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A COUNTERPART OF BESSEL'S INEQUALITY IN INNER PRODUCT SPACES AND SOME GRÜSS TYPE RELATED RESULTS

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ABSTRACT. A counterpart of the famous Bessel's inequality for orthornormal families in real or complex inner product spaces is given. Applications for some Grüss type inequalities are also provided.

1. Introduction

In [1], the author has proved the following Grüss type inequality in real or complex inner product spaces.

THEOREM 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\phi, \Phi, \gamma, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4}|\Phi - \phi|\,|\Gamma - \gamma|. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In [2], the following refinement of (1.2) has been pointed out.

THEOREM 2. *Let H, $\mathbb{K}$ and e be as in Theorem 1. If $\phi, \Phi, \gamma, \Gamma, x, y$ satisfy* (1.1) *or, equivalently* (see [2, Lemma 1])

$$\left\|x - \frac{\phi + \Phi}{2}e\right\| \leq \frac{1}{2}|\Phi - \phi|, \quad \left\|y - \frac{\gamma + \Gamma}{2}e\right\| \leq \frac{1}{2}|\Gamma - \gamma|, \tag{1.3}$$

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then

$$\begin{aligned} &|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \\ &\le \frac{1}{4}|\Phi-\phi||\Gamma-\gamma| - [\operatorname{Re}\langle \Phi e - x, x-\phi e\rangle]^{\frac{1}{2}}[\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}. \end{aligned} \tag{1.4}$$

In [3], N. Ujević has generalised Theorem 1 for the case of real inner product spaces as follows.

THEOREM 3. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real numbers field $\mathbb{R}$, and $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthornormal family in H, i.e., we recall that $\langle e_i,e_j\rangle = 0$ if $i\neq j$ and $\|e_i\| = 1$, $i,j\in\{1,\dots,n\}$. If $\phi_i,\gamma_i,\Phi_i,\Gamma_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$ satisfy the condition*

$$\left\langle \sum_{i=1}^n \Phi_i e_i - x, x - \sum_{i=1}^n \phi_i e_i\right\rangle \ge 0,\quad \left\langle \sum_{i=1}^n \Gamma_i e_i - y, y - \sum_{i=1}^n \gamma_i e_i\right\rangle \ge 0, \tag{1.5}$$

then one has the inequality:

$$\left|\langle x,y\rangle - \sum_{i=1}^n \langle x,e_i\rangle\langle e_i,y\rangle\right| \le \frac{1}{4}\left[\sum_{i=1}^n(\Phi_i-\phi_i)^2\cdot\sum_{i=1}^n(\Gamma_i-\gamma_i)^2\right]^{\frac{1}{2}}. \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

We note that the key point in his proof is the following identity:

$$\begin{aligned} &\sum_{i=1}^n(\langle x,e_i\rangle-\phi_i)(\Phi_i-\langle x,e_i\rangle) - \left\langle x-\sum_{i=1}^n\phi_i e_i, \sum_{i=1}^n \Phi_i e_i - x\right\rangle \\ &= \|x\|^2 - \sum_{i=1}^n \langle x,e_i\rangle^2, \end{aligned} \tag{1.7}$$

holding for $x\in H$, $\phi_i,\Phi_i\in\mathbb{R}$, $i\in\{1,\dots,n\}$ and $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthornormal family of vectors in the real inner product space H.

In this paper we point out a counterpart of Bessel's inequality in both real and complex inner product spaces. This result will then be employed to provide a refinement of the Grüss type inequality (1.6) for real or complex inner products. Related results as well as integral inequalities for general measure spaces are also given.

2. A counterpart of Bessel's inequality

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i \in I}$ a countable family of *orthornormal vectors* in H, i.e.,

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}, \quad i, j \in I, \tag{2.1}$$

where I is the set of indices.

It is well known that, the following inequality due to Bessel holds

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \|x\|^2 \quad \text{for any } x \in H, \tag{2.2}$$

where the meaning of the sum is:

$$\sum_{i \in I} |\langle x, e_i \rangle|^2 := \sup_{F \subset I} \left\{ \sum_{i \in F} |\langle x, e_i \rangle|^2 , \quad F \text{ is a finite part of } I \right\}. \tag{2.3}$$

The following lemma holds.

LEMMA 1. *Let $\{e_i\}_{i \in I}$ be a family of orthornormal vectors in H, F a finite part of I and ϕ_i, Φ_i $(i \in F)$, real or complex numbers. The following statements are equivalent for $x \in H$*

(i) $\operatorname{Re} \left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle \geq 0$

(ii) $\left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}$.

Proof. It is easy to see that for $y, a, A \in H$, the following are equivalent (see [2, Lemma 1])

(a) $\operatorname{Re} \langle A - y, y - a \rangle \geq 0$ and

(aa) $\left\| y - \frac{a+A}{2} \right\| \leq \frac{1}{2} \|A - a\|$.

Now, for $a = \sum_{i \in F} \phi_i e_i$, $A = \sum_{i \in F} \Phi_i e_i$, we have

$$\|A - a\| = \left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\| = \left(\left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \right\|^2 \right)^{\frac{1}{2}}$$
$$= \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \|e_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}},$$

giving, for $y = x$, the desired equivalence. □

The following counterpart of Bessel's inequality holds.

THEOREM 4. *Let* $\{e_i\}_{i\in I}$, F, ϕ_i, Φ_i, $i \in F$ *and* $x \in H$ *be so that either* (i) *or* (ii) *of Lemma 1 holds. Then we have the inequality:*

$$
\begin{aligned}
(2.4)\qquad 0 &\leq \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 \\
&\leq \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \\
&\leq \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 .
\end{aligned}
$$

The constant $\frac{1}{4}$ *is best in both inequalities.*

Proof. Define

$$I_1 := \sum_{i\in F} \operatorname{Re}\left[(\Phi_i - \langle x, e_i\rangle) \left(\overline{\langle x, e_i\rangle} - \overline{\phi_i} \right) \right]$$

and

$$I_2 := \operatorname{Re}\left[\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \right].$$

Observe that

$$I_1 = \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} \right] + \sum_{i\in F} \operatorname{Re}\left[\overline{\phi_i} \langle x, e_i\rangle \right] - \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\phi_i} \right] - \sum_{i\in F} |\langle x, e_i\rangle|^2$$

and

$$
\begin{aligned}
I_2 &= \operatorname{Re}\left[\sum_{i\in F} \Phi_i \overline{\langle x, e_i\rangle} + \sum_{i\in F} \overline{\phi_i} \langle x, e_i\rangle - \|x\|^2 - \sum_{i\in F}\sum_{j\in F} \Phi_i \overline{\phi_i} \langle e_i, e_j\rangle \right] \\
&= \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\langle x, e_i\rangle} \right] + \sum_{i\in F} \operatorname{Re}\left[\overline{\phi_i} \langle x, e_i\rangle \right] - \|x\|^2 - \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{\phi_i} \right].
\end{aligned}
$$

Consequently, subtracting I_2 from I_1, we deduce the following equality that is useful in its turn

$$
\begin{aligned}
(2.5)\qquad \|x\|^2 - \sum_{i\in F} |\langle x, e_i\rangle|^2 &= \sum_{i\in F} \operatorname{Re}\left[(\Phi_i - \langle x, e_i\rangle) \left(\overline{\langle x, e_i\rangle} - \overline{\phi_i} \right) \right] \\
&\quad - \operatorname{Re}\left[\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \right].
\end{aligned}
$$

Using the following elementary inequality for complex numbers

$$\operatorname{Re}\left(a\bar{b}\right) \leq \frac{1}{4}|a+b|^2, \qquad a, b \in \mathbb{K},$$

for the choices $a = \Phi_i - \langle x, e_i \rangle$, $b = \langle x, e_i \rangle - \phi_i$ $(i \in F)$, we deduce

$$\sum_{i \in F} \operatorname{Re}\left[(\Phi_i - \langle x, e_i \rangle) \left(\overline{\langle x, e_i \rangle} - \overline{\phi_i} \right) \right] \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 . \tag{2.6}$$

Making use of (2.5), (2.6) and the assumption (i), we deduce (2.4).

The sharpness of the constant $\frac{1}{4}$ was proved for a single element e, $\|e\| = 1$ in [1], or for the real case in [3].

We can give here a simple proof as follows.

Assume that there is a $c > 0$ such that

$$\begin{aligned} 0 &\leq \|x\|^2 - \sum_{i \in F} |\langle x, e_i \rangle|^2 \\ &\leq c \sum_{i \in F} |\Phi_i - \phi_i|^2 - \operatorname{Re}\left\langle \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right\rangle , \end{aligned} \tag{2.7}$$

provided ϕ_i, Φ_i, x and F satisfy (i) or (ii).

We choose $F = \{1\}$, $e_1 = e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \in \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\Phi_1 = \Phi = m > 0$, $\phi_1 = \phi = -m$, $H = \mathbb{R}^2$ to get from (2.7) that

$$\begin{aligned} &0 \leq x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} \\ &\leq 4cm^2 - \left(\frac{m}{\sqrt{2}} - x_1 \right) \left(x_1 + \frac{m}{\sqrt{2}} \right) - \left(\frac{m}{\sqrt{2}} - x_2 \right) \left(x_2 + \frac{m}{\sqrt{2}} \right) , \end{aligned} \tag{2.8}$$

provided

$$\begin{aligned} 0 &\leq \langle me - x, x + me \rangle \\ &= \left(\frac{m}{\sqrt{2}} - x_1 \right) \left(x_1 + \frac{m}{\sqrt{2}} \right) + \left(\frac{m}{\sqrt{2}} - x_2 \right) \left(x_2 + \frac{m}{\sqrt{2}} \right) . \end{aligned} \tag{2.9}$$

If we choose $x_1 = \frac{m}{\sqrt{2}}$, $x_2 = -\frac{m}{\sqrt{2}}$, then (2.9) is fulfilled and by (2.8) we get $m^2 \leq 4cm^2$, giving $c \geq \frac{1}{4}$. □

3. A refinement of the Grüss inequality

The following result holds.

THEOREM 5. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and $\phi_i, \Phi_i, \gamma_i, \Gamma_i \in \mathbb{K}$, $i \in F$ and $x, y \in H$. If either*

$$\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \geq 0, \tag{3.1}$$
$$\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \tag{3.2}$$
$$\left\| y - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} e_i \right\| \leq \frac{1}{2} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}},$$

hold, then we have the inequalities

$$\begin{aligned} &\left| \langle x, y \rangle - \sum_{i\in F} \langle x, e_i \rangle \langle e_i, y \rangle \right| \\ \leq\ & \frac{1}{4} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}} \\ &- \left[\operatorname{Re}\left\langle \sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i \right\rangle \right]^{\frac{1}{2}} \\ &\cdot \left[\operatorname{Re}\left\langle \sum_{i\in F} \Gamma_i e_i - y, y - \sum_{i\in F} \gamma_i e_i \right\rangle \right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{4} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Using Schwartz's inequality in the inner product space $(H, \langle\cdot,\cdot\rangle)$ one has

$$\left|\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right|^2 \leq \left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2\left\|y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \tag{3.4}$$

and since a simple calculation shows that

$$\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle = \langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle$$

and

$$\left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i\right\|^2 = \|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2$$

for any $x, y \in H$, then by (3.4) and by the counterpart of Bessel's inequality in Theorem 4, we have

$$\begin{aligned}
&\left|\langle x,y\rangle-\sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right|^2 \\
\leq\ & \left(\|x\|^2-\sum_{i\in F}|\langle x,e_i\rangle|^2\right)\left(\|y\|^2-\sum_{i\in F}|\langle y,e_i\rangle|^2\right) \\
\leq\ & \left[\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2-\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-x, x-\sum_{i\in F}\phi_i e_i\right\rangle\right] \\
&\times\left[\frac{1}{4}\sum_{i\in F}|\Gamma_i-\gamma_i|^2-\operatorname{Re}\left\langle\sum_{i\in F}\Gamma_i e_i-y, y-\sum_{i\in F}\gamma_i e_i\right\rangle\right] \\
\leq\ & \left[\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}\right. \\
&-\left[\operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-x, x-\sum_{i\in F}\phi_i e_i\right\rangle\right]^{\frac{1}{2}} \\
&\left.\cdot\left[\operatorname{Re}\left\langle\sum_{i\in F}\Gamma_i e_i-y, y-\sum_{i\in F}\gamma_i e_i\right\rangle\right]^{\frac{1}{2}}\right]^2
\end{aligned} \tag{3.5}$$

where, for the last inequality, we have made use of the inequality

$$\left(m^2-n^2\right)\left(p^2-q^2\right) \le (mp-nq)^2,$$

where $m,n,p,q>0$.

Taking the square root in (3.5) and observing that the quantity in the last square bracket is nonnegative (see for example (2.4)), we deduce the desired result (3.3).

The best constant has been proved in [1] for one element and we omit the details. □

4. Some companion inequalities

The following companion of the Grüss inequality also holds.

THEOREM 6. *Let $\{e_i\}_{i\in I}$ be a family of orthornormal vectors in H, F a finite part of I and $\phi_i, \Phi_i \in \mathbb{K}$, $i \in F$ and $x,y \in H$ such that*

$$\operatorname{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i\in F}\phi_i e_i \right\rangle \ge 0 \tag{4.1}$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i \right\| \le \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}, \tag{4.2}$$

then we have the inequality

$$\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \le \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2. \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Start with the well known inequality

$$\operatorname{Re}\langle z,u\rangle \le \frac{1}{4}\|z+u\|^2, \quad z,u\in H. \tag{4.4}$$

Since

$$\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle = \left\langle x - \sum_{i\in F}\langle x,e_i\rangle e_i, y - \sum_{i\in F}\langle y,e_i\rangle e_i \right\rangle,$$

for any $x, y \in H$, then, by (4.4), we get

$$\begin{aligned} &\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \\ (4.5)\qquad &= \operatorname{Re}\left[\left\langle x-\sum_{i\in F}\langle x,e_i\rangle e_i,\, y-\sum_{i\in F}\langle y,e_i\rangle e_i\right\rangle\right] \\ &\le \frac{1}{4}\left\|x-\sum_{i\in F}\langle x,e_i\rangle e_i + y - \sum_{i\in F}\langle y,e_i\rangle e_i\right\|^2 \\ &= \left\|\frac{x+y}{2}-\sum_{i\in F}\left\langle\frac{x+y}{2},e_i\right\rangle e_i\right\|^2 \\ &= \left\|\frac{x+y}{2}\right\|^2-\sum_{i\in F}\left|\left\langle\frac{x+y}{2},e_i\right\rangle\right|^2. \end{aligned}$$

If we apply the counterpart of Bessel's inequality in Theorem 4 for $\frac{x+y}{2}$, we may state that

$$(4.6)\qquad \left\|\frac{x+y}{2}\right\|^2-\sum_{i\in F}\left|\left\langle\frac{x+y}{2},e_i\right\rangle\right|^2 \le \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $\frac{1}{4}$ is the best constant in (4.3) follows by the fact that if in (4.1) we choose $x = y$, then it becomes (i) of Lemma 1, implying (2.4), for which, we have shown that $\frac{1}{4}$ was the best constant. □

The following corollary may be of interest if we wish to evaluate the absolute value of

$$\operatorname{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right].$$

COROLLARY 1. *With the assumptions of Theorem 6 and if*

$$(4.7)\qquad \operatorname{Re}\left\langle\sum_{i\in F}\Phi_i e_i-\frac{x\pm y}{2},\frac{x\pm y}{2}-\sum_{i\in F}\phi_i e_i\right\rangle\ge 0$$

or, equivalently

$$(4.8)\qquad \left\|\frac{x\pm y}{2}-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i\right\|\le\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

then we have the inequality

$$\left|\mathrm{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right]\right| \le \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2. \tag{4.9}$$

Proof. We only remark that, if

$$\mathrm{Re}\left\langle \sum_{i\in F}\Phi_i e_i - \frac{x-y}{2}, \frac{x-y}{2} - \sum_{i\in F}\phi_i e_i\right\rangle \ge 0$$

holds, then by Theorem 6 for $(-y)$ instead of y, we have

$$\mathrm{Re}\left[-\langle x,y\rangle + \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \le \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2,$$

showing that

$$\mathrm{Re}\left[\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right] \ge -\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2. \tag{4.10}$$

Making use of (4.3) and (4.10), we deduce the desired inequality (4.9). □

REMARK 1. If H is a real inner product space and $m_i, M_i \in \mathbb{R}$ with the property that

$$\left\langle \sum_{i\in F}M_i e_i - \frac{x\pm y}{2}, \frac{x\pm y}{2} - \sum_{i\in F}m_i e_i\right\rangle \ge 0 \tag{4.11}$$

or, equivalently,

$$\left\|\frac{x\pm y}{2} - \sum_{i\in F}\frac{M_i+m_i}{2}\cdot e_i\right\| \le \frac{1}{2}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{\frac{1}{2}}, \tag{4.12}$$

then we have the Grüss type inequality

$$\left|\langle x,y\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,y\rangle\right| \le \frac{1}{4}\sum_{i\in F}(M_i-m_i)^2. \tag{4.13}$$

5. Integral inequalities

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Let $\rho\ge 0$ be a $\mu-$measurable function on Ω. Denote by $L^2_\rho(\Omega,\mathbb{K})$ the Hilbert space of all real or complex valued functions defined on Ω and $2-\rho-$integrable on Ω, i.e.,

$$\int_\Omega \rho(s)\,|f(s)|^2\,d\mu(s) < \infty. \tag{5.1}$$

Consider the family $\{f_i\}_{i\in I}$ of functions in $L^2_\rho(\Omega,\mathbb{K})$ with the properties that

$$\int_\Omega \rho(s)\,f_i(s)\,\overline{f_j}(s)\,d\mu(s) = \delta_{ij}, \quad i,j\in I, \tag{5.2}$$

where δ_{ij} is 0 if $i\neq j$ and $\delta_{ij}=1$ if $i=j$. $\{f_i\}_{i\in I}$ is an orthornormal family in $L^2_\rho(\Omega,\mathbb{K})$.

The following proposition holds.

Proposition 1. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in $L^2_\rho(\Omega,\mathbb{K})$, F a finite subset of I, $\phi_i,\Phi_i\in\mathbb{K}$ $(i\in F)$ and $f\in L^2_\rho(\Omega,\mathbb{K})$, so that either*

$$\int_\Omega \rho(s)\,\mathrm{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right]d\mu(s)\geq 0 \tag{5.3}$$

or, equivalently,

$$\int_\Omega \rho(s)\left|f(s) - \sum_{i\in F}\frac{\Phi_i+\phi_i}{2}f_i(s)\right|^2 d\mu(s)\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2. \tag{5.4}$$

Then we have the inequality

$$\begin{aligned}0 &\leq \int_\Omega \rho(s)\,|f(s)|^2\,d\mu(s) - \sum_{i\in F}\left|\int_\Omega \rho(s)\,f(s)\,\overline{f_i}(s)\,d\mu(s)\right|^2 \\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2 \\ &\quad - \int_\Omega \rho(s)\,\mathrm{Re}\left[\left(\sum_{i\in F}\Phi_i f_i(s) - f(s)\right)\right. \\ &\quad \left.\cdot\left(\overline{f}(s) - \sum_{i\in F}\overline{\phi_i}\,\overline{f_i}(s)\right)\right]d\mu(s) \\ &\leq \frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2.\end{aligned} \tag{5.5}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

The proof follows by Theorem 4 applied for the Hilbert space $L_\rho^2(\Omega$ $\mathbb{K})$ and the orthornormal family $\{f_i\}_{i\in I}$.

The following Grüss type inequality also holds.

PROPOSITION 2. *Let $\{f_i\}_{i\in I}$ and F be as in Proposition 1. If ϕ_i, Φ_i γ_i, $\Gamma_i \in \mathbb{K}$ $(i \in F)$ and $f, g \in L_\rho^2(\Omega, \mathbb{K})$ so that either*

(5.6)
$$\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Phi_i f_i(s) - f(s)\right)\left(\overline{f}(s) - \sum_{i\in F} \overline{\phi_i}\, \overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$
$$\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Gamma_i f_i(s) - g(s)\right)\left(\overline{g}(s) - \sum_{i\in F} \overline{\gamma_i}\, \overline{f_i}(s)\right)\right] d\mu(s) \geq 0,$$

or, equivalently,

$$\int_\Omega \rho(s) \left| f(s) - \sum_{i\in F} \frac{\Phi_i + \phi_i}{2} f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i\in F} |\Phi_i - \phi_i|^2, \tag{5.7}$$
$$\int_\Omega \rho(s) \left| g(s) - \sum_{i\in F} \frac{\Gamma_i + \gamma_i}{2} f_i(s)\right|^2 d\mu(s) \leq \frac{1}{4} \sum_{i\in F} |\Gamma_i - \gamma_i|^2,$$

then we have the inequalities

$$\begin{aligned} &\left|\int_\Omega \rho(s) f(s) g(s)\, d\mu(s)\right. \\ &\quad \left. - \sum_{i\in F} \int_\Omega \rho(s) f(s) \overline{f_i}(s)\, d\mu(s) \int_\Omega \rho(s) f_i(s) \overline{g(s)} d\mu(s)\right| \\ \leq\ & \frac{1}{4}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in F} |\Gamma_i - \gamma_i|^2\right)^{\frac{1}{2}} \\ &\quad - \left[\int_\Omega \rho(s) \operatorname{Re}\left[\left(\sum_{i\in F} \Phi_i f_i(s) - f(s)\right)\right.\right. \\ &\quad \left.\left. \cdot \left(\overline{f}(s) - \sum_{i\in F} \overline{\phi_i} \overline{f_i}(s)\right)\right] d\mu(s)\right]^{\frac{1}{2}} \end{aligned} \tag{5.8}$$

$$\times\left[\int_{\Omega}\rho(s)\operatorname{Re}\left[\left(\sum_{i\in F}\Gamma_i f_i(s)-g(s)\right)\right.\right.$$
$$\left.\left.\cdot\left(\overline{g}(s)-\sum_{i\in F}\overline{\gamma_i}\overline{f_i}(s)\right)\right]d\mu(s)\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is the best possible.

The proof follows by Theorem 5 and we omit the details.

REMARK 2. Similar results may be stated if we apply the inequalities in Section 4. We omit the details.

In the case of real spaces, the following corollaries provide much simpler sufficient conditions for the counterpart of Bessel's inequality (5.5) or for the Grüss type inequality (5.8) to hold.

COROLLARY 2. *Let $\{f_i\}_{i\in I}$ be an orthornormal family of functions in the real Hilbert space $L^2_\rho(\Omega,\mathbb{R})$, F a finite part of I, $M_i, m_i\in\mathbb{R}$ $(i\in F)$ and $f\in L^2_\rho(\Omega,\mathbb{R})$ so that*

$$\sum_{i\in F}m_i f_i(s)\leq f(s)\leq\sum_{i\in F}M_i f_i(s)\quad\text{for }\mu-\text{a.e. } s\in\Omega. \tag{5.9}$$

Then we have the inequalities

(5.10)

$$0\leq\int_{\Omega}\rho(s)f^2(s)\,d\mu(s)-\sum_{i\in F}\left[\int_{\Omega}\rho(s)f(s)f_i(s)\,d\mu(s)\right]^2$$
$$\leq\frac{1}{4}\sum_{i\in F}(M_i-m_i)^2$$
$$-\int_{\Omega}\rho(s)\left(\sum_{i\in F}M_i f_i(s)-f(s)\right)\left(f(s)-\sum_{i\in F}m_i f_i(s)\right)d\mu(s)$$
$$\leq\frac{1}{4}\sum_{i\in F}(M_i-m_i)^2.$$

The constant $\frac{1}{4}$ is best possible.

COROLLARY 3. *Let $\{f_i\}_{i\in I}$ and F be as in Corollary 2. If $M_i, m_i, N_i, n_i \in \mathbb{R}$ $(i \in F)$ and $f, g \in L^2_\rho(\Omega, \mathbb{R})$ such that*

$$\sum_{i\in F} m_i f_i(s) \le f(s) \le \sum_{i\in F} M_i f_i(s) \tag{5.11}$$

and

$$\sum_{i\in F} n_i f_i(s) \le g(s) \le \sum_{i\in F} N_i f_i(s) \quad \text{for } \mu-\text{a.e. } s\in\Omega.$$

Then we have the inequalities

$$\begin{aligned}
&\left|\int_\Omega \rho(s) f(s) g(s)\, d\mu(s) - \sum_{i\in F}\int_\Omega \rho(s) f(s) f_i(s)\, d\mu(s)\int_\Omega \rho(s) g(s) f_i(s)\, d\mu(s)\right| \\
&\le \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{\frac{1}{2}} \\
&\quad -\left[\int_\Omega \rho(s)\left(\sum_{i\in F} M_i f_i(s) - f(s)\right)\cdot\left(f(s)-\sum_{i\in F} m_i f_i(s)\right) d\mu(s)\right]^{\frac{1}{2}} \\
&\quad \times\left[\int_\Omega \rho(s)\left(\sum_{i\in F} N_i f_i(s) - g(s)\right)\cdot\left(g(s)-\sum_{i\in F} n_i f_i(s)\right) d\mu(s)\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\left(\sum_{i\in F}(M_i-m_i)^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}(N_i-n_i)^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.12}$$

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