geq 0]$. Now we have the following morphisms of fibrations:

$$\begin{array}{ccccc}
\Omega^5 Spin/Spin(n) & \xrightarrow{f} & \Omega_0^4 Spin(n) & \longrightarrow & \Omega_0^4 Spin \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^5 Spin/Spin(n+1) & \longrightarrow & \Omega_0^4 Spin(n+1) & \longrightarrow & \Omega_0^4 Spin \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^4 S^n & \longrightarrow & \Omega^4 S^n & \longrightarrow & *
\end{array} \tag{3}$$

Consider the Serre spectral sequence for the first column fibration. Since the connectivity of the space $\Omega^5 Spin/Spin(n)$ is $(n-6)$, the element ι_{n-4} in $E^{n-4}_{n-4,0} \cong H_*(\Omega^4 S^n; \mathbb{F}_2)$ should transgress to the element of dimension $n-5$ in $E^{n-4}_{0,n-5} \cong H_*(\Omega^5 Spin/Spin(n); \mathbb{F}_2)$.

Now consider the Serre spectral sequence for the top row fibration converging to $H_*(\Omega_0^4 Spin(n); \mathbb{F}_2)$. By the Bott periodicity, there is a homotopy equivalence between $\Omega_0^4 Spin$ and Sp. Note that $H_*(Sp; \mathbb{F}_2) = E(u_{4k+3} : k \geq 0)$. Hence if $n-5$ is not of the form $4k+2$, $0 \leq k \leq n-1$, that is, n is not 3 modulo 4, then the element of dimension $n-5$ in $H_*(\Omega^5 Spin/Spin(n); \mathbb{F}_2)$ can not be the target of the first non-trivial transgression. Therefore

$$f_* : H_{n-5}(\Omega^5 Spin/Spin(n); \mathbb{F}_2) \to H_{n-5}(\Omega_0^4 Spin(n); \mathbb{F}_2)$$

is not zero if $n \neq 4k+3$ for some $k \geq 0$. Now we consider the morphism between the first column fibration and the second column fibration. Then by the naturality of differentials, we get that

$$\tau(\iota_{n-4}) : E_{n-4,0} \cong H_{n-4}(\Omega^4 S^n; \mathbb{F}_2) \to E_{0,n-5} \cong H_{n-5}(\Omega_0^4 Spin(n); \mathbb{F}_2)$$

is not zero if $n \neq 4k+3$ for some $k \geq 0$. Hence the Serre spectral sequence for the second column fibration converging to $H_*(\Omega_0^4 Spin(n+1); \mathbb{F}_2)$ does not collapse at E^2 if $n \neq 4k+3$ for some $k \geq 0$.

For the case of $n = 4k+3$, we consider the following morphisms of fibrations:

$$\begin{array}{ccccc} \Omega_0^5 Spin & \longrightarrow & * & \longrightarrow & \Omega_0^4 Spin \\ {\scriptstyle \Omega^5 h}\downarrow & & \downarrow & & \| \\ \Omega^5 Spin/Spin(4k+3) & \xrightarrow{f} & \Omega_0^4 Spin(4k+3) & \longrightarrow & \Omega_0^4 Spin \end{array}$$

Since $\Omega_0^4 Spin \simeq Sp$ and $\Omega_0^5 Spin \simeq \Omega Sp$, every transgression is nontrivial for the Serre spectral sequence for the top path loop fibration. Consider the map: $h : Spin \to Spin/Spin(4k+3)$. Then

$$h^* : H^{4k+3}(Spin/Spin(4k+3); \mathbb{F}_2) \to H^{4k+3}(Spin; \mathbb{F}_2)$$

is nonzero. Therefore

$$(\Omega^5 h)^* : H^{4k-2}(\Omega_0^5 Spin/Spin(4k+3); \mathbb{F}_2) \to H^{4k-2}(\Omega_0^5 Spin; \mathbb{F}_2), k \geq 1$$

is also nonzero. From this we have that

$$(\Omega^5 h)_* : H_{4k-2}(\Omega_0^5 Spin; \mathbb{F}_2) \to H_{4k-2}(\Omega_0^5 Spin/Spin(4k+3); \mathbb{F}_2)$$

is nonzero. Hence we have nontrivial transgression which is from the generator of degree $4k-1$ for the Serre spectral sequence for the bottom row fibration converging to $H_*(\Omega_0^4 Spin(4k+3); \mathbb{F}_2)$, that is,

$$f_* : H_{4k-2}(\Omega^5 Spin/Spin(4k+3); \mathbb{F}_2) \to H_{4k-2}(\Omega_0^4 Spin(4k+3); \mathbb{F}_2)$$

is zero. Hence by the naturality of differentials, the Serre spectral sequence for the second column fibration in diagram (3) collapses at E^2 if $n = 4k+3$ for some $k \geq 0$. □

Since $\mathcal{G}_0^b(Spin(n)) \simeq \Omega_0^4 Spin(n)$, we compute $H_*(\Omega_0^4 Spin(n); \mathbb{F}_2)$ to get $H_*(\mathcal{G}_0^b(Spin(n)); \mathbb{F}_2)$. The mod 2 homology of $\Omega_0^3 Spin(n)$ is computed in [6]. Note that $Spin(3) \cong S^3$ and $Spin(4) \cong S^3 \times S^3$. Hence from (1) and (2), we get the result for $n = 3, 4, 5$. So we exclude these cases for the following theorem.

THEOREM 3.3. *As an algebra, $H_*(\mathcal{G}^b_0(Spin(4n));\mathbb{F}_2)$, $n > 1$, is isomorphic to*

$$\begin{aligned}&\mathbb{F}_2[Q_1^a u_{4k-1} : 1 \le k < [n/2]]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b v_{8k+4} : a,b \ge 0, [(n-1)/2] \le k \le n-2]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c u_{4k-1} : a,b,c \ge 0, [n/2] \le k \le n-1]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c w_{4n-5+2k} : a,b,c \ge 0, 0 \le k < 2n-1\\ &\quad \text{and } 2n+k \not\equiv 3 \mod 4],\end{aligned}$$

$H_*(\mathcal{G}^b_0(Spin(4n+1));\mathbb{F}_2)$, $n > 1$, is isomorphic to

$$\begin{aligned}&\mathbb{F}_2[Q_1^a u_{4k-1} : 1 \le k < [n/2]]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b v_{8k+4} : a,b \ge 0, [(n-1)/2] \le k \le n-2]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c u_{4k-1} : a,b,c \ge 0, [n/2] \le k \le n-1]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c w_{4n-3+2k} : a,b,c \ge 0, 0 \le k < 2n\\ &\quad \text{and } 2n+k \not\equiv 2 \mod 4],\end{aligned}$$

$H_*(\mathcal{G}^b_0(Spin(4n+2));\mathbb{F}_2)$, $n > 0$, is isomorphic to

$$\begin{aligned}&\mathbb{F}_2[Q_1^a u_{4k-1} : 1 \le k < [n/2]]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b v_{8k+4} : a,b \ge 0, [n/2] \le k \le n-2]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c u_{4k-1} : a,b,c \ge 0, [(n+1)/2] \le k \le n-1]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c w_{4n-1+2k} : a,b,c \ge 0, 0 \le k < 2n-1\\ &\quad \text{and } 2n+k \not\equiv 1 \mod 4]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_4^c v_{8n-4} : a,b,c \ge 0]\\ &\otimes\begin{cases}\mathbb{F}_2[Q_1^a Q_3^b w_{4n-3} : a,b \ge 0] & \text{if } n \text{ is even}\\ \mathbb{F}_2[Q_1^a Q_3^b u_{4[n/2]-1} : a,b \ge 0] & \text{if } n \text{is odd}\end{cases},\end{aligned}$$

$H_*(\mathcal{G}^b_0(Spin(4n+3));\mathbb{F}_2)$, $n > 0$, is isomorphic to

$$\begin{aligned}&\mathbb{F}_2[Q_1^a u_{4k-1} : 1 \le k < [(n+1)/2]]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b v_{8k+4} : a,b \ge 0, [n/2] \le k \le n-1]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c u_{4k-1} : a,b,c \ge 0, [(n+1)/2] \le k \le n-1]\\ &\otimes\mathbb{F}_2[Q_1^a Q_2^b Q_3^c w_{4n-1+2k} : a,b,c \ge 0, 0 \le k < 2n+1\\ &\quad \text{and } 2n+k \not\equiv 1 \mod 4],\end{aligned}$$

where $[n]$ denotes the greatest number less than or equal to n.

Proof. Consider the Serre spectral sequence for the following fibration:

$$\Omega^4_0 Spin(4n+i) \to \Omega^4_0 Spin(4n+i+1) \to \Omega^4 S^{4n+i}.$$

(Case 1) $i = 0$. $n > 1$. We can express $H_*(\Omega^4 S^{4n};\mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b Q_3^c \iota_{4n-4} : a,b,c \geq 0]$ as follows:

$$\mathbb{F}_2[Q_1^a Q_2^b \iota_{4n-4} : a,b \geq 0] \otimes \mathbb{F}_2[Q_1^a Q_2^b Q_3^c (Q_3 \iota_{4n-4}) : a,b,c \geq 0].$$

By Theorem 3.2, we have non-trivial transgression from ι_{4n-4} to the primitive element w_{4n-5}. Hence by Theorem 2.1, we have

$$\tau(Q_0^a Q_1^b Q_2^c \iota_{4n-4}) = Q_1^a Q_2^b Q_3^c w_{4n-5}, a,b,c \geq 0.$$

Now we check whether $\tau(Q_3 \iota_{4n-4})$ is trivial or not, that is, $Q_4 w_{4n-5}$ is zero or not. By the dimension reason, only possible primitive element of that degree is

$$\begin{cases} Q_2 v_{4n-4} & \text{if } n \text{ is even} \\ Q_2^2 u_{2n-3} & \text{if } n \text{ is odd}. \end{cases}$$

By the Nishida relation, Sq_*^1 action on $Q_2 v_{4n-4}$ is $Q_1 v_{4n-4}$ and Sq_*^1 action on $Q_2^2 u_{2n-3}$ is $Q_1 Q_2 u_{2n-3}$, i.e., $Sq_*^1 Q_2 v_{4n-4} \neq 0$ and $Sq_*^1 Q_2^2 u_{2n-3} \neq 0$. On the other hand, the Nishida relation implies

$$\begin{aligned} Sq_*^1 Q_4 w_{4n-5} &= \textstyle\sum_j \binom{4n-2}{1-2j} Q_{3+2j} Sq_*^j w_{4n-5} \\ &= (4n-2) Q_3 w_{4n-5} \\ &= 0. \end{aligned}$$

Hence $Q_4 w_{4n-5}$ is neither $Q_2 v_{4n-4}$ nor $Q_2^2 u_{2n-3}$. Hence $\tau(Q_3 \iota_{4n-4}) = 0$. Let $Q_3 \iota_{4n-4} = w_{8n-5}$. Then $Q_1^a Q_2^b Q_3^c w_{8n-5}$ are permanent cycles for $a,b,c \geq 0$ and we get the conclusion.

(Case 2) $i = 1$. $n > 1$. By Theorem 3.2, there exists non-trivial transgression from ι_{4n-3} to the primitive element of degree $4n-4$. But the only possible primitive element of that degree is v_{4n-4} if n is even and $Q_2 u_{2n-3}$ if n is odd. Note that $[(n-1)/2] = [(n-2)/2]$ for even n and $[n/2] = [(n-1)/2]$ for odd n. So we have

$$\tau(\iota_{4n-3}) = \begin{cases} v_{4n-4} & \text{if } n \text{ is even} \\ Q_2 u_{2n-3} & \text{if } n \text{ is odd.} \end{cases}$$

For even n, by Theorem 2.1, we get

$$\begin{aligned} &\tau(Q_0^a Q_1^b \iota_{4n-3}) = Q_1^a Q_2^b v_{4n-4},\ a,b \geq 0, \\ &\tau(Q_0^a Q_1^b Q_3^c \iota_{4n-3}) = Q_1^a Q_2^{b+1} Q_3^c (w_{4n-3}),\ a,b,c \geq 0. \end{aligned}$$

Note that $Q_4(v_{4n-4}) = Q_2(w_{4n-3})$. On the other hand $Q_0^aQ_1^bQ_3^c(w_{4n-3})$ are permanent cycles for $a, b, c \geq 0$. For odd n, we have

$$\begin{aligned}\tau(Q_0^aQ_1^bQ_3^c\iota_{4n-3}) &= Q_1^aQ_2^bQ_4^c(Q_2u_{2n-3}) \\ &= Q_1^aQ_2^{b+1}Q_3^c(u_{2n-3}),\ a,b,c \geq 0.\end{aligned}$$

On the other hand, $Q_0^aQ_1^bQ_3^c(u_{2n-3})$ are permanent cycles for $a, b, c \geq 0$. For each n, $Q_0^aQ_1^bQ_2^{c+1}Q_3^d(\iota_{4n-3})$ are also permanent cycles for $a, b, c \geq 0$. Let $Q_2\iota_{4n-3} = v_{8n-4}$. Then $Q_0^aQ_1^bQ_2^cQ_4^d(v_{8n-4})$ are permanent cycles for $a, b, c \geq 0$. Moreover by the Bockstein lemma, we have that

$$d^2Q_4^{a+1}v_{8n-4} = Q_1Q_3^{a+1}w_{4n-3}, a \geq 0 \tag{4}$$

where d^r is the r-th order differential in the homology Bockstein spectral sequence $\{B^r, d^r\}$ [8].

(Case 3) $i = 2$, $n \geq 1$.

$$\tau(Q_0^aQ_2^b\iota_{4n-2}) = \begin{cases} Q_1^aQ_3^bw_{4n-3} & \text{if } n \text{ is even} \\ Q_1^aQ_3^{b+1}u_{4[n/2]-1} & \text{if } n \text{ is odd}\ . \end{cases}$$

Moreover $Q_0^aQ_1^{b+1}Q_2^c(\iota_{4n-2})$ are also permanent cycles for $a, b, c \geq 0$. By the Nishida relation, $Sq_*^1Q_1Q_2^{a+1}\iota_{4n-2} = Q_0Q_2^{a+1}\iota_{4n-2}$, $a \geq 0$. By the generalized Bockstein lemma [8] together with (4), we get

$$Sq_*^1Q_1Q_2^{a+1}\iota_{4n-2} = Q_4^{a+1}v_{8n-4}, a \geq 0\,.$$

By the Nishida relation, we have $Q_2Q_1\iota_{4n-2} = Q_4v_{8n-4}$. If we put $Q_1\iota_{4n-2} = w_{8n-3}$, $Q_0^aQ_1^{b+1}Q_2^c(\iota_{4n-2})$ can be expressed as $Q_0^aQ_1^bQ_3^cw_{8n-3}$. Then we have the following:

$$\begin{aligned}&\mathbb{F}_2[Q_1^aQ_3^bw_{8n-3} : a, b \geq 0] \otimes \mathbb{F}_2[Q_1^aQ_2^bQ_4^{c+1}v_{8n-4} : a, b, c \geq 0] \\ &= \mathbb{F}_2[Q_1^aQ_2^bQ_3^cw_{8n-3} : a, b, c \geq 0]\,.\end{aligned}$$

Moreover $Q_0^aQ_1^bQ_2^cQ_3^{d+1}(\iota_{4n-2})$ are also permanent cycles for $a, b, c \geq 0$. So if we put $Q_3\iota_{4n-2} = w_{8n-1}$, then $Q_0^aQ_1^bQ_2^cQ_c^d(w_{8n-1})$ are permanent cycles for $a, b, c \geq 0$.

(Case 4) $i = 3$, $n \geq 1$. By Theorem 3.2, the Serre spectral sequence converging to $H_*(\Omega_0^4 Spin(n+4); \mathbb{F}_2)$ collapses at the E_2-term and we get the conclusion. □

4. Mod p homology of based gauge groups

Given a path-loop fibration, $\Omega X \to PX \to X$, there also exists a second quadrant Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of bicommutative and biassociative Hopf algebras where

(1) $E_2 = \mathrm{Tor}_{H^*(X;R)}(R,R)$ as Hopf algebras
(2) $E_\infty = E_0(H^*(\Omega X;R))$ as Hopf algebras
(3) d_r has bidegree $(r, -r+1)$.

We denote the primitives and the indecomposables of $H^*(X;\mathbb{F}_p)$ by $PH^*(X;\mathbb{F}_p)$ and $QH^*(X;\mathbb{F}_p)$, respectively. In the Eilenberg-Moore spectral sequence, we have a suspension map

$$\begin{aligned}\sigma : QH^*(X;\mathbb{F}_p) &\cong \mathrm{Tor}^{-1,*}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\\ &= E_2^{-1,*} \to E_\infty^{-1,*} \subset H^{*-1}(\Omega X;\mathbb{F}_p)\,.\end{aligned}$$

Since the elements of $\mathrm{Tor}^{-1,*}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)$ are primitive and permanent cycles in the Eilenberg-Moore spectral sequence, the above map induces the suspension homomorphism $\sigma : QH^*(X;\mathbb{F}_p) \to PH^{*-1}(\Omega X;\mathbb{F}_p)$.

THEOREM 4.1. [4, Theorem 5.14] *Let X be a path connected H-space. Then the following is true.*

(a) *The Eilenberg-Moore spectral sequence collapses at E_2 if and only if* $\ker \sigma = 0$.

(b) *The suspension $\sigma : QH^k(X;\mathbb{F}_p) \to PH^{k-1}(\Omega X;\mathbb{F}_p)$ is injective if* $k \not\equiv 2 \bmod 2p$.

(c) *The suspension $\sigma : QH^k(X;\mathbb{F}_p) \to PH^{k-1}(\Omega X;\mathbb{F}_p)$ is surjective if $k-1 \not\equiv -2 \bmod 2p$.*

THEOREM 4.2. [13, Theorem 1] *The Eilenberg-Moore spectral sequences for the path loop fibrations converging to the mod p (co)homology of the single, the double, and the triple loop spaces of any simply connected finite H-space collapse at the E_2-term.*

From now on we denote $H_*(\Omega^i S^n;\mathbb{F}_p)$ by $\Omega_i(n)$ and $\otimes_{k=1}^r H_*(\Omega^i S^{n_k};\mathbb{F}_p)$ by $\Omega_i(n_1,\cdots,n_r)$ for $i = 2,3,4$.

THEOREM 4.3. *For p an odd prime, as an algebra*

$$
\begin{aligned}
&H_*(\mathcal{G}_0^b(Spin(2n+1));\mathbb{F}_p) = E(Q_{p-1}^a Q_{3(p-1)}^b(u_{2p-3}) : a, b \geq 0)\\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a Q_{3(p-1)}^b(u_{2p-3}) : a \geq 0, b > 0]\\
&\otimes E(Q_{p-1}^a Q_{3(p-1)}^b(u_{2i-3}) : a, b \geq 0, 1 < i \leq [\tfrac{2n-1}{p}], i \not\equiv 0 \bmod p, i : odd)\\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a Q_{3(p-1)}^b(u_{2i-3}) : a \geq 0, b > 0, 1 < i \leq [\tfrac{2n-1}{p}],\\
&\quad i \not\equiv 0 \bmod p, i : odd]\\
&\otimes \mathbb{F}_p[Q_{2(p-1)}^a Q_{4(p-1)}^b v_{2pi-4} : a, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 \bmod p]\\
&\otimes E(Q_{p-1}^a \beta Q_{2(p-1)}^b Q_{4(p-1)}^c v_{2pi-4} : a \geq 0, b > 0, c \geq 0,\\
&\quad [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a \beta Q_{2(p-1)}^b Q_{4(p-1)}^c v_{2pi-4} : a, b > 0, c \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1,\\
&\quad i \equiv 0 \bmod p]\\
&\otimes \Omega_4(2i+1)([\tfrac{2n-1}{p}] < i \leq 2n-1, i \not\equiv 0 \bmod p, i : odd),
\end{aligned}
$$

$$H_*(\mathcal{G}_0^b(Spin(2n+2));\mathbb{F}_p) = H_*(\mathcal{G}_0^b(Spin(2n+1));\mathbb{F}_p) \otimes \Omega_4(2n+1).$$

Proof. From [3], as an algebra $H_*(\Omega_0^3 SU(n+1);\mathbb{F}_p)$ is isomorphic to

$$
\begin{aligned}
&\mathbb{F}_p[Q_{2(p-1)}^a(Q_{2(p-1)}[1] * [-p]) : a \geq 0]\\
&\otimes \mathbb{F}_p[Q_{2(p-1)}^a(u_{2i-2}) : a \geq 0, 1 < i \leq n, i \not\equiv 0 \bmod p]\\
&\otimes E(Q_{p-1}^a \beta Q_{2(p-1)}^b u_{2i-2} : a \geq 0, b > 0, [\tfrac{n}{p}] < i \leq n, i \not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a \beta Q_{2(p-1)}^b u_{2i-2} : a, b > 0, [\tfrac{n}{p}] < i \leq n, i \not\equiv 0 \bmod p]\\
&\otimes E(Q_{p-1}^a Q_{3(p-1)}^b v_{2pi-3} : a, b \geq 0, [\tfrac{n}{p}] < i \leq n, i \equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q_{p-1}^a Q_{3(p-1)}^b v_{2pi-3} : a > 0, b \geq 0, [\tfrac{n}{p}] < i \leq n, i \equiv 0 \bmod p].
\end{aligned}
$$

For an odd prime p, Harris [10] proved that there is mod p cross sections $s : SU(2n+1)/SO(2n+1) \to SU(2n+1)$ and get the following decomposition

$$SU(2n+1) \simeq_{(p)} SU(2n+1)/SO(2n+1) \times SO(2n+1).$$

Hence we have

$$\Omega^3 SU(2n+1) \simeq_{(p)} \Omega^3 SU(2n+1)/SO(2n+1) \times \Omega^3 SO(2n+1).$$

So the mod p homology of $\Omega^3 SO(2n+1)$ for odd primes p is the one part of the direct summands of the mod p homology of $\Omega^3 SU(2n+1)$.

From this and the fact $\Omega^3 SO(2n+1) \simeq \Omega^3 Spin(2n+1)$, we get

$$\begin{aligned} H_*(\Omega_0^3 Spin(2n+1);\mathbb{F}_p) &= \mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1]*[-p]) : a \geq 0] \\ &\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a \geq 0, 1 < i \leq 2n-1, i \not\equiv 0 \bmod p, i : odd] \\ &\otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a \geq 0, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, \\ &\quad i \not\equiv 0 \bmod p, i : odd) \\ &\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)} u_{2i-2} : a, b > 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, \\ &\quad i \not\equiv 0 \bmod p, i : odd] \\ &\otimes E(Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, i \equiv 0 \bmod p) \\ &\otimes \mathbb{F}_p[\beta Q^a_{p-1} Q^b_{3(p-1)} v_{2pi-3} : a > 0, b \geq 0, [\tfrac{2n-1}{p}] < i \leq 2n-1, \\ &\quad i \equiv 0 \bmod p]. \end{aligned}$$

Now we study the Eilenberg-Moore spectral sequence converging to $H^*(\Omega_0^4 Spin(2n+1);\mathbb{F}_p)$ with

$$E_2 \cong \mathrm{Tor}_{H^*(\Omega_0^3 Spin(2n+1));\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)\,. \tag{5}$$

Then by Theorem 4.1, the collapse condition of above spectral sequence depends on whether

$$\sigma : QH^{2kp+2}(\Omega_0^3 Spin(2n+1);\mathbb{F}_p) \to PH^{2kp+1}(\Omega_0^4 Spin(2n+1);\mathbb{F}_p)$$

is injective or not. By the exact sequence of Milnor -Moore and Theorem 4.1, we have that

$$\begin{aligned} QH^{2kp+2}(\Omega_0^3 Spin(2n+1);\mathbb{F}_p) &\cong PH^{2kp+2}(\Omega_0^3 Spin(2n+1);\mathbb{F}_p) \\ &\cong QH^{2kp+3}(\Omega^2 Spin(2n+1);\mathbb{F}_p)\,. \end{aligned} \tag{6}$$

From Theorem 4.2 and the above information of $H_*(\Omega_0^3 Spin(2n+1);\mathbb{F}_p)$, we can obtain that $H_*(\Omega^2 Spin(2n+1);\mathbb{F}_p)$ is, as an algebra, isomorphic to

$$\begin{aligned} &E(Q^a_{(p-1)} x_{2i-1} : a \geq 0, 1 < i \leq 2n+1, i \not\equiv 0 \bmod p, i : odd) \\ &\otimes \mathbb{F}_p[\beta Q^a_{(p-1)} x_{2i-1} : a > 0, [\frac{2n+1}{p}] < i \leq 2n+1, i \not\equiv 0 \bmod p, i : odd] \\ &\otimes \mathbb{F}_p[Q^a_{2(p-1)} y_{2i-2} : a > 0, [\frac{2n+1}{p}] < i \leq 2n+1, i \equiv 0 \bmod p]\,. \end{aligned}$$

Since $|Q^a_{(p-1)} x_{2i-1}| = 2p^a i - 1$ and $|Q^a_{2(p-1)} y_{2i-2}| = 2p^a i - 2$, there is no indecomposable element of degree $2kp+3$ for some $k \geq 0$ in

$H_*(\Omega^2 Spin(2n+1); \mathbb{F}_p)$. So by duality, there is no primitive element of degree $2kp+3$ for some $k \geq 0$ in $H^*(\Omega^2 Spin(2n+1); \mathbb{F}_p)$. Then in (6), $QH^{2kp+2}(\Omega_0^3 Spin(2n+1); \mathbb{F}_p) = 0$. Hence $\ker \sigma = 0$. So by Theorem 4.1.(a), the Eilenberg-Moore spectral sequence in (5) collapses at E_2 and there is no coalgebra extension problem in such a spectral sequence [12]. Hence by duality, the Eilenberg-Moore spectral sequence converging to $H_*(\Omega_0^4 Spin(2n+1); \mathbb{F}_p)$ with

$$E^2 \cong \mathrm{Cotor}_{H_*(\Omega_0^3 Spin(2n+1); \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

collapses at E^2 and there is no algebra extension problem. Hence we get the conclusion for $H_*(\Omega_0^4 Spin(2n+1); \mathbb{F}_p)$ by the formal Cotor calculation.

Now we consider the following fibration

$$\Omega^3 Spin(2n+1) \longrightarrow \Omega^3 Spin(2n+2) \longrightarrow \Omega^3 S^{2n+1}.$$

Over the rationals, we have $Spin(2n+2)) \simeq_Q Spin(2n+1) \times S^{2n+1}$ and we have $H_*(Spin(2n+2); \mathbb{F}_p) = H_*(Spin(2n+1); \mathbb{F}_p) \otimes H_*(S^{2n+1}; \mathbb{F}_p)$. Then by Theorem 4.2, we have that

$$H_*(\Omega_0^3 Spin(2n+2); \mathbb{F}_p) = H_*(\Omega_0^3 Spin(2n+1); \mathbb{F}_p) \otimes H_*(\Omega_0^3 S^{2n+1}; \mathbb{F}_p).$$

Like the case of $Spin(2n+1)$, there is no primitive element of degree $2kp+3$ for some $k \geq 0$ in $H_*(\Omega^2 Spin(2n+2); \mathbb{F}_p)$. So the Eilenberg-Moore spectral sequence converging to $H_*(\Omega_0^4 Spin(2n+2); \mathbb{F}_p)$ collapses at E^2 and we get the conclusion. □

Remark. Spinor groups play a crucial role in understanding exceptional Lie groups. For examples, G_2 is a subgroup of $Spin(7)$ fixing a point $z \in S^7$ where $Spin(7)$ acts transitively on $(x, y, z) \in S^6 \times S^6 \times S^7$, where $x \perp y$ and E_8 can be constructed from $Spin(16)$[1]. Moreover there is a sequence of the following subgroups such that every subgroup is mod 2 totally non-homologous to zero in the group containing it [14]:

$$G_2 \subset Spin(7) \subset Spin(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

Hence the homology of based gauge groups associated with spinor group have a rich information for those of exceptional Lie groups.

References

[1] J. F. Adams, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, University of Chicago Press, 1996.

[2] M. F. Atiyah and J. D. S. Jones, *Topological aspects of Yang-Mills theory*, Comm. Math. Phys. **61** (1978), no. 2, 97–118.

[3] C. P. Boyer, B. M. Mann and D. Waggoner, *On the homology of $SU(n)$ instantons*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 529–561.

[4] W. Browder, *On differential Hopf algebra*, Trans. Amer. Math. Soc. **107** (1963), 153–176.

[5] H. Cartan, *Démonstration homologique des théorèmes de périodicitè de Bott II*, Séminaire Cartan et Moore 1959/1960, Exposé 17 Ecole Norm. Sup. Paris.

[6] Y. Choi, *Homology of the double and triple loop space of $SO(n)$*, Math. Z. **222** (1996), no. 1, 59–80.

[7] ———, *The mod 2 homology of $Sp(n)$ instantons and the classifying space of the gauge group*, Bull. Austral. Math. Soc. **55** (1997), no. 3, 503–512.

[8] ———, *On the Bockstein lemma*, Topology Appl. **106** (2000), no. 2, 217–224.

[9] F. R. Cohen, T. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics Vol. 533, Springer, 1976.

[10] B. Harris, *On the homotopy group of the classical groups*, Ann. of Math.(2) **74** (1961), 407–413.

[11] T. Hunter, *On $H_*(\Omega^{n+2}S^{n+1};\mathbb{F}_2)$*, Trans. Amer. Math. Soc. **314** (1989), no. 1, 405–420.

[12] R. Kane, *On loop spaces without p torsion*, Pacific J. Math. **60** (1975), no. 1, 189–201.

[13] J. P. Lin, *On the collapses of certain Eilenberg–Moore spectral sequence*, Topology Appl. **132** (2003), no. 1, 29–35.

[14] M. Mimura and H. Toda, *Topology of Lie groups, I, II*, Translations of Mathematical Monographs vol. 91, Amer. Math. Soc., 1991.

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