

## EIGENVALUES OF A RANDOM WALK ON ORIENTED MATROIDS

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ABSTRACT. We generalize the results about eigenvalues of random walks on central hyperplane arrangements computed by Bidigare, Hanlon and Rockmore to the cases of random walks on oriented matroids.

### 1. Introduction

An *affine hyperplane* in  $V = \mathbb{R}^n$  is an  $(n - 1)$  dimensional affine subspace of  $\mathbb{R}^n$ . A finite set  $\mathcal{A}$  of affine hyperplane in  $\mathbb{R}^n$  is called an *affine hyperplane arrangement*. Then  $\mathcal{A}$  cuts  $V$  into regions called *chambers*. We denote by  $\mathcal{F}$  the collection of all faces of the chambers. The arrangement  $\mathcal{A}$  is called *central* if

$$\bigcap_{H \in \mathcal{A}} H \neq \emptyset.$$

Let  $\mathcal{C}$  be the set of all chambers of an arrangement  $\mathcal{A}$ . For  $F \in \mathcal{F}$  and  $C \in \mathcal{C}$ , the product  $FC$  is defined to be the nearest chamber to  $C$  having  $F$  as a face. Here “nearest” is defined in terms of the number of hyperplanes in  $\mathcal{A}$  separating  $C$  from  $FC$ . Let  $\omega$  be a probability measure on  $\mathcal{F}$ . Then a step in the walk is given by “From  $C \in \mathcal{C}$ , choose  $F$  from the measure  $\omega$  and move to  $FC$ ”. One can also describe the walk on  $\mathcal{C}$  by giving its transition matrix  $K$ :

$$(1.1) \quad K(C, C') = \sum_{FC=C'} \omega(F), \text{ for each } C, C' \in \mathcal{C}.$$

Let  $\mathcal{S}$  be the set of all nonempty affine subspaces  $W \subseteq V$  of the form  $W = \bigcap_{H \in \mathcal{A}'} H$ , where  $\mathcal{A}' \subset \mathcal{A}$ .  $\mathcal{S}$  is called the *intersection lattice*.

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Bidigare, Hanlon and Rockmore [1] proved the following theorem. Moreover, K. S. Brown and P. Diaconis [3] generalized it to non-central cases.

**THEOREM 1.1.** [1] *Let  $\mathcal{A}$  be a central hyperplane arrangement in  $V$ , let  $\mathcal{F}$  be the set of faces, let  $\mathcal{S}$  be the intersection poset and let  $\omega$  be a probability measure on  $\mathcal{F}$ . Then the matrix  $K$  is diagonalizable. For each  $W \in \mathcal{S}$ , there is an eigenvalue*

$$\lambda_W = \sum_{\substack{F \in \mathcal{F} \\ F \subseteq W}} \omega(F)$$

with multiplicity  $m_W = |\mu(W, V)| = (-1)^{\text{codim}(W, V)} \mu(W, V)$ , where  $\mu$  is the Möbius function of  $\mathcal{S}$  and  $\text{codim}(W, V)$  is the codimension of  $W$  in  $V$ . See [4] for the definition of Möbius function.

The following construction follows [1]. For any finite set  $\mathcal{S}$ , let  $\mathbb{R}\mathcal{S}$  denote the vector space of all real linear combinations  $\sum_{S \in \mathcal{S}} \alpha(S)S$  ( $\alpha(S) \in \mathbb{R}$ ) of elements of  $\mathcal{S}$ . In particular, we have vector spaces  $\mathbb{R}\mathcal{C}$  and  $\mathbb{R}\mathcal{F}$  generated by the chambers and faces of a hyperplane arrangement. Then  $\mathbb{R}\mathcal{F}$  is an  $\mathbb{R}$ -algebra and  $\mathbb{R}\mathcal{C}$  is an  $\mathbb{R}\mathcal{F}$ -module via the action of faces on chambers. Given a probability measure  $\omega$  of  $\mathcal{F}$ , we have an element

$$T = T_\omega = \sum_{F \in \mathcal{F}} \omega(F)F$$

of  $\mathbb{R}\mathcal{F}$ , which therefore acts as an operator on  $\mathbb{R}\mathcal{C}$ . Explicitly, given an element  $\alpha = \sum_{C \in \mathcal{C}} \alpha(C)C \in \mathbb{R}\mathcal{C}$ ,

$$T(\alpha) = \sum_{F \in \mathcal{F}, C \in \mathcal{C}} \omega(F) \alpha(C)FC = \sum_{C' \in \mathcal{C}} \beta(C')C',$$

where  $\beta(C') = \sum_{\substack{FC=C' \\ F, C}} \omega(F) \alpha(C) = \sum_{C \in \mathcal{C}} \alpha(C) K(C, C')$ .

Here

$$K(C, C') = \sum_{FC=C'} \omega(F).$$

Thus, if elements of  $\mathbb{R}\mathcal{C}$  are row vectors indexed by  $\mathcal{C}$ ,  $T$  acts as right multiplication by the matrix  $K$ . In particular, the eigenvectors of  $T$  on  $\mathbb{R}\mathcal{C}$  are the left eigenvectors of  $K$ .

For a hyperplane  $H$  in  $V$ ,  $H^+$  and  $H^-$  denote the open half-spaces determined by  $H$ . The choice of which one to call  $H^+$  is arbitrary. Then a face is a nonempty set  $F \subseteq V$  of the form

$$F = \bigcap_{i \in I} H_i^{\sigma_i},$$

where  $\sigma_i \in \{+, -, 0\}$  and  $H_i^0 = H_i$ .

The sequence  $\sigma = (\sigma_i)_{i \in I}$  which encodes the definition of  $F$  is called the *sign sequence* of  $F$  and is denoted by  $\sigma(F)$ . In particular, the faces such that  $\sigma_i \neq 0$  for all  $i$  are called *chambers*. For a face  $F$ , the support of  $F$  is defined to be the affine subspace

$$\text{supp } F = \bigcap_{\sigma_i(F)=0} H_i.$$

In fact, the face  $F$  with a given support  $W$  form the chambers of the hyperplane arrangement  $\mathcal{A}_W$  in  $W$  consisting of the intersections  $H_i \cap W$  for those  $i$  such that  $\sigma_i(F) \neq 0$ . The arrangement  $\mathcal{A}_W$  is called the *restriction* of  $\mathcal{A}$  to  $W$ .

The *face poset* of  $\mathcal{A}$  is the set  $\mathcal{F}$  of faces, ordered as follows: Given  $F, G \in \mathcal{F}$ , we say that  $F$  is a *face* of  $G$  and we will denote by  $F \leq G$  if for each  $i \in I$  either  $\sigma_i(F) = 0$  or  $\sigma_i(F) = \sigma_i(G)$ . Given  $F, G \in \mathcal{F}$ , their product  $FG$  is the face with sign sequence

$$\sigma_i(FG) = \begin{cases} \sigma_i(F) & \text{if } \sigma_i(F) \neq 0, \\ \sigma_i(G) & \text{if } \sigma_i(F) = 0. \end{cases}$$

Now let  $\mathcal{S}$  be the set of all supports of elements of  $\mathcal{F}$  and let  $\mathcal{S}_p = \{W \in \mathcal{S} \mid \text{codim}(W, V) = p\}$ . Also let  $\mathcal{C}_W = \{F \in \mathcal{F} \mid \text{supp } F = W\}$  and let  $\mathcal{F}_p = \{F \in \mathcal{F} \mid \text{codim}(\text{supp } F, V) = p\}$ . Then

$$\mathbb{R}\mathcal{F}_p = \bigoplus_{W \in \mathcal{S}_p} \mathbb{R}\mathcal{C}_W.$$

In [3], the  $\mathbb{R}\mathcal{F}$ -module homomorphism

$$\partial_p : \mathbb{R}\mathcal{F}_p \longrightarrow \mathbb{R}\mathcal{F}_{p-1}$$

was defined by using an orientation of  $V$  and it was proved that  $\partial_p$  is a boundary map and the sequence

$$(1.2) \quad 0 \cdots \longrightarrow \mathbb{R}\mathcal{F}_p \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_2} \mathbb{R}\mathcal{F}_1 \xrightarrow{\partial_1} \mathbb{R}\mathcal{C} \xrightarrow{\partial_0} \mathbb{R} \longrightarrow 0$$

is an exact chain complex. Also, the operator  $T$  is a chain homomorphism between the chain complexes (1.2)'s. By using the result that the

sequence (1.2) is exact, Bidigare, Hanlon and Rockmore proved Theorem 1.1 by computing eigenvalues of  $T : \mathbb{R}\mathcal{C} \rightarrow \mathbb{R}\mathcal{C}$  (See [1], [3]).

In this paper, we prove that an oriented matroid is a generalization of a central hyperplane arrangement. And we generalize the boundary map  $\partial_p$  to the oriented matroids and we prove that generalized one of the sequence (1.2) is also exact. From this we have a generalization of Theorem 1.1 as follows, because the remaining proof of this generalized theorem is the same as the proof of Theorem 1.1.

**THEOREM 1.2.** *Let  $\mathcal{V}$  be an oriented matroid. And let  $\omega$  be a probability measure on the set of faces  $\mathcal{F}$  associated to an oriented matroid  $\mathcal{V}$ . Let  $\mathcal{S}$  be the set of all supports. Then the matrix  $K$  defined in (1.1) is diagonalizable. For each  $W \in \mathcal{S}$ , there is an eigenvalue*

$$\lambda_W = \sum_{\substack{F \in \mathcal{F} \\ F \subseteq W}} \omega(F)$$

with multiplicity  $m_W = |\mu(W, V)| = (-1)^{\text{codim}(W, V)} \mu(W, V)$ .

## 2. Generalization

We first prove that the oriented matroid is a generalization of the face poset of central hyperplane arrangement. Let  $E$  denote the finite set  $\{1, 2, 3, \dots, n\}$ . Then  $\mathcal{V} \subset \{-, 0, +\}^E$  is the set of vectors of an *oriented matroid* if  $\mathcal{V}$  satisfies the following axioms:

- (V<sub>0</sub>)  $0 \in \mathcal{V}$ .
- (V<sub>1</sub>) If  $X \in \mathcal{V}$ , then  $-X \in \mathcal{V}$ .
- (V<sub>2</sub>) If  $X, Y \in \mathcal{V}$ , then  $X \circ Y \in \mathcal{V}$ , where the coordinates of  $X \circ Y$  are those of  $X$  replaced the zero coordinate of it by the corresponding coordinates of  $Y$ .
- (V<sub>3</sub>) Given  $X, Y \in \mathcal{V}$ , let  $S(X, Y) = \{i \mid x_i = -y_i \neq 0\}$ . For every  $j \in S(X, Y)$ , there is a  $Z \in \mathcal{V}$  with  $Z_j = 0$  and  $Z_i = (X \circ Y)_i = (Y \circ X)_i$  for  $i \notin S(X, Y)$ .

In this case we call  $X \in \mathcal{V}$  a *vectors* of an oriented matroid  $\mathcal{V}$ .

**PROPOSITION 2.1.** *Let  $\mathcal{A}$  be a central hyperplane arrangement and let  $\mathcal{F}$  be the face poset. Also let  $\mathcal{V} = \{\sigma = (\sigma_i(F))_{i \in I} \mid F \in \mathcal{F}\}$  be the set of all sign sequences of the elements of  $\mathcal{F}$ . Then  $\mathcal{V}$  is the set of vectors of an oriented matroid.*

*Proof.* It is clear that (V<sub>0</sub>) and (V<sub>1</sub>) are true. Let  $\sigma(F) = (\sigma_i(F))_{i \in I}$  and  $\sigma(G) = (\sigma_i(G))_{i \in I}$  be elements in  $\mathcal{V}$ . Since  $\sigma(FG) = \sigma(F) \circ \sigma(G)$ ,

$\sigma(F) \circ \sigma(G) \in \mathcal{V}$ . Thus  $(V_2)$  holds. Let  $\sigma(F), \sigma(G) \in \mathcal{V}$  and let  $S(\sigma(F), \sigma(G)) = \{i \mid \sigma_i(F) = -\sigma_i(G) \neq 0\}$ . For  $j \in S(\sigma(F), \sigma(G))$ , we can find a face  $K \in \mathcal{F}$  satisfying the following conditions:

(1)  $\sigma_j(K) = 0$ .

(2) For  $i \notin S(\sigma(F), \sigma(G))$  if  $\sigma_i(F) = \sigma_i(G) = 0$ , then  $\sigma_i(K) = 0$ , otherwise  $0 \neq \sigma_i(K) = \sigma_i(F)$  or  $\sigma_i(G)$ .

Then  $\sigma(K)$  satisfies the condition  $(V_3)$ .  $\square$

Proposition 2.1 shows that the oriented matroids are generalization of the face poset of central hyperplane arrangements.

From now on, we show that some lemmas for the proof Theorem 1.2.

The oriented matroids arising from hyperplane arrangements are said to be *realizable*. A *chamber* of an oriented matroid  $\mathcal{V}$  is an element  $Y \in \mathcal{V}$  with no zero coordinates. Note that if  $Y$  is a chamber of  $\mathcal{V}$  and  $X \in \mathcal{V}$  then  $X \circ Y$  is a chamber. Thus if  $\omega(\cdot)$  is a probability distribution on  $\mathcal{V}$ , we may define a Markov chain  $K(X, Y)$  on the chambers of  $\mathcal{V}$  via

$$(2.1) \quad K(X, Y) = \sum_{Z \circ X = Y} \omega(Z).$$

Let  $\mathcal{F}$  be the set whose elements  $F$  (called faces) are in one-to-one correspondence with the sign sequences in  $\mathcal{V}$ . We denote the correspondence by  $F \mapsto \sigma(F) = (\sigma_i(F))_{i \in I}$ , where  $I$  is a finite index set. Then the set  $\mathcal{F}$  of faces of an oriented matroid is a poset under the face relation defined before.

Each face  $F$  has a support, determined by the zero set  $\mathcal{Z}(F) = \{i \in I \mid \sigma_i(F) = 0\}$ . The set  $\mathcal{S}$  of all supports is a lattice in a natural way, which we call the *intersection lattice*. For any  $W \in \mathcal{S}$ , we write  $\mathcal{Z}(W)$  for the zero set of any face  $F$  with support  $W$ . We denote by  $V$  the largest element of  $\mathcal{S}$ . This is the support of any maximal element  $C \in \mathcal{F}$ . These maximal elements of  $\mathcal{F}$  are called *chambers*, and the set of all of them is denoted by  $\mathcal{C}$ . For any  $W \in \mathcal{S}$ , the set  $F \in \mathcal{F}$  with support  $W$  is again the set of chambers of an oriented matroid  $\mathcal{V}_W$ , said to be obtained by restriction to  $W$ . Its face poset is  $\mathcal{F}_W = \{F \in \mathcal{F} \mid \text{supp } F \leq W\}$ . The *rank* of an oriented matroid is the length of the interval  $[0, V]$  in  $\mathcal{S}$ . The length of the interval  $[W, V]$  is the codimension of  $W$ ; it is equal to  $\text{rank } \mathcal{V} - \text{rank } \mathcal{V}_W$ .

We wish to define the notion of orientation for an element  $W \in \mathcal{S}$ . It suffices to consider  $W = V$ , since this then applies to arbitrary  $W$  by the restriction operation described above. By an orientation for  $V$  we will mean a rule that associates to each maximal chain  $0 = A_0 < \cdots < A_n$  in  $\mathcal{F}$  a sign  $\varepsilon = \pm 1$  in such a way that adjacent maximal chains get

opposite signs. Here two maximal chains are adjacent if they differ in exactly one position. We will also say in this situation that one maximal chain is obtained from the other by an elementary move.

Give an orientation for each  $W \in \mathcal{S}$ . That is, attach a sign  $\varepsilon = \pm 1$  to any chain in  $\mathcal{F}$  of the form

$$0 = A_0 < A_1 < \cdots < A_r$$

in such a way that sign changes if an elementary move is performed. Given  $A < B$  in  $\mathcal{F}$ , take a chain

$$0 = A_0 < \cdots < A_r = A.$$

Define

$$[A : B] = \begin{cases} +1 & \text{if } \varepsilon(A_0, \dots, A_r) = \varepsilon(A_0, \dots, A_r, B), \\ -1 & \text{if } \varepsilon(A_0, \dots, A_r) = -\varepsilon(A_0, \dots, A_r, B). \end{cases}$$

Then it is easy to see that  $[A : B]$  is independent of the choice of the chain from 0 to  $A$ . Also we have the following diamond condition.

**LEMMA 2.2.** *Let  $C$  be a chamber in an oriented matroid  $\mathcal{V}$  and  $B_1, B_2$  be faces of  $C$  with  $\text{supp } B_1 = \text{supp } B_2$  and let  $A$  be a face of  $B_1$  and  $B_2$  with codimension 2. Then  $[A : B_1][B_1 : C] = -[A : B_2][B_2 : C]$ .*

*Proof.* It is sufficient to show that the result is true for the following four cases:

- (i)  $[B_1 : C] = +1, [A : B_1] = +1.$
- (ii)  $[B_1 : C] = +1, [A : B_1] = -1.$
- (iii)  $[B_1 : C] = -1, [A : B_1] = +1.$
- (iv)  $[B_1 : C] = -1, [A : B_1] = -1.$

Case (i) Let  $[B_1 : C] = +1$ . Then for a chain  $A_0 < A_1 < \cdots < A_r = A$

$$\begin{aligned} \varepsilon(A_0, \dots, A_r, B_1) &= \varepsilon(A_0, \dots, A_r, B_1, C), \\ \varepsilon(A_0, \dots, A_r, B_1) &= -\varepsilon(A_0, \dots, A_r, B_2), \\ \varepsilon(A_0, \dots, A_r, B_1, C) &= -\varepsilon(A_0, \dots, A_r, B_2, C). \end{aligned}$$

Thus  $\varepsilon(A_0, \dots, A_r, B_2) = \varepsilon(A_0, \dots, A_r, B_2, C)$ . Hence  $[B_2 : C] = +1$ . Also, since  $[A : B_1] = +1$ ,

$$\varepsilon(A_0, \dots, A_r) = \varepsilon(A_0, \dots, A_r, B_1) = -\varepsilon(A_0, \dots, A_r, B_2).$$

Thus  $[A : B_2] = -1$ . Therefore we have the equality

$$[A : B_1][B_1 : C] = -[A : B_2][B_2 : C].$$

Case (ii) If  $[B_1 : C] = +1$ , then  $[B_2 : C] = +1$  by case (i). Since  $[A : B_1] = -1$ ,  $[A : B_2] = +1$ . Therefore we have the equality in this case. The proofs of the cases (iii) and (iv) are entirely analogous.  $\square$

Let  $\sigma : \mathcal{F} \rightarrow \mathcal{V}$  be an order preserving isomorphism. For  $F \in \mathcal{F}$  and  $C \in \mathcal{C}$ , let  $\sigma(F) = x, \sigma(C) = y$ . If we define  $FC \in \mathcal{C}$  by  $\sigma^{-1}(xy)$ , then  $\mathbb{R}\mathcal{C}$  can be an  $\mathbb{R}\mathcal{F}$ -module. We have the linear map  $\partial_0 : \mathbb{R}\mathcal{C} \rightarrow \mathbb{R}$  given by  $\partial_0(C) = 1$  for all  $C \in \mathcal{C}$ . For each  $F \in \mathcal{F}$ ,  $F$  acts as the identity on  $\mathbb{R}$ . Thus  $\partial_0$  is an  $\mathbb{R}\mathcal{F}$ -module homomorphism and  $\ker \partial_0$  is an  $\mathbb{R}\mathcal{F}$ -module.

Let  $\mathcal{F}_1 \subset \mathcal{F}$  be the set of codimension 1 face, that is,

$$\mathcal{F}_1 = \{F \mid \text{codim } F = 1\}.$$

Define  $\partial_1 : \mathbb{R}\mathcal{F}_1 \rightarrow \ker \partial_0$  by the following. For each  $A \in \mathcal{F}_1$  and for two chambers  $C$  and  $C'$  having  $A$  as a face,

$$\partial_1(A) = [A : C]C + [A : C']C'.$$

Define an action of  $\mathcal{F}$  on  $\mathbb{R}\mathcal{F}_1$  as follows. Given  $F \in \mathcal{F}$  and  $A \in \mathcal{F}_1$ ,  $F * A \in \mathcal{F}_1$  is defined by the following. Similarly to the realizable case, if  $\text{supp } F \subseteq H = \text{supp } A$ , then  $FA \in \mathcal{F}_1$ . Thus, in this case  $F * A = FA$ . If  $\text{supp } F \not\subseteq \text{supp } A$ , then  $FA$  is a chamber and we set  $F * A = 0$ .

**LEMMA 2.3.** *Let  $F \in \mathcal{F}$  and  $A \in \mathcal{F}_1$ . Also, let  $C$  and  $C'$  be chambers having  $A$  as a face. Then*

(1) *If  $\text{supp } F \not\subseteq \text{supp } A$ , then  $FA = FC = FC'$ .*

(2) *If  $\text{supp } F \subseteq \text{supp } A$ ,  $FC$  and  $FC'$  are two chambers having  $FA$  as a face.*

*Proof.* (1) If  $i \in \text{supp } F = \{i \mid \sigma_i(F) \neq 0\}$  and  $i \notin \text{supp } A$ , then  $\sigma_i(FA) = \sigma_i(F) = \sigma_i(FC) = \sigma_i(FC')$ . If  $\sigma_j(F) = 0$  and  $\sigma_j(A) \neq 0$ , then

$$\sigma_j(FA) = \sigma_j(A) = \sigma_j(C) = \sigma_j(FC) = \sigma_j(FC').$$

Thus (1) is proved.

(2) Let  $\text{supp } F \subset \text{supp } A$ . If  $\sigma_j(F) = 0$  and  $\sigma_j(A) \neq 0$ ,

$$\sigma_j(FA) = \sigma_j(A) = \sigma_j(C) = \sigma_j(FC) = \sigma_j(FC').$$

Thus for each  $j \in \text{supp } A$ ,  $\sigma_j(FA) = \sigma_j(FC) = \sigma_j(FC')$ . If  $i \notin \text{supp } A$ ,  $\sigma_i(FC) = \sigma_i(C)$  and  $\sigma_i(FC') = \sigma_i(C')$ . Therefore  $FA$  is a face of both  $FC$  and  $FC'$ .  $\square$

**LEMMA 2.4.** *For each  $F \in \mathcal{F}$  and for each  $A \in \mathcal{F}_1$ ,  $\partial_1(F * A) = F \cdot \partial_1(A)$ . That is,  $\partial_1$  is a chain homomorphism.*

*Proof.* Let  $C$  and  $C'$  be the chambers having  $A$  as a face. Let  $\text{supp } F \not\subseteq \text{supp } A$ . By Lemma 2.3 (1),  $FA = FC = FC'$ . In this case,

$$\begin{aligned} F\partial_1 A &= F([A : C]C + [A : C']C') = [A : C]FC + [A : C']FC' \\ &= \pm FC \mp FC' = 0. \end{aligned}$$

Also  $F * A = 0$  and so  $\partial_1(F * A) = 0$ .

If  $\text{supp } F \subset \text{supp } A$ ,

$$\begin{aligned} \partial_1(F * A) &= \partial_1(FA) \quad (\text{Lemma 2.3 (2)}) \\ &= [FA : FC]FC + [FA : FC']FC' \\ \text{and } F\partial_1(A) &= F([A : C]C + [A : C']C') \\ &= [A : C]FC + [A : C']FC'. \end{aligned}$$

Since  $\text{supp } FA = \text{supp } A$ ,  $[FA : FC] = \pm\sigma_i(FC) = \pm\sigma_i(C) = [A : C]$  and  $[FA : FC'] = \pm\sigma_i(FC') = \pm\sigma_i(C') = [A : C']$  by Lemma 2 in [3],  $\partial_1(F * A) = F\partial_1(A)$ .  $\square$

Let  $\mathcal{S}_p = \{W \in \mathcal{S} : \text{codim}(W, V) = p\}$  and let  $\mathcal{C}_W$  be the set of faces with support  $W$ . Let  $\mathcal{F}_p = \{F \mid \text{codim}(F, V) = p\}$ . For each  $F \in \mathcal{F}$  and  $A \in \mathcal{F}_p$ , set

$$F * A = \begin{cases} FA, & \text{if } F \subseteq \text{supp } A \\ 0, & \text{otherwise.} \end{cases}$$

This makes  $\mathbb{R}\mathcal{F}_p$  an  $\mathbb{R}\mathcal{F}$ -module. Let

$$\mathbb{R}\mathcal{F}_p = \bigoplus_{W \in \mathcal{S}_p} \mathbb{R}\mathcal{C}_W.$$

Define

$$\partial_p : \mathbb{R}\mathcal{F}_p \rightarrow \mathbb{R}\mathcal{F}_{p-1}$$

by  $\partial_p(A) = \sum_{B > A} [A : B]B$ . Then by Lemma 2.2,  $\partial_{p-1}\partial_p = 0$ .

LEMMA 2.5. For each  $F \in \mathcal{F}$  and  $A \in \mathcal{F}_p$ ,  $\partial_p(F * A) = F\partial_p(A)$ .

*Proof.* Let  $B$  be the face of codimension  $p - 1$  with  $A < B$ . Let  $W = \text{supp } B$ . Then  $A$  is a codimension 1 face of the oriented matroid  $\mathcal{V}_W$ . Thus we have the result by Lemma 2.4.  $\square$

THEOREM 2.6. The sequence

$$(2.2) \quad \cdots \longrightarrow \bigoplus_{W \in \mathcal{S}_2} \mathbb{R}\mathcal{C}_W \longrightarrow \bigoplus_{H \in \mathcal{S}_1} \mathbb{R}\mathcal{C}_H \longrightarrow \mathbb{R}\mathcal{C} \longrightarrow \mathbb{R} \longrightarrow 0$$

is an exact chain complex of  $\mathbb{R}\mathcal{F}$ -modules.



*Proof.* By Lemma 2.5,  $\partial_p$  is an  $\mathbb{R}\mathcal{F}$ -module homomorphism. Also, by Lemma 2.2, (2.2) is a chain complex. Also the opposite poset  $\mathcal{F}^{op}$  is isomorphic to a cell complex  $\Delta$ , which is topologically  $n$ -ball ([2], corollary 4.3.4), where 0 corresponds to the  $n$ -cell. Thus the sequence (2.2) is an exact chain complex.  $\square$

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