Bull. Korean Math. Soc. **43** (2006), No. 1, pp. 1–7

BOUNDED MATRICES OVER REGULAR RINGS

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Abstract. In this paper, we investigate bounded matrices over regular rings. We observe that every bounded matrix over a regular ring can be described by idempotent matrices and invertible matrices. Let $A, B \in M_n(R)$ be bounded matrices over a regular ring R. We prove that $(AB)^d = U(BA)^d U^{-1}$ for some $U \in GL_n(R)$.

Let I be an ideal of a ring R. We say that I is a bounded ideal of R in case there exists a positive integer m such that $x^m = 0$ for all nilpotent $x \in I$. We say that $A \in M_n(R)$ is a bounded matrix provided that $M_n(R)AM_n(R)$ is a bounded ideal of $M_n(R)$. An element $x \in R$ is regular if there exists $y \in R$ such that $x = xyx$. A ring R is said to be regular in case every element in R is regular. In this paper, we investigate bounded matrices over regular rings. We observe that every bounded matrix over a regular ring can be described by idempotent matrices and invertible matrices. Let $A, B \in M_n(R)$ be bounded matrices over a regular ring R. It is shown that $(AB)^d = U(BA)^d U^{-1}$ for some $U \in GL_n(R)$.

Throughout, all rings are associative rings with identities. $U(R)$ denotes the set of units of R, $M_n(R)$ denotes the ring of $n \times n$ matrices over R and $GL_n(R)$ stands for the n dimensional general linear group of R.

Lemma 1. *Let $A \in M_n(R)$ be a bounded matrix over a regular ring R. Then there exists a bounded ideal I of R such that $A \in M_n(I)$.*

Proof. Since A is a bounded matrix, $M_n(R)AM_n(R)$ is a bounded ideal of $M_n(R)$. Let e_{ij} be a usual matrix units($1 \leq i, j \leq n$), i.e., in the (i, j)-position its entry is 1; otherwise, its entries are 0. One easily checks that $e_{ij}M_n(R)AM_n(R)e_{ij} \cong Ra_{ij}R$ and $e_{ij}M_n(R)e_{ij} \cong R$. That is, $Ra_{ij}R$ is a bounded ideal of R. By [9, Corollary 6.7], the sum of two ideals with index at most m must have index at most m; hence, we see

Received August 23, 2005.
2000 Mathematics Subject Classification: 16E50, 16U99.
Key words and phrases: bounded matrix, idempotent matrix, invertible matrix.

that $I = \sum\limits_{1\leq i,j\leq n} Ra_{ij}R$ is a bounded ideal of R. Clearly, $A \in M_n(I)$. Therefore we complete the proof. □

A square matrix A over a ring R is said to admit a diagonal reduction if there exist some invertible matrices P and Q such that PAQ is a diagonal matrix. It is well known that every square matrix over unit-regular rings admits a diagonal reduction(cf. [10, Theorem 3]). P. Ara et al. have extended this result to separative exchange rings (cf. [2, Theorem 2.4]). On the other hand, Menal and Moncasi [11] showed that the diagonalizability for some rectangular matrix over some regular rings fails. Now we observe the following result.

THEOREM 2. *Every bounded matrix over a regular ring admits a diagonal reduction.*

Proof. Let $A = (a_{ij}) \in M_n(R)$ be a bounded matrix over a regular ring R. By Lemma 1, there exists a bounded ideal I of R such that $A \in M_n(I)$. Using [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $a_{ij} \in eRe$, and so $A \in M_n(eRe)$. As $e \in I$, we deduce that eRe is unit-regular. Applying [10, Theorem 3], there exist some $U', V' \in GL_n(eRe)$ such that $U'AV' = diag(r_1, \ldots, r_n)$ for some $r_1, \ldots, r_n \in eRe$. Set $U = U' + diag(1-e, \ldots, 1-e)$ and $V = V' + diag(1-e, \ldots, 1-e)$. Then $U, V \in GL_n(R)$. Furthermore, we have $UAV = U'AV' = diag(r_1, \ldots, r_n)$, as asserted. □

COROLLARY 3. *Every $n \times n (n \geq 2)$ bounded matrix over a regular ring is a sum of two invertible matrices.*

Proof. Let $A = (a_{ij}) \in M_n(R)(n \geq 2)$ be a bounded matrix over a regular ring R. In view of Theorem 2, there exist $U, V \in GL_n(R)$ such that $UAV = diag(r_1, \ldots, r_n)$ for some $r_1, \ldots, r_n \in R$. Clearly, $diag(r_1, r_2, \ldots, r_n)$ is a sum of two invertible matrices, i.e., we have

$$diag(r_1, r_2, \ldots, r_n) = \begin{pmatrix} r_1 & 1 & \cdots & 0 & 0 \\ 0 & r_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{n-1} & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & r_n \end{pmatrix}.$$

Therefore we get

$$A = U^{-1}\begin{pmatrix} r_1 & 1 & \cdots & 0 & 0 \\ 0 & r_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{n-1} & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}V^{-1}$$

$$+U^{-1}\begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \cdots & 0 & r_n \end{pmatrix}V^{-1},$$

as desired. □

COROLLARY 4. *Let R be a regular ring with $1/2 \in R$. Then every $n \times n$ bounded matrix over R is a sum of two invertible matrices.*

Proof. If $n \geq 2$, then the result holds by Corollary 3. We now assume that $n = 1$. Let $x \in R$ such that RxR is a bounded ideal of R. In view of [11, Lemma 1.1], we have an idempotent $e \in RxR$ such that $x \in eRe$. As $eRe \subseteq RxR$, we deduce that eRe is a regular ring of bounded index; hence, it is unit-regular. Thus we have an idempotent $f \in eRe$ and a unit $v \in eRe$ such that $x = fv$. Let $u = v + (1 - e)$. Then we have $x = f\big(v + (1 - e)\big) = \big(1/2 + (2f - 1)/2\big)u = u/2 + (2f - 1)u/2$. Clearly, $u/2, (2f - 1)u/2 = \big(u^{-1}(4f - 2)\big)^{-1} \in U(R)$. Therefore we get the result. □

A ring R is said to be a clean ring in case every element in R is a sum of an idempotent and a unit. We know that every strongly π-regular ring is a clean ring (cf. [15, Theorem 1]). That author proved that every exchange ring with artinian primitive factors is a clean ring (see [5, Theorem 1]). A natural problem is how to extend this fact to matrices over a ring which is not a clean ring.

THEOREM 5. *Every bounded matrix over a regular ring is a sum of an idempotent matrix and an invertible matrix.*

Proof. Let $A = (a_{ij}) \in M_n(R)$ be a bounded matrix over a regular ring R. In view of Lemma 1, there exists a bounded ideal I of R such that $A \in M_n(I)$. Since all $a_{ij} \in I$, by [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $a_{ij} \in eRe$; hence, $A \in M_n(eRe)$. Clearly, eRe is a regular ring of bounded index. It follows by [9, Theorem 7.12] that $M_n(eRe)$ is a regular ring of bounded index. Using [9, Theorem 7.15], we know that $M_n(eRe)$ is a strongly π-regular ring; hence, it is a clean ring by [15, Theorem 1]. Thus we have an idempotent matrix $E' \in M_n(eRe)$ and an invertible $U' \in M_n(eRe)$ such that $A = E + U$. Therefore $A = \big(E' + diag(1-e,\dots,1-e)\big) + \big(U' - diag(1-e,\dots,1-e)\big)$. Let $E = E' + diag(1-e,\dots,1-e)$ and $U = U' - diag(1-e,\dots,1-e)$. Then $E = E^2 \in M_n(R)$ and $U \in GL_n(R)$. In addition, we have $A = E + U$. Thus the result follows. □

Analogously, we deduce that every bounded matrix over a regular ring is a product of an idempotent matrix and an invertible matrix. We denote the set of all lower triangular matrices by $\mathfrak{L}$, i.e., $\mathfrak{L} = \{(a_{ij}) \mid a_{ij} = 0 \text{ whenever } i < j\}$, and denote the set of all upper triangular matrices by $\mathfrak{U}$, i.e., $\mathfrak{U} = \{(a_{ij}) \mid a_{ij} = 0 \text{ whenever } i > j\}$.

LEMMA 6. *Let $A \in M_n(R)$ be a matrix over a unit-regular ring R. Then A can be written as $A = LUM, L \in \mathfrak{L}, U \in \mathfrak{U}, M \in \mathfrak{L}$ and in U and M all the diagonal entries are equal to* 1.

Proof. Obviously, R is a Hermite ring. On the other hand, R has stable range one. Therefore we get the result by [14, Theorem 3.1]. □

THEOREM 7. *Every bounded matrix over a regular ring is a product of at most three triangular matrices.*

Proof. Let $A = (a_{ij}) \in M_n(R)$. According to Lemma 1, there exists a bounded ideal I of R such that all $A \in M_n(I)$. By [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $a_{ij} \in eRe$; hence, $A \in M_n(eRe)$. As eRe is a regular ring of bounded index, it follows from [9, Corollary 7.11] that eRe is unit-regular. Thus, by Lemma 6, A can be written as $A = LUM, L \in \mathfrak{L}, U \in \mathfrak{U}, M \in \mathfrak{L}$ and in U and M all the diagonal entries are equal to e. One directly checks that $A = \big(L + diag(1-e,\dots,1-e)\big)\big(U + diag(1-e,\dots,1-e)\big)M$ and in $L + diag(1-e,\dots,1-e)$ and $U + diag(1-e,\dots,1-e)$ all the diagonal entries are equal to 1. □

COROLLARY 8. *Let $A = (a_{ij}) \in M_n(R)$. If all $Ra_{ij}R$ are bounded ideals of R, then A is a product of at most three triangular matrices.*

Proof. Let $I = \sum\limits_{1\leq i,j\leq n} Ra_{ij}R$. By [9, Corollary 7.8], I is a bounded ideal of R. It follows by [11, Lemma 1.1] that $A \in M_n(eRe)$ for some idempotent $e \in I$. Clearly, $M_n(eRe)$ is a regular ring of bounded index from [9, Theorem 7.12]. As a result, A is a bounded matrix over eRe. Using Theorem 7, A can be written as $A = LUM, L \in \mathfrak{L}, U \in \mathfrak{U}, M \in \mathfrak{L}$ and in U and M all the diagonal entries are equal to e. Similarly to Theorem 7, we get $A = \big(L+diag(1-e,\dots,1-e)\big)\big(U+diag(1-e,\dots,1-e)\big)M$ and in $L+diag(1-e,\dots,1-e)$ and $U+diag(1-e,\dots,1-e)$ all the diagonal entries are equal to 1. $\square$

Let $A = (a_{ij}) \in M_n(R)$. If all $Ra_{ij}R$ are nil ideals of bounded index, by Corollary 8, we see that A is a product of at most three triangular matrices.

Recall that a matrix $A \in M_n(R)$ has the Drazin inverse in case there exist a positive integer m and a matrix $X \in M_n(R)$ such that $A^m = A^{m+1}X, AX = XA$ and $X = XAX$. Clearly, the solution X is unique, and we say that X is the Drazin inverse A^d of A.

THEOREM 9. *Let $A, B \in M_n(R)$ be bounded matrices over a regular ring R. Then there exists an invertible matrix U such that $(AB)^d = U(BA)^dU^{-1}$.*

Proof. Assume that $AB = (c_{ij}), BA = (d_{ij}) \in M_n(R)$. Set $I = \sum\limits_{1\leq i,j\leq n} Rc_{ij}R + \sum\limits_{1\leq i,j\leq n} Rd_{ij}R$. Since A and B are both bounded matrices, so are AB and BA. Similarly to Lemma 1, we show that all $Rc_{ij}R$ and all $Rd_{ij}R$ are bounded ideal of R. It follows by [9, Corollary 7.8] that I is a bounded ideal of R. Using [11, Lemma 1.1], we have an idempotent $e \in I$ such that all $c_{ij} \in eRe$ and all $d_{ij} \in eRe$. Clearly, eRe is a regular ring of bounded index; hence, so is $M_n(eRe)$ by [9, Theorem 7.12]. It follows by [9, Theorem 7.15] that $M_n(eRe)$ is strongly π-regular. That is, $AB, BA \in M_n(eRe)$ have the Drazin inverses. In addition, $M_n(eRe)$ has stable range one by [1, Theorem 4]. Therefore there exists some $V \in GL_n(eRe)$ such that $(AB)^d = V(BA)^dV^{-1}$ by [7, Theorem 1.2]. Set $U = V + diag(1-e,\dots,1-e)$. As $AB, BA \in M_n(eRe)$, by the uniqueness of the Drazin inverses of AB and BA, we deduce that $(AB)^d, (BA)^d \in M_n(eRe)$. Therefore $(AB)^d = U(BA)^dU^{-1}$, as asserted. $\square$

COROLLARY 10. *Let $A, B \in M_n(R)$ be bounded matrices over a regular ring R. Then the following are equivalent:*

(1) $AM_n(R) \cong BM_n(R)$.
(2) *There exist some $U, V \in GL_n(R)$ such that $A = UBV$.*

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) Since R is a regular ring, so is $M_n(R)$. Since $A = (a_{ij})$ is a bounded matrix, by Lemma 1, there exists a bounded ideal I of R such that $A \in M_n(I)$. According to [11, Lemma 1.1], we have an idempotent $e \in I$ such that $A \in M_n(eRe)$. Clearly, eRe is a regular ring of bounded index, and so it is unit-regular. Using [9, Corollary 4.7], $M_n(eRe)$ is unit-regular. Thus we have some $C' \in GL_n(eRe)$ such that $A = AC'A$. Set $C = C' + diag(1-e, \dots, 1-e)$. Then $A = ACA$ with $C \in GL_n(R)$. Similarly, we have some $D \in GL_n(R)$ such that $B = BDB$. Set $E = AC$ and $F = BD$. Then $E, F \in M_n(R)$ are idempotent matrices and $EM_n(R) \cong FM_n(R)$. Thus we get $G \in EM_n(R)F$ and $H \in FM_n(R)E$ such that $E = GH$ and $F = HG$. One easily checks that $M_n(R)GM_n(R) \subseteq M_n(R)EM_n(R) \subseteq M_n(R)AM_n(R)$; hence, G is a bounded matrix. Likewise, H is a bounded index. By virtue of Theorem 9, we have $U, V' \in GL_n(R)$ such that $E^d = UF^dV'$. That is, $E = UFV'$. Set $V = DV'C^{-1}$. Therefore $A = EC^{-1} = UFV'C^{-1} = UBDV'C^{-1} = UBV$, as asserted. □

COROLLARY 11. *Let A and B be $n \times n$ matrices over a bounded ideal of a regular ring R. Then the following are equivalent:*

(1) $AM_n(R) \cong BM_n(R)$.
(2) *There exist some $U, V \in GL_n(R)$ such that $A = UBV$.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) Suppose that $A = (a_{ij}), B = (b_{ij}) \in M_n(I)$ and I is a bounded ideal of a regular ring R. By [11, Lemma 1.1], there exists an idempotent $e \in I$ such that all $a_{ij}, b_{ij} \in eRe$; hence, $A, B \in M_n(eRe)$. Clearly, eRe is a regular ring of bounded index, so is $M_n(eRe)$. Thus we see that A and B are both bounded matrices over eRe. It follows from $AM_n(R) \cong BM_n(R)$ that $AM_n(eRe) \cong BM_n(eRe)$. Using Corollary 10, we have $U', V' \in GL_n(eRe)$ such that $A = U'BV'$. Set $U = U' + diag(1-e, \dots, 1-e)$ and $V = V' + diag(1-e, \dots, 1-e)$. Then $A = UBV$ and $U, V \in GL_n(R)$, as asserted. □

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