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ON A q-FOCK SPACE AND ITS UNITARY DECOMPOSITION

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ABSTRACT. A Fock representation of q-commutation relation is studied by constructing a q-Fock space as the space of the representation, the q-creation and q-annihilation operators $(-1 < q < 1)$. In the case of $0 < q < 1$, the q-Fock space is interpolated between the Boson Fock space and the full Fock space. Also, a unitary decomposition of the q-Fock space $(q \neq 0)$ is studied.

1. Introduction

The existence of a Fock representation of q-commutation relation (introduced by Greenberg [6], Bożejko and Speicher [3]) was first studied in [4] by constructing a q-Fock space as the space of representation, see also [2]. A representation of the q-commutation relation $(-1 \leq q \leq 1)$ is given as the form:

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \cdot \mathbf{1}, \quad \zeta, \eta \in H,$$

where H is a Hilbert space and a is an operator(on a Hilbert space $\mathcal{K}$)-valued linear map. Here the Hilbert space $\mathcal{K}$ is called the space of the representation of the relation. The q-commutation relation $(-1 < q < 1)$ provides an interpolation between the fermionic and bosonic commutation relations which correspond to $q = -1$ and $q = 1$, respectively. The spaces of the representation of the fermionic and bosonic commutation relations are called the Fermion and Boson Fock spaces, respectively. Also, the full Fock space corresponds to $p = 0$. It seems natural to construct a q-Fock space as the space of the representation of the q-commutation relation $(0 < q < 1)$ which is interpolated between the full Fock space and the Boson Fock space.

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Main purpose of this paper is to construct a q-Fock space as the space of the representation of the q-commutation relation such that for $0 < q < 1$, the q-Fock space is interpolated between the full Fock space and the Boson Fock space. Then we study a unitary decomposition of the q-Fock space. We hope that the decomposition will be useful for the study of quantum martingales. We refer to [7, 10], for the case of Boson.

The paper is organized as follows. In Section 2 we study a Fock representation of q-commutation relation as constructing a q-Fock space. In Section 3 we prove a unitary decomposition of the q-Fock space.

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2. Fock representation of q-commutation relation

Let $\Gamma_0(H)$ be the full Fock space over a complex Hilbert space H with the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$. Let $\Gamma_0^{\text{finite}}(H)$ be the linear span of vectors of the forms $\xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes n}$, $n = 0, 1, 2, \ldots$, where $H^{\otimes 0} = \mathbb{C}\Omega$ for the vacuum vector $\Omega \in \Gamma_0(H)$.

Let $q \in (-1, 1)$ be fixed. For each $n = 0, 1, 2, \ldots$, we put

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0.$$

The q-factorial is defined as

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad [0]_q! = 1.$$

Let S_n denote the symmetric group of all permutations on $\{1, \ldots, n\}$ and $I(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$ defined by

$$I(\sigma) = \#\{(i,j) | 1 \le i < j \le n, \sigma(i) > \sigma(j)\}.$$

The operator P_q is defined on $\Gamma_0^{\text{finite}}(H)$ by a linear extension of

$$P_q \Omega = \Omega;$$
$$P_q(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_n} q^{I(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

Put

$$\xi_1 \otimes_q \cdots \otimes_q \xi_n := P_q(\xi_1 \otimes \cdots \otimes \xi_n), \quad \xi_i \in H, \quad i = 1, \ldots, n.$$

Then we have

$$(2.1)\qquad \xi_1 \otimes_q \cdots \otimes_q \xi_n = \sum_{i=1}^{n} q^{i-1} \xi_i \otimes \left(\xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n\right).$$

Let $\Gamma_q^{\text{finite}}(H)$ be the linear span of vectors of the forms $\xi_1 \otimes_q \cdots \otimes_q \xi_n \in H^{\otimes n}$, $n = 0, 1, 2, \ldots$. Now, we consider the sesquilinear form $\langle\langle \cdot, \cdot \rangle\rangle_q$ defined on $\Gamma_q^{\text{finite}}(H)$ by a sesquilinear extension of

$$(2.2)\qquad \begin{aligned} &\langle\langle \xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_m \rangle\rangle_q \\ := \;& \delta_{nm} [n]_q! \, \langle\langle \xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes \cdots \otimes \eta_m \rangle\rangle_0 . \end{aligned}$$

Then by applying Theorem 2.2 in [4], we can easily see that the sesquilinear form $\langle\langle \cdot, \cdot \rangle\rangle_q$ is the strictly positive, i.e., $\langle\langle \xi, \xi \rangle\rangle_q > 0$ for $0 \neq \xi \in \Gamma_q^{\text{finite}}(H)$.

LEMMA 2.1. (1) *For any $\sigma \in S_n$, we have $I(\sigma) = I(\sigma^{-1})$;*
(2) *For any $\xi_i, \eta_i \in H$, $i = 1, \ldots, n$, we have*

$$\begin{aligned} &\langle\langle \xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle\rangle_0 \\ = \;& \langle\langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle\rangle_0 . \end{aligned}$$

Proof. The proof is straightforward. □

The completion of $\Gamma_q^{\text{finite}}(H)$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_q$ is called the *q-Fock space* and denoted by $\Gamma_q(H)$.

For each $\zeta \in H$, we define the *q-creation operator* $a^*(\zeta)$ and the *q-annihilation operator* $a(\zeta)$ on the dense subspace $\Gamma_0^{\text{finite}}(H)$ of the q-Fock space $\Gamma_q(H)$ as follows:

$$a^*(\zeta)\Omega = \zeta;$$

$$(2.3)\qquad a^*(\zeta) \xi_1 \otimes_q \cdots \otimes_q \xi_n = \frac{1}{\sqrt{[n+1]_q}} \zeta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_n$$

and

$$a(\zeta)\Omega = 0;$$

$$(2.4)\qquad a(\zeta) \xi_1 \otimes_q \cdots \otimes_q \xi_n = \sqrt{[n]_q} \zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n),$$

where $f \otimes^1 g$ is the left 1-contraction of $f \in H$ and $g \in H^{\otimes m}$, see [8]. From (2.1), we have

$$(2.5)\quad \zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n) = \sum_{i=1}^{n} q^{i-1} \langle \zeta, \xi_i \rangle \, \xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n,$$

where the symbol $\check{\xi}_i$ means that ξ_i has to be deleted in the tensor product and $\langle\cdot,\,\cdot\rangle$ denotes the inner product on H.

THEOREM 2.2. *Let $\zeta \in H$.*

(1) *The operators $a^*(\zeta)$ and $a(\zeta)$ are adjoints of each other on $\Gamma_q^{\text{finite}}(H)$ with respect to $\langle\!\langle\cdot,\,\cdot\rangle\!\rangle_q$.*

(2) *The operators $a^*(\zeta)$ and $a(\zeta)$ are bounded on $\Gamma_q(H)$.*

Proof. (1) From (2.3), (2.2) and (1) in Lemma 2.1, for any $\xi_i, \eta_j \in H$, $i = 1, \dots, n-1$, $j = 1, \dots, n$, we have

$$
\begin{aligned}
(2.6)\quad & \langle\!\langle a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_{n-1},\, \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle\!\rangle_q \\
&= \frac{[n]_q!}{\sqrt{[n]_q}} \sum_{\sigma \in S_n} q^{I(\sigma)} \left\langle f_{\sigma(1)},\, \eta_1 \right\rangle \cdots \left\langle f_{\sigma(n)},\, \eta_n \right\rangle \\
&= \sqrt{[n]_q}\,[n-1]_q! \sum_{\sigma \in S_n} q^{I(\sigma^{-1})} \left\langle f_1,\, \eta_{\sigma^{-1}(1)} \right\rangle \cdots \left\langle f_n,\, \eta_{\sigma^{-1}(n)} \right\rangle \\
&= \sqrt{[n]_q}\,[n-1]_q! \sum_{\tau \in S_n} q^{I(\tau)} \left\langle \zeta,\, \eta_{\tau(1)} \right\rangle \left\langle \xi_1,\, \eta_{\tau(2)} \right\rangle \cdots \left\langle \xi_{n-1},\, \eta_{\tau(n)} \right\rangle \\
&= \sqrt{[n]_q}\,[n-1]_q!\, \langle\!\langle \xi_1 \otimes \cdots \otimes \xi_{n-1},\, \zeta \otimes^1 \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle\!\rangle_0,
\end{aligned}
$$

where $f_1 = \zeta$ and $f_i = \xi_{i-1}$, $i = 2, \dots, n$. For convenience, we put

$$
\overrightarrow{\xi} = \xi_1 \otimes \cdots \otimes \xi_{n-1}, \qquad \overrightarrow{\xi}_q = \xi_1 \otimes_q \cdots \otimes_q \xi_{n-1}.
$$

Then by (2.5) and (2) in Lemma 2.1 we have

$$
\begin{aligned}
(2.7)\quad & \left\langle\!\!\left\langle \overrightarrow{\xi},\, \zeta \otimes^1 \eta_1 \otimes_q \cdots \otimes_q \eta_n \right\rangle\!\!\right\rangle_0 \\
&= \left\langle\!\!\left\langle \overrightarrow{\xi},\, P_q \left(\sum_{i=1}^n q^{i-1} \langle \zeta,\, \eta_i \rangle\, \eta_1 \otimes \cdots \otimes \check{\eta}_i \otimes \cdots \otimes \eta_n \right) \right\rangle\!\!\right\rangle_0 \\
&= \left\langle\!\!\left\langle \overrightarrow{\xi}_q,\, \sum_{i=1}^n q^{i-1} \langle \zeta,\, \eta_i \rangle\, \eta_1 \otimes \cdots \otimes \check{\eta}_i \otimes \cdots \otimes \eta_n \right\rangle\!\!\right\rangle_0 \\
&= \frac{1}{[n-1]_q!} \left\langle\!\!\left\langle \overrightarrow{\xi}_q,\, \sum_{i=1}^n q^{i-1} \langle \zeta,\, \eta_i \rangle\, \eta_1 \otimes_q \cdots \otimes_q \check{\eta}_i \otimes_q \cdots \otimes_q \eta_n \right\rangle\!\!\right\rangle_q .
\end{aligned}
$$

Therefore, by (2.6), (2.7) and (2.4) we have

$$
\begin{aligned}
& \langle\!\langle a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_{n-1},\, \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle\!\rangle_q \\
&= \langle\!\langle \xi_1 \otimes_q \cdots \otimes_q \xi_{n-1},\, a(\zeta)\eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle\!\rangle_q
\end{aligned}
$$

which follows the proof.

(2) By a simple modification of the proof of (ii) in Theorem 6 in [1], we prove that for any $\zeta \in H$ and $\xi_i \in H$, $i = 1, \dots, n$,

$$\begin{aligned}\langle\!\langle P_q(\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n), \zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n \rangle\!\rangle_0 \\ \leq \frac{1}{1-q} |\zeta|_0^2 \langle\!\langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle\!\rangle_0 .\end{aligned}$$

Therefore, for any $\zeta \in H$ and $\xi_i \in H$, $i = 1, \dots, n$, we obtain that

$$\begin{aligned}&\|a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_q^2 \\ &\quad = [n]_q! \langle\!\langle P_q(\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n), \zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n \rangle\!\rangle_0 \\ &\quad \leq \frac{1}{1-q} |\zeta|_0^2 [n]_q! \langle\!\langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle\!\rangle_0 \\ &\quad = \frac{1}{1-q} |\zeta|_0^2 \|\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_q^2 .\end{aligned}$$

Hence we prove that $\|a^*(\zeta)\|_{\mathrm{OP}} \leq 1/\sqrt{1-q}\,|\zeta|_0$ which proves (2). □

THEOREM 2.3. *The q-creation and q-annihilation operators fulfill the q-commutation relation, i.e.,*

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \cdot \mathbf{1}, \quad \zeta, \eta \in H. \tag{2.8}$$

Proof. By (2.5), for any $\xi_i \in H$, $i = 1, \dots, n$ we have

$$\begin{aligned}&a(\zeta)a^*(\eta)\xi_1 \otimes_q \cdots \otimes_q \xi_n \\ &\quad = \frac{1}{\sqrt{[n+1]_q}} a(\zeta)\eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_n \\ &\quad = \langle \zeta, \eta \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_n \\ &\qquad + \sum_{i=1}^n q^i \langle \zeta, \xi_i \rangle \eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n .\end{aligned} \tag{2.9}$$

On the other hand, we have

$$\begin{aligned}&\sum_{i=1}^n q^i \langle \zeta, \xi_i \rangle \eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n \\ &\quad = q\sqrt{[n]_q}\, a^*(\eta) \left(\sum_{i=1}^n q^{i-1} \langle \zeta, \xi_i \rangle \xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n \right) \\ &\quad = qa^*(\eta)a(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n .\end{aligned} \tag{2.10}$$

Therefore, by (2.9) and (2.10) we have

$$\begin{aligned} & a(\zeta)a^*(\eta)\xi_1 \otimes_q \cdots \otimes_q \xi_n \\ & \quad = \langle \zeta, \eta \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_n + q a^*(\eta) a(\zeta) \xi_1 \otimes_q \cdots \otimes_q \xi_n \end{aligned}$$

which proves (2.8). □

The *Boson Fock space* is defined by

$$\begin{aligned} \Gamma_1(H) &= \bigoplus_{n=0}^{\infty} H^{\widehat{\otimes} n} \\ &= \left\{ \phi = (f_n)_{n=0}^{\infty} | f_n \in H^{\widehat{\otimes} n}, n = 0, 1, \ldots, \|\phi\|_1 < \infty \right\}, \end{aligned}$$

where $H^{\widehat{\otimes} n}$ is the symmetric n-tensor product and $\|\phi\|_1^2 = \sum_{n=0}^{\infty} |f_n|^2$. Then we have the following.

THEOREM 2.4. *For any $0 \le q \le 1$ we have the following continuous inclusions:*

$$\Gamma_1(H) \subset \Gamma_q(H) \subset \Gamma_0(H).$$

In particular, $\Gamma_1(H)$ is isometrically embedded into $\Gamma_q(H)$ and the second inclusion is contraction.

Proof. For any $\xi_i \in H$, $i = 1, \ldots, n$, $\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n$ denotes the symmetric n tensor product of ξ_i, $i = 1, \ldots, n$, i.e.,

$$\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

Then we have

$$P_q(\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n) = [n]_q! \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n$$

and

$$\begin{aligned} \|\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n\|_q^2 &= \frac{1}{[n]_q!^2} \|P_q(\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n)\|_q^2 \\ &= \frac{1}{[n]_q!} \left\langle\!\left\langle P_q\left(\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n\right), \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n \right\rangle\!\right\rangle_0 \\ &= \left\langle\!\left\langle \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n, \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n \right\rangle\!\right\rangle_0 \\ &= \|\xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n\|_1^2. \end{aligned}$$

Hence $\Gamma_1(H)$ is isometrically embedded into $\Gamma_q(H)$.

To prove the second inclusion, we first note that for $0 \le q < 1$

$$P_q^{[n]} > 0 \quad \text{and} \quad \|P_q^{[n]}\|_{\mathrm{OP}} \le [n]_q!,$$

see Theorem 2 in [1], where $P_q^{[n]}$ is the restriction of P_q to $H^{\otimes n}$. Therefore, for $0 \leq q < 1$ we have

$$\left(P_q^{[n]}\right)^2 \leq \|P_q^{[n]}\|_{\mathrm{OP}} P_q^{[n]} \leq [n]_q! P_q^{[n]},$$

see Proposition 2.2.13 in [5]. Hence for any $\xi_i \in H$, $i = 1, \ldots, n$ we have

$$\begin{aligned}\|\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_0^2 = & \langle\!\langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), P_q(\xi_1 \otimes \cdots \otimes \xi_n)\rangle\!\rangle_0 \\ & \leq [n]_q! \langle\!\langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n\rangle\!\rangle_0 \\ & = \|\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_q^2\end{aligned}$$

which proves the second inclusion is contraction. □

3. Unitary decomposition of q-Fock space

Let $S_{m;n}$ denote the symmetric group of all permutations on $\{m, m+1, \ldots, n\}$ for $n \geq m$.

LEMMA 3.1. *Let $\sigma \in S_{1;m+n}$ with $\sigma = \tau \cup \lambda$ for some $\tau \in S_{1;m}$ and $\lambda \in S_{m+1;m+n}$. Then we have*

$$I(\sigma) = I(\tau) + I(\lambda).$$

The proof is obvious.

THEOREM 3.2. *Let H_1, H_2 be Hilbert spaces and let $H = H_1 \oplus H_2$. Suppose that $q \neq 0$. There exists a unique unitary isomorphism between*

$$U : \Gamma_q(H_1 \oplus H_2) \longrightarrow \Gamma_q(H_1) \otimes \Gamma_q(H_2)$$

satisfying the relations:

$$\begin{aligned}U\left(\frac{1}{\sqrt{[m+n]_q!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n)\right) \\ = \frac{1}{\sqrt{[m]_q! [n]_q!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m) \otimes P_q(\eta_1 \otimes \cdots \otimes \eta_n)\end{aligned} \tag{3.1}$$

for all $\xi_i \in H_1$, $\eta_j \in H_2$, $1 \leq i \leq m$, $1 \leq j \leq n$, $m = 1, 2, \ldots, n = 1, 2, \ldots$ and

$$U\Omega = \Omega_1 \otimes \Omega_2,$$

where Ω_1 and Ω_2 are vacuum vectors in $\Gamma_q(H_1)$ and $\Gamma_q(H_2)$, respectively.

Proof. The proof is a simple modification of the proof of Proposition 19.7 in [9]. We may assume without loss of generality that H_1 and H_2 are mutually orthogonal subspaces of H. Put

$$S = \{\Omega, \frac{1}{\sqrt{[m+n]_q!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n) | \xi_i \in H_1,$$
$$\eta_j \in H_2, 1 \leq i \leq m, 1 \leq j \leq n, m = 1, 2, \ldots, n = 1, 2, \ldots\}$$
$$S_1 = \{\Omega_1, \frac{1}{\sqrt{[m]_q!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m) | \xi_i \in H_1, 1 \leq i \leq m, m = 1, 2, \ldots\}$$
$$S_2 = \{\Omega_2, \frac{1}{\sqrt{[n]_q!}} P_q(\eta_1 \otimes \cdots \otimes \eta_n) | \eta_j \in H_2, 1 \leq j \leq n, n = 1, 2, \ldots\}.$$

Then S, S_1, S_2 are total in $\Gamma_q(H), \Gamma_q(H_1), \Gamma_q(H_2)$, respectively. Also, the set $\{\xi \otimes \eta | \xi \in S_1, \eta \in S_2\}$ is total in $\Gamma_q(H)$. Thus it is enough to show that the map defined by (3.1) preserves the scalar product. Let $\xi_i, \xi'_k \in H_1$, $1 \leq i \leq m$, $1 \leq k \leq m'$, and $\eta_j, \eta'_l \in H_2$, $1 \leq j \leq n$, $1 \leq l \leq n'$. We now consider the following three different cases:

Case 1: $m + n \neq m' + n'$:
By (2.2), we see that $P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n)$ and $P_q(\xi'_1 \otimes \cdots \otimes \xi'_{m'} \otimes \eta'_1 \otimes \cdots \otimes \eta'_{n'})$ are orthogonal.

Case 2: $m + n = m' + n'$ and $m \neq m'$:
Without loss of generality let $m < m'$ and $n > n'$. Put

$$(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = (\zeta_1, \ldots, \zeta_{m+n}) \tag{3.2}$$

and

$$(\xi'_1, \ldots, \xi'_{m'}, \eta'_1, \ldots, \eta'_{n'}) = (\zeta'_1, \ldots, \zeta'_{m+n}). \tag{3.3}$$

Note that for any $\sigma \in S_{m+n}$, $\{\zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(m')}\}$ contains at least one $\eta_j \in H_2$. Since H_1 and H_2 are orthogonal, by (2) in Lemma 2.1 we have

$$\begin{aligned}
&\langle\langle P_q(\zeta'_1 \otimes \cdots \otimes \zeta'_{m+n}), P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \rangle\rangle_q \\
= \;& [m+n]_q! \, \langle\langle P_q(\zeta'_1 \otimes \cdots \otimes \zeta'_{m+n}), \zeta_1 \otimes \cdots \otimes \zeta_{m+n} \rangle\rangle_0 \\
= \;& [m+n]_q! \sum_{\sigma \in S_{m+n}} q^{I(\sigma)} \langle \xi'_1, \zeta_{\sigma(1)} \rangle \cdots \langle \xi'_{m'}, \zeta_{\sigma(m')} \rangle \\
& \times \langle \eta'_1, \zeta_{\sigma(m'+1)} \rangle \cdots \langle \eta'_{n'}, \zeta_{\sigma(m+n)} \rangle \\
= \;& 0.
\end{aligned}$$

Case 3: $m = m'$ and $n = n'$:
Define $(\zeta_1, \ldots, \zeta_{m+n})$ and $(\zeta'_1, \ldots, \zeta'_{m+n})$ as in (3.2) and (3.3), respectively. Note that if there exists $1 \leq i \leq m$ such that $\sigma(i) \in \{m +$

$1, \ldots, m+n\}$ for $\sigma \in S_{m+n}$, then

$$\left\langle \zeta_1, \zeta'_{\sigma(1)} \right\rangle \cdots \left\langle \zeta_{m+n}, \zeta'_{\sigma(m+n)} \right\rangle = 0.$$

Therefore, for any σ such that $\sigma = \tau \cup \lambda$ with some $\tau \in S_{1;m}$ and $\lambda \in S_{m+1;m+n}$,

$$\left\langle \zeta_1, \zeta'_{\sigma(1)} \right\rangle \cdots \left\langle \zeta_{m+n}, \zeta'_{\sigma(m+n)} \right\rangle$$

can be a non-zero value. Hence we have

$$\begin{aligned}
& \langle\!\langle P_q(\zeta'_1 \otimes \cdots \otimes \zeta'_{m+n}), P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \rangle\!\rangle_q \\
= \; & [m+n]_q! \, \langle\!\langle \zeta'_1 \otimes \cdots \otimes \zeta'_{m+n}, P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \rangle\!\rangle_0 \\
= \; & [m+n]_q! \sum_{\sigma \in S_{m+n}} q^{I(\sigma)} \left\langle \xi'_1, \zeta_{\sigma(1)} \right\rangle \cdots \left\langle \xi'_{m'}, \zeta_{\sigma(m')} \right\rangle \\
& \times \left\langle \eta'_1, \zeta_{\sigma(m'+1)} \right\rangle \cdots \left\langle \eta'_{n'}, \zeta_{\sigma(m+n)} \right\rangle \\
= \; & [m+n]_q! \sum_{\sigma = \tau \cup \lambda; \tau \in S_{1;m}, \lambda \in S_{m+1;m+n}} q^{I(\sigma)} \\
& \times \left\langle \xi'_1, \zeta_{\sigma(1)} \right\rangle \cdots \left\langle \xi'_{m'}, \zeta_{\sigma(m')} \right\rangle \\
& \times \left\langle \eta'_1, \zeta_{\lambda(m'+1)} \right\rangle \cdots \left\langle \eta'_{n'}, \zeta_{\lambda(m+n)} \right\rangle .
\end{aligned}$$

Therefore, by Lemma 3.1 we have

$$\begin{aligned}
& \langle\!\langle P_q(\zeta'_1 \otimes \cdots \otimes \zeta'_{m+n}), P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \rangle\!\rangle_q \\
= \; & [m+n]_q! \sum_{\tau \in S_{1;m}, \lambda \in S_{m+1;m+n}} q^{I(\tau)} q^{I(\lambda)} \\
& \times \left\langle \xi'_1, \xi_{\tau(1)} \right\rangle \cdots \left\langle \xi'_{m'}, \xi_{\tau(m)} \right\rangle \\
& \times \left\langle \eta'_1, \zeta_{\lambda(m+1)} \right\rangle \cdots \left\langle \eta'_{n'}, \zeta_{\lambda(m+n)} \right\rangle \\
= \; & [m+n]_q! \sum_{\tau \in S_m, \lambda \in S_n} q^{I(\tau)} q^{I(\lambda)} \left\langle \xi'_1, \xi_{\tau(1)} \right\rangle \cdots \left\langle \xi'_{m'}, \xi_{\tau(m)} \right\rangle \\
& \times \left\langle \eta'_1, \eta_{\lambda(1)} \right\rangle \cdots \left\langle \eta'_{n'}, \eta_{\lambda(n)} \right\rangle \\
= \; & \frac{[m+n]_q!}{[m]_q! [n]_q!} \langle\!\langle P_q(\xi'_1 \otimes \cdots \otimes \xi'_m), P_q(\xi_1 \otimes \cdots \otimes \xi_m) \rangle\!\rangle_q \\
& \times \langle\!\langle P_q(\eta'_1 \otimes \cdots \otimes \eta'_n), P_q(\eta_1 \otimes \cdots \otimes \eta_n) \rangle\!\rangle_q
\end{aligned}$$

which completes the proof. □

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