J. Korean Math. Soc. **43** (2006), No. 2, pp. 383–398

APPROXIMATIONS OF OPTION PRICES FOR A JUMP-DIFFUSION MODEL

IN-SUK WEE

ABSTRACT. We consider a geometric Lévy process for an underlying asset. We prove first that the option price is the unique solution of certain integro-differential equation without assuming differentiability and boundedness of derivatives of the payoff function. Second result is to provide convergence rate for option prices when the small jumps are removed from the Lévy process.

1. Introduction

We consider a geometric Lévy process for an underlying asset;

$$dS_t = S_{t-}dY_t, \tag{1.1}$$

where Y is a Lévy process satisfying some additional conditions. Unlike the Black-Scholes model, it is well-known that the market is incomplete, hence there are many martingale measures from which we can choose. We select the minimal martingale measure which was introduced in [3] and the price of an European contingent claim is expressed as the conditional expectation of discounted payoff with respect to the minimal martingale measure. In the Black-Scholes model, the price can be computed explicitly and satisfies certain differential equation if the payoff function is continuous and satisfies certain polynomial growth condition. For the model considered here in (1.1), the corresponding statements for computation of the price are no longer true. First the valuation formula for the price cannot be explicitly given in general except when the Lévy process is compound Poisson. Second it is implicitly assumed

Received December 30, 2004. Revised February 23, 2005.
2000 Mathematics Subject Classification: Primary 91B28, 60H30.
Key words and phrases: Black-Scholes model, jump-diffusion model, Lévy process, option price.
This work was supported by grant No. R04-2000-000-00005-0 from the Korea Science Foundation and by the Korea Research Foundation Grant KRF 2003-070-C00008.

that the price satisfies the corresponding integro-differential equation. To justify rigorously, one may prove that a nice solution to the integro-differential equation exists, and that the standard Feynman-Kac type result holds. But to my best knowledge, the existence of solution to the integro-differential equation can be proved under more restrictive conditions such as existence of bounded first and second derivatives of the payoff function which can not be applied even to an European call and put (see [4]).

There are two-fold aims of this paper. The first purpose is to provide a rigorous proof that the price is a solution of the integro-differential equation when the conditions on the payoff function are relaxed to the level in the case of the differential equation. This solves the difficultly not only in pricing but also in finding the hedging strategy, for which it is required to find the derivatives of the price formula (see [7]). The second purpose lies in justifying the practical simulation, in which Lévy process is often approximated to compound Poisson precess by removing the small jumps. It is then intuitively clear that the option price for (1.1) can be approximated by the option prices with compound Poisson processes replacing Lévy process. This is often useful, since when the model is generated by a compound Poisson process, the option price can be expressed explicitly as a series of Black-Scholes formulas incorporated with Poisson distributions. We verify the convergence of the option prices and provide the convergence rates in terms of Lévy measure near zero.

Throughout the paper, unless otherwise specified, we denote C to be a generic positive constant depending on insignificant variables whose value may vary from line to line.

2. Preliminaries

Suppose that the price of a risky asset is described by (1.1), where Y is a Lévy process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. The characteristic function of Y is given by

$$E\exp(iuY_t) = \exp(t\psi(u)),$$

where

$$\begin{aligned}\psi(u) &= i\beta u - \frac{\sigma^2}{2}u^2 + \int_{\{|x|\le 1\}}(e^{iux} - 1 - iux)\nu(dx)\\ &\quad + \int_{\{|x|>1\}}(e^{iux} - 1)\nu(dx)\end{aligned}$$

and

$$\int \min\{1, x^2\}\nu(dx) < \infty.$$

The notations and definition are basically from [7]. We assume that the filtration $\{\mathcal{F}_t\}$ is the minimal one generated by Y and satisfies the usual conditions. For the purposes of our work, we make the following assumptions:

$$(A1) \qquad \int_{\{|x|\geq 1\}} x^2\nu(dx) < \infty,$$

$$(A2) \qquad \int (\ln(1+x))^2\nu(dx) < \infty,$$

$(A3)$ The support of ν is contained in $(-1,\infty)$.

From the Lévy decomposition of Y and (A1), we have

$$(2.1) \qquad \begin{aligned} Y_t &= \sigma B_t + \int_0^t\int y(\mu(ds,dy) - \nu(dy)ds) + tE(Y_1) \\ &\equiv \sigma B_t + M_t + at, \end{aligned}$$

where $\mu(dt,dy)$ is a Poisson measure on $R^+ \times R - \{0\}$ with intensity measure $dt\nu(dy)$.

We denote by r the riskeless interest rate and let $\hat{S}_t = e^{-rt}S_t$. In [2], it was shown that the minimal martingale measure Q is given by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = Z_t,$$

where

$$Z_t = 1 + \int_0^t \gamma Z_{s-}(\sigma dB_t + dM_t),$$

and

$$\gamma = \frac{r-a}{\sigma^2 + \int y^2\nu(dy)}.$$

Moreover under Q,

$$\tilde{B}_t = B_t - \gamma\sigma t$$

is a Brownian motion and the process

$$\tilde{M}_t = M_t - t\int \gamma y^2\nu(dy) = \int_0^t\int y(\mu(ds,dy) - \tilde{\nu}(dy)ds)$$

is a martingale where

$$\tilde{\nu}(dy) = (1 + \gamma y)\nu(dy).$$

Finally we need the following condition to ensure $Z_t > 0$ almost surely for all t.

(A4) Either (i) or (ii) holds;

(i) $0 < r - a < \sigma^2 + \int y^2 \nu(dy)$,

(ii) $r - a < 0$ and the support of ν is contained in $(-1, -\frac{1}{\gamma})$.

Now we consider an European contingent claim with payoff $\varphi(S_T)$ at the maturity T. Let the price with respect to the minimal martingale measure Q be given by

$$E_Q \left(e^{-r(T-t)} \varphi(S_T) \middle| \mathcal{F}_t \right) = u(t, S_t). \tag{2.2}$$

3. Solution of integro-differential equation

For the Black-Scholes model, it is well known that the option price is the unique solution to Cauchy problem of valuation PDE under some growth condition on φ. For a more general diffusion model, Feynman-Kac type result was proved under less restrictive analytic conditions on the SDE coefficients in [6]. In this section, we prove that the option price is the unique solution to the Cauchy problem of the corresponding integro-differential equation under a polynomial growth condition on φ. One possible approach seems to formulate and prove the Feynman-Kac type result. But one must show that there exists a solution to the Cauchy problem of the integro-differential equation beforehand so that Ito's formula can be applied. But to my best knowledge, a solution for the integro-differential equation exists under very restrictive conditions such as differentiability and boundedness of derivatives of φ which does not hold even for an European call option. (e.g. [4])

In this section, we prove that $u(t, x)$ in (2.2) is a unique solution of the Cauchy problem of the given integro-differential equation under a polynomial growth condition on φ without requiring differentiability. The result seems to be not very surprising, but it has been overlooked in the literatures (e.g. [2]) and needs to be verified. We confine to deal with a linear integro-differential equation. The following is the main result of this section.

THEOREM 3.1. *Suppose that φ is continuous for $x > 0$ and there exist $k \geq 2$, $p > 1$ and $\epsilon > 0$ such that*

$$|\varphi(x)| \leq C(|x|^k + 1), \tag{3.1}$$

$$\int_{\{|y|\geq 1\}} |y|^{kp}\tilde{\nu}(dy) < \infty, \tag{3.2}$$

$$\int_{\{|1+y|<\epsilon\}} (1+y)^{-2p}\tilde{\nu}(dy) < \infty. \tag{3.3}$$

Then $u(t,x)$ satisfies the Cauchy problem of the following integro-differential equation;

$$\begin{aligned} \partial_t u(t,x) &+ \frac{1}{2}\sigma^2 x^2 \partial_{xx} u(t,x) + rx\partial_x u(t,x) - ru(t,x) \\ &+ \int (u(t,x(1+y)) - u(t,x) - xy\partial_x u(t,x))\tilde{\nu}(dy) \\ &= 0 \text{ on } (0,T) \times R^+, \qquad (3.4) \\ u(T,x) &= \varphi(x) \text{ on } R^+. \end{aligned}$$

Moreover u is the unique solution which satisfies, for any $0 \leq t \leq T$, and $x > 0$,

$$|u(t,x)| \leq C(x^k + 1). \tag{3.5}$$

In the rest of this section, we prove the theorem by presenting and combining a few lemmas. Let for $0 < t < T$,

$$U(t,x) = e^{-r(T-t)} E_Q \varphi \left(x \exp \left\{ \left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma(\tilde{B}_T - \tilde{B}_t) \right\} \right),$$
$$V(t,x) = U(T-t, e^x).$$

Then

$$u(t,x) = E_Q U(t, \xi_{t,x}(T)) = E_Q V(T-t, \eta_{t,\ln x}(T)),$$

where for $t < s < T$,

$$\xi_{t,x}(s) = x + \int_t^s \int \xi_{t,x}(\tau) y(\mu(d\tau, dy) - \tilde{\nu}(dy)d\tau),$$
$$\eta_{t,\ln x}(s) = \ln \xi_{t,x}(s).$$

From elementary calculation,

$$V(t,x) = \int \Gamma(t, x-y)\tilde{\varphi}(y)dy,$$

where

$$\Gamma(t,x) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\left(-\frac{x^2}{2\sigma^2 t}\right) e^{\alpha x + \beta t},$$

$$\alpha = \frac{r - \frac{1}{2}\sigma^2}{\sigma^2}, \quad \beta = -\frac{(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2},$$

$$\tilde{\varphi}(x) = \varphi(e^x).$$

It is also easy to check that if (3.1) holds, then for any x, t, and $l = 0, 1, 2$,

$$|\partial^l V(t,x)| \le C t^{-\frac{1+l}{2}} (e^{kx} + 1), \tag{3.6}$$

where ∂^l denotes differentiation of order l with respect to x and t.

LEMMA 3.1. *If for some* $k > 1$, (3.1) *holds and*

$$\int_{\{|y|>1\}} |y|^k \tilde{\nu}(dy) < \infty, \tag{3.7}$$

then for $x > 0$,

$$\partial_x u(t,x) = \frac{1}{x} E_Q \left(\partial_x V(T-t, \eta_{t,\ln x}(T)) \right), \tag{3.8}$$

$$\begin{aligned} \partial_{xx} u(t,x) = &\ \frac{1}{x^2} E_Q \left(\partial_{xx} V(T-t, \eta_{t,\ln x}(T)) \right) \\ &- \frac{1}{x^2} E_Q \left(\partial_x V(T-t, \eta_{t,\ln x}(T)) \right). \end{aligned} \tag{3.9}$$

Proof. Note that for $|z| \le |\ln(1 + h/x)|$,

$$\begin{aligned} & \left| V(T-t, \eta_{t,\ln(x+h)}(T)) - V(T-t, \eta_{t,\ln x}(T)) \right| / |h| \\ = &\ |\partial_x V(T-t, \eta_{t,\ln x}(T) + z)| \, |\ln(1+h/x)| / |h| \\ \le &\ C(t) \left((\xi_{t,x}(T))^k (1+h/x)^k + 1 \right) |\ln(1+h/x)| / |h|. \end{aligned}$$

Applying Ito's formula (e.g. [5]), we obtain

$$\begin{aligned} & E_Q(\xi_{t,x}(T))^k \\ = &\ x^k + \left(\int_t^T E_Q(\xi_{t,x}(s))^k ds \right) \int ((1+y)^k - 1 - ky) \tilde{\nu}(dy), \end{aligned}$$

hence

$$E_Q(\xi_{t,x}(T))^k = x^k \exp\left\{ (T-t) \int ((1+y)^k - 1 - ky) \tilde{\nu}(dy) \right\}. \tag{3.10}$$

Using this and applying dominated convergence theorem, (3.8) follows, and (3.9) can be proved analogously. □

LEMMA 3.2. *Under the assumptions of Theorem 3.1,*

$$\partial_t u(t,x) = -E_Q\left(\partial_t V(T-t,\eta_{t,\ln x}(T))\right) - \int (u(t,x(1+y)) - u(t,x) - \partial_x u(t,x)xy)\tilde{\nu}(dy).$$

Proof. We write for $h > 0$,

$$\begin{aligned} & u(t,x) - u(t-h,x) \\ (3.11) \quad & = u(t,x) - E_Q V(T-t,\eta_{t-h,\ln x}(T)) \\ & + E_Q(V(T-t,\eta_{t-h,\ln x}(T)) - V(T-t+h,\eta_{t-h,\ln x}(T))). \end{aligned}$$

First we show that

(3.12)
$$\begin{aligned} & \lim_{h\to 0} E_Q\left(V(T-t,\eta_{t-h,\ln x}(T)) - V(T-t+h,\eta_{t-h,\ln x}(T))\right)/h \\ = & -E_Q\left(\partial_t V(T-t,\eta_{t,\ln x}(T))\right). \end{aligned}$$

By (3.6) and (3.10), it is easy to calculate that

$$\begin{aligned} & E_Q\left|V(T-t,\eta_{t-h,\ln x}(T)) - V(T-t+h,\eta_{t-h,\ln x}(T))\right|^p/h^p \\ \le & C\left(E_Q(\xi_{t-h,x}(T))^{pk} + 1\right) \\ = & C\left(x^{pk}\exp\left\{(T-t+h)\int((1+y)^{pk} - 1 - pky)\tilde{\nu}(dy)\right\} + 1\right). \end{aligned}$$

Hence

$$\{(V(T-t,\eta_{t-h,\ln x}(T)) - V(T-t+h,\eta_{t-h,\ln x}(T)))/h,\ 0 < h < h_0\}$$

are uniformly integrable, for some $h_0 > 0$.

Since $\eta_{t-h,\ln x}(T) \xrightarrow{L^2} \eta_{t,\ln x}(T)$ as $h \to 0$, (3.12) is proved. It remains to deal with the first term in (3.11). We write

$$\begin{aligned} & E_Q V(T-t,\eta_{t-h,\ln x}(T)) - u(t,x) \\ & = E_Q E_Q(V(T-t,\eta_{t-h,\ln x}(T))|\mathcal{F}_t) - u(t,x) \\ (3.13) \quad & = E_Q u(t,\xi_{t-h,x}(t)) - u(t,x) \\ & = E_Q \int_{t-h}^{t}\int [u(t,\xi_{t-h,x}(s)(1+y)) - u(t,\xi_{t-h,x}(s)) \\ & - \partial_x u(t,\xi_{t-h,x}(s))\xi_{t-h,x}(s)y]\,\tilde{\nu}(dy)ds. \end{aligned}$$

Now we observe that

$$
\begin{aligned}
&E_Q\bigg|\frac{1}{h}\int_{t-h}^{t}\int [u(t,\xi_{t-h,x}(s)(1+y)) - u(t,\xi_{t-h,x}(s)) \\
&\quad - \partial_x u(t,\xi_{t-h,x}(s))\xi_{t-h,x}(s)y]\,\tilde{\nu}(dy)ds\bigg|^p \\
&\le E_Q\frac{1}{h}\int_{t-h}^{t}\bigg|\int [u(t,\xi_{t-h,x}(s)(1+y)) - u(t,\xi_{t-h,x}(s)) \\
&\quad - \partial_x u(t,\xi_{t-h,x}(s))\xi_{t-h,x}(s)y]\,\tilde{\nu}(dy)\bigg|^p ds \\
&\le \frac{C}{h}E_Q\int_{t-h}^{t}\int |\partial_{xx}u(t,\Delta)|^p|\xi_{t-h,x}(s)|^{2p}y^2\tilde{\nu}(dy)ds,
\end{aligned}
\tag{3.14}
$$

where Δ lies between $\xi_{t-h,x}(s)$ and $\xi_{t-h,x(1+y)}(s)$. By Lemma 3.1, we have, for $0 < h \le h_0$

(3.15)

$$
\begin{aligned}
&|\partial_{xx}u(t,\Delta)|(\xi_{t-h,x}(s))^2 \\
&\le \Delta^{-2}E_Q\,|\partial_{xx}V(T-t,\eta_{t,\ln\Delta}(T)) - \partial_x V(T-t,\eta_{t,\ln\Delta}(T))|\,(\xi_{t-h,x}(s))^2 \\
&\le Cx^2\Delta^{-2}\left(\Delta^k E_Q(\xi_{t,1}(T))^k + 1\right)(\xi_{t-h,1}(s))^2 \\
&\le Cx^2\left(\Delta^{k-2}e^{(T-t)\int(1+y)^k-1-ky)\tilde{\nu}(dy)} + \Delta^{-2}\right)(\xi_{t-h,1}(s))^2 \\
&\le Cx^k(\xi_{t-h,1}(s))^k(1+(1+y)^{k-2}) \\
&\quad\times \exp\left\{(T-t)\int(1+y)^k - 1 - ky)\tilde{\nu}(dy)\right\} + C(1+(1+y)^{-2}),
\end{aligned}
$$

where C is independent of t. Getting back to (3.14) and using (3.15) we have that for $0 < h \le h_0$, the last term in (3.14) is less than

$$
\begin{aligned}
&Cx^{kp}e^{(T-t)p\int((1+y)^k-1-ky)\tilde{\nu}(dy)}\int(1+(1+y)^{(k-2)p})y^2\tilde{\nu}(dy) \\
&\quad + C\int(1+y)^{-2p}y^2\tilde{\nu}(dy).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\bigg\{h^{-1}\int_{t-h}^{t}\int [u(t,\xi_{t-h,x}(s)(1+y)) - u(t,\xi_{t-h,x}(s)) \\
&\quad - \partial_x u(t,\xi_{t-h,x}(s))\xi_{t-h,x}(s)y]\tilde{\nu}(dy)ds,\ 0 < h \le h_0\bigg\}
\end{aligned}
$$

are uniformly integrable, and as $h \to 0$, converge to

$$\int (u(t, x(1+y) - u(t,x) - \partial_x u(t,x)xy)\, \tilde{\nu}(dy).$$

Dealing with $u(t+h,x) - u(t,x)$ for $h > 0$ analogously, we complete the proof. □

Proof of Theorem 3.1. Using Lemma 3.1, and Lemma 3.2 and the definition of V, it is easy to obtain that the integro-differential equation in (3.4) holds. The boundary condition in (3.4) follows by the continuity of φ, $\xi_{t,x}(T) \xrightarrow{L^2} x$ as $t \to T$ and

$$E_Q \,|\, U(t, \xi_{t,x}(T))| \le C\left(x^k \exp\left\{T \int ((1+y)^k - 1 - ky)\tilde{\nu}(dy)\right\} + 1\right).$$

To prove (3.5), we note that (3.6) implies

$$\begin{aligned} |u(t,x)| &= |E_Q V(T-t, \eta_{t,\ln x}(T))| \\ &\le C\left(E_Q(\xi_{t,x}(T))^k + 1\right) \\ &= C\left(x^k \exp\left\{(T-t) \int ((1+y)^k - 1 - ky)\tilde{\nu}(dy)\right\} + 1\right). \end{aligned}$$

Finally, we employ the well-known Feynman-Kac type argument to the prove the uniqueness. □

4. Convergence rates for option prices

We remove the small jumps from the Lévy process Y_t in (1.1) and approximate it by the resulting compound Poisson process. Convergence in distribution function with the convergence rate in this regard was verified in [1]. In this section we study the convergence rate of the option prices when the Lévy process is approximated by the compound Poisson processes. It is interesting to note that the convergence rate is determined by $k(\delta)$, where

$$k(\delta) = \int_{\{|y| \le \delta\}} y^2 \nu(dy).$$

It is a meaningful quantity which determines the Berry-Essen type estimate for the difference of distribution functions in [1]. Most often in practice we approximate Y_t by ${Y_t}^\delta$, where

$$\begin{aligned}
{Y_t}^\delta &= \sigma^\delta B_t + \int_0^t \int_{\{|y|>\delta\}} y(\mu(ds,dy) - \nu(dy)ds) + at \\
&\equiv \sigma^\delta B_t + {M_t}^\delta + at, \\
(\sigma^\delta)^2 &= \sigma^2 + \int_{\{|y|\le\delta\}} y^2 \nu(dy).
\end{aligned}$$

It seems natural to consider a stock price ${S_t}^\delta$ derived by ${Y_t}^\delta$ in the place of Y_t. Let

$$d{S_t}^\delta = S_{t-}^\delta d{Y_t}^\delta.$$

The Radon-Nikodym derivative of the minimal martingale measure Q^δ with respect to P is given by

$$Z_t^\delta = 1 + \int_0^t \gamma Z_{s-}^\delta(\sigma^\delta dB_s + d{M_s}^\delta),$$

under which

$$\tilde{B}_t^\delta = B_t - \gamma\sigma^\delta t,$$

and

$$\tilde{M}_t^\delta = \int_0^t \int_{\{|y|>\delta\}} y(\mu(ds,dy) - \tilde{\nu}(dy)ds)$$

are Brownian motion and compensated compound Poisson process respectively. The price of option with the payoff $\varphi({S_T}^\delta)$ on $\{S_t^\delta\}$ is given by

$$E_{Q^\delta}\left(e^{-r(T-t)}\varphi({S_T}^\delta)\middle|\mathcal{F}_t\right) \equiv u^\delta(t, {S_t}^\delta),$$

where

$$\begin{aligned}
&u^\delta(t,x) = E_{Q^\delta} U^\delta(t, \xi_{t,x}^\delta(T)), \\
&U^\delta(t,x) = e^{-r(T-t)} \\
&\qquad \times E_{Q^\delta}\varphi\left(x\exp\left\{\left(r - \frac{(\sigma^\delta)^2}{2}\right)(T-t) + \sigma^\delta\left(\tilde{B}_T{}^\delta - \tilde{B}_t{}^\delta\right)\right\}\right), \\
&\xi_{t,x}^\delta(T) = x + \int_t^T \int_{\{|y|>\delta\}} \xi_{t,x}^\delta(s) y(\mu(ds,dy) - \tilde{\nu}(dy)ds).
\end{aligned}$$

By changing the variables,

$$V^\delta(t,x) = U^\delta(T-t, e^x),$$
$$\eta^\delta_{t,\ln x}(T) = \ln \xi^\delta_{t,x}(T),$$

we have

$$u^\delta(t,x) = E_{Q^\delta} V^\delta(T-t, \eta^\delta_{t,\ln x}(T)),$$
$$V^\delta(t,x) = \int \Gamma^\delta(t, x-y)\tilde{\varphi}(y)dy,$$

where Γ^δ has the analogous expression to Γ with σ^δ replacing σ. It is well-known that $u^\delta(t,x)$ can be explicitly expressed as a sum of modified Black-Scholes formulas. Since the most of Lévy processes are not easily simulated, it is an usual practice to neglect small jumps. Hence with $u^\delta(t,x)$ as an approximation to option price $u(t,x)$, it is meaningful to have an estimate for $u^\delta(t,x) - u(t,x)$. We present the main result of this section about the convergence rate for $u^\delta(t,x) - u(t,x)$ up to the constant. Although the result seems to be new, the proof is routine but not simple which consists of three steps.

THEOREM 4.1. *Suppose that for $k > 1$, (3.1) holds, and $\int_{\{|y|>1\}} |y|^{2k} \tilde{\nu}(dy) < \infty$. Then for fixed $t \in (0,T]$, we have that for small $\delta > 0$, and any $x > 0$*

$$|u^\delta(t,x) - u(t,x)| \le C\sqrt{k(\delta)}(x^k + 1).$$

The following equality allows us to proceed the proof in three steps;

$$\begin{aligned} u^\delta(t,x) - u(t,x) &= E_{Q^\delta} U^\delta(t, \xi^\delta_{t,x}(T)) - E_Q U(t, \xi_{t,x}(T)) \\ &= E_{Q^\delta}[U^\delta(t, \xi^\delta_{t,x}(T)) - U(t, \xi^\delta_{t,x}(T))] \\ &\quad + E_{Q^\delta}[U(t, \xi^\delta_{t,x}(T)) - U(t, \xi_{t,x}(T))] \\ &\quad + E_{Q^\delta} U(t, \xi_{t,x}(T)) - E_Q U(t, \xi_{t,x}(T)). \end{aligned} \tag{4.1}$$

We provide two lemmas for the necessary estimates.

LEMMA 4.1. *Suppose that (3.1) holds for $k > 1$. Then for fixed $0 < t \le T$ we have, for δ small and any $x > 0$,*

$$|U^\delta(t,x) - U(t,x)| \le Ck(\delta)(x^k + 1).$$

Proof. We write

$$V^\delta(t,x) - V(t,x) = \int (\Gamma^\delta(t,x-y) - \Gamma(t,x-y))\tilde{\varphi}(y)dy.$$

Note that for some $0 < \epsilon < 1$, and δ small,

$$|\Gamma^\delta(t,x) - \Gamma(t,x)| \le Ck(\delta)t^{-1/2}\exp\left(-(1-\epsilon)x^2/2\sigma^2 t\right).$$

Then it is easy to check that

$$|V^\delta(t,x) - V(t,x)| \le Ck(\delta)(e^{kx}+1),$$

which implies the assertion. □

LEMMA 4.2. *If* (3.1) *holds for* $k > 1$, *then for fixed* $t \in (0,T]$, *we have for any* $x, y > 0$,

$$|U(t,x) - U(t,y)| \le C(x^k + y^k + 1)\left|\ln x/y\right|.$$

Proof. Note that for some $0 < \epsilon < 1$,

$$\begin{aligned}|\Gamma(t,x) - \Gamma(t,y)| \le C|x-y|t^{-\frac{3}{2}} \\ \times \left[\exp\left(-(1-\epsilon)x^2/2\sigma^2 t\right) + \exp\left(-(1-\epsilon)y^2/2\sigma^2 t\right)\right]\end{aligned}$$

which implies that

$$|V(t,x) - V(t,y)| \le \frac{C}{\sqrt{t}}(e^{kx} + e^{ky} + 1)|x-y|.$$

□

Proof of Theorem 4.1. We begin with the equality (4.1) and develop the proof in the following three steps.

STEP 1. By Lemma 4.1 and (3.10), we have

$$\begin{aligned}&\left|E_{Q^\delta}(U^\delta(t,\xi^\delta_{t,x}(T)) - U(t,\xi^\delta_{t,x}(T)))\right| \\ &\le Ck(\delta)E_{Q^\delta}((\xi^\delta_{t,x}(T))^k + 1) \\ &= Ck(\delta)\left(x^k\exp\left((T-t)\int_{\{|y|>\delta\}}((1+y)^k - 1 - ky)\tilde{\nu}(dy)\right) + 1\right).\end{aligned}$$

STEP 2. Lemma 4.2 implies that

$$
\begin{aligned}
&\left|E_{Q^\delta}\left[U(t,\xi^\delta_{t,x}(T)) - U(t,\xi_{t,x}(T))\right]\right| \\
&\le CE_{Q^\delta}\left(\left|\ln\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right|\left[(\xi^\delta_{t,x}(T))^k + (\xi_{t,x}(T))^k + 1\right]\right) \\
&= CE_{Q^\delta}\left(\left|\ln\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right|\left[1+\left(\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right)^k\right]\right)E_{Q^\delta}(\xi^\delta_{t,x}(T))^k \qquad (4.2)\\
&\quad + CE_{Q^\delta}\left|\ln\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right| \\
&\le Cx^k\exp\left\{(T-t)\int_{\{|y|>\delta\}}((1+y)^k - 1 - ky)\tilde{\nu}(dy)\right\} \\
&\quad \times E_Q\left(\left|\ln\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right|\left(1+\left(\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right)^k\right)\frac{1}{\zeta_\delta(T)}\right) \\
&\quad + CE_Q\left(\left|\ln\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right|\frac{1}{\zeta_\delta(T)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
d\zeta_\delta(t) &= \int_{\{|y|\le\delta\}}\gamma y\zeta_\delta(t)(\mu(dt,dy) - \tilde{\nu}(dy)dt), \\
\zeta_\delta(0) &= 1.
\end{aligned}
$$

By using the similar argument as in (3.10), we obtain

$$
E_Q(\zeta_\delta(T))^{-4} = \exp\left\{T\int_{\{|y|\le\delta\}}((1+\gamma y)^{-4} - 1 + 4\gamma y)\tilde{\nu}(dy)\right\}
$$

and

$$
\begin{aligned}
&E_Q\left(\frac{\xi_{t,x}(T)}{\xi^\delta_{t,x}(T)}\right)^{4k} \\
&= \exp\left\{(T-t)\int_{\{|y|\le\delta\}}((1+y)^{4k} - 1 - 4ky)\tilde{\nu}(dy)\right\}.
\end{aligned}
$$

Note that

$$\ln \frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)} = \int_t^T \int_{\{|y|\le\delta\}} \ln(1+y)(\mu(ds,dy) - \tilde{\nu}(dy)ds) + (T-t)\int_{\{|y|\le\delta\}} (\ln(1+y) - y)\tilde{\nu}(dy).$$

By Ito's formula, we obtain

$$E_Q\left(\ln \frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)}\right)^2 = (T-t)\int_{\{|y|\le\delta\}} (\ln(1+y))^2\tilde{\nu}(dy) + (T^2-t^2)\left(\int_{\{|y|\le\delta\}} (\ln(1+y) - y)\tilde{\nu}(dy)\right)^2 \le C(T-t)k(\delta).$$

Continuing to work on the expectations in (4.2), we have

$$E_Q\left(\left|\ln \frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)}\right| \left(1 + \left(\frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)}\right)^k\right)\frac{1}{\zeta_\delta(T)}\right) \tag{4.3}$$

$$\le C\left[E_Q\left(\ln \frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)}\right)^2\right]^{\frac{1}{2}} \left[1 + E_Q\left(\frac{\xi_{t,x}(T)}{\xi^{\delta}_{t,x}(T)}\right)^{4k}\right]^{\frac{1}{4}} \left[E_Q(\zeta_\delta(T))^{-4}\right]^{\frac{1}{4}}$$

$$\le C\sqrt{k(\delta)}\left(1 + \exp\left\{4^{-1}(T-t)\int_{\{|y|\le\delta\}} ((1+y)^{4k} - 1 - 4ky)\tilde{\nu}(dy)\right\}\right) \times \exp\left\{4^{-1}T\int_{\{|y|\le\delta\}} ((1+\gamma y)^{-4} - 1 + 4\gamma y)\tilde{\nu}(dy)\right\},$$

and analogously

$$E_Q\left(\left|\ln \frac{\xi_{t,x}(T)}{\zeta^{\delta}_{t,x}(T)}\right| \frac{1}{\zeta_\delta(T)}\right) \le C\sqrt{k(\delta)}\exp\left\{2^{-1}T\int_{\{|y|\le\delta\}} ((1+\gamma y)^{-2} - 1 + 2\gamma y)\tilde{\nu}(dy)\right\}. \tag{4.4}$$

Combining (4.2), (4.3), and (4.4), we have for small $\delta > 0$,

$$\begin{aligned}&\left|E_{Q^\delta}U(t,\xi_{t,x}(T)) - E_Q U(t,\xi_{t,x}(T))\right| \\ &\leq C\sqrt{k(\delta)}\left(x^k \exp\left\{(T-t)\int((1+y)^k - 1 - ky)\tilde{\nu}(dy)\right\} + 1\right).\end{aligned}$$

Therefore we have, for δ small, and $x > 0$,

$$\begin{aligned}&\left|E_{Q^\delta}(U(t,\xi^\delta_{t,x}(T)) - U(t,\xi_{t,x}(T)))\right| \\ &\leq Cx^k\sqrt{k(\delta)}e^{(T-t)\int((1+y)^k-1-ky)\tilde{\nu}(dy)}.\end{aligned}$$

STEP 3. We write

$$\begin{aligned}&\left|E_{Q^\delta}U(t,\xi_{t,x}(T)) - E_Q U(t,\xi_{t,x}(T))\right| \\ &\leq \left[E_Q\left(U(t,\xi_{t,x}(T))\right)^2\right]^{\frac{1}{2}}\left[E_Q\left(\frac{Z^\delta(T)}{Z(T)} - 1\right)^2\right]^{\frac{1}{2}} \\ &\leq C\left[E_Q(\xi_{t,x}(T))^{2k} + 1\right]^{\frac{1}{2}}\left[E_Q\left(\frac{Z^\delta(T)}{Z(T)} - 1\right)^2\right]^{\frac{1}{2}} \\ &= C\left(x^k\exp\left(2^{-1}(T-t)\int\left((1+y)^{2k} - 1 - 2ky\right)\tilde{\nu}(dy)\right) + 1\right) \\ &\quad\times\left[E_Q\left(\frac{Z^\delta(T)}{Z(T)} - 1\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Note that

$$\begin{aligned}E_Q\frac{Z^\delta(T)}{Z(T)} &= 1, \\ E_Q\left(\frac{Z^\delta(T)}{Z(T)}\right)^2 &= \exp\Bigg\{\gamma^2(\sigma^\delta - \sigma)^2 T \\ &\quad + T\int_{\{|y|\leq\delta\}}((1+\gamma y)^{-2} - 1 + 2\gamma y)\tilde{\nu}(dy) \\ &\quad - 2T\int_{\{|y|\leq\delta\}}\gamma^2 y^2\nu(dy)\Bigg\},\end{aligned}$$

imply

$$E_Q\left(\frac{Z^\delta(T)}{Z(T)} - 1\right)^2 \leq Ck(\delta),$$

which completes the proof of Theorem 4.1. □

References

[1] S. Asmussen and J. Rosinski, *Approximations of small jumps of Lévy processes with a view towards simulation*, J. Appl. Probab. **38** (2001), no. 1, 482–493.

[2] T. Chan, *Pricing contingent claims on stocks driven by Lévy process*, Ann. Appl. Probab. **9** (1999), no. 2, 504–528.

[3] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, Applied stochastic analysis (London, 1989), 389–414, Stochastics Monogr., 5, Gordon and Breach, New York, 1991.

[4] M. G. Garroni and J. L. Menaldi, *Green Functions for Second Order Parabolic Integro-differential Problems*, Longman Scientific and Technical, England, 1992.

[5] I. I. Gihman and A. V. Skorohod, *Stochastic Differential Equations,* Springer-Verlag, New York, 1972.

[6] D. Heath and M. Schewizer, *Martingales versus PDEs in finance: An equivalence result with examples*, J. Appl. Prob. **37** (2000), 947–957.

[7] D. Hong and I.-S. Wee, *Convergence of jump-diffusion models to the Black-Scholes model*, Stoch. Anal. Appl. **21** (2003), no. 1, 141–160.

Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail: iswee@korea.ac.kr