

## NOTE ON COMMUTING TOEPLITZ OPERATORS ON THE PLURIHARMONIC BERGMAN SPACE

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ABSTRACT. We obtain a characterization of commuting Toeplitz operators with holomorphic symbols acting on the pluriharmonic Bergman space of the polydisk. We also obtain a characterization of normal Toeplitz operators with pluriharmonic symbols. In addition, some results for special types of semi-commutators are included.

### 1. Introduction

For a fixed integer  $n \geq 2$ , we shall let  $D^n$  denote the unit polydisk which is the cartesian product of  $n$  copies of the unit disk  $D$  in the complex plane  $\mathbb{C}$ . Also, let  $L^p(D^n) = L^p(D^n, dV_n)$  denote the usual Lebesgue space where  $dV_n$  is the volume measure on  $D^n$  normalized to have total mass 1.

Recall that a complex-valued function  $f \in C^2(D^n)$  is said to be pluriharmonic if

$$\partial_j \bar{\partial}_k f = 0, \quad j, k = 1, 2, \dots, n.$$

Here and elsewhere,  $\partial_j$  denotes the complex partial differentiation with respect to the  $j$ -th variable. The pluriharmonic Bergman space  $b^2(D^n)$  is then the space of all pluriharmonic functions in  $L^2(D^n)$ . It is well known that  $b^2(D^n)$  is a closed subspace of  $L^2(D^n)$ , and hence is a Hilbert space. Each point evaluation is easily verified to be a bounded linear functional on  $b^2(D^n)$ . Hence, for each  $z \in D^n$ , there exists a unique function  $R_z \in b^2(D^n)$ -called the pluriharmonic Bergman kernel-that has

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the reproducing property:

$$f(z) = \int_{D^n} f(w) \overline{R_z(w)} dV_n(w)$$

for every  $f \in b^2(D^n)$ . From this reproducing formula, it follows that the Hilbert space orthogonal projection  $Q$  from  $L^2(D^n)$  onto  $b^2(D^n)$  is realized as an integral operator

$$(1.1) \quad Q(\varphi)(z) = \int_{D^n} \varphi(w) \overline{R_z(w)} dV_n(w), \quad z \in D^n$$

for  $\varphi \in L^2(D^n)$ .

As is well-known, a function  $f \in C^2(D^n)$  is pluriharmonic if and only if it admits a decomposition  $f = g + \bar{h}$ , where  $g$  and  $h$  are holomorphic. See Chapter 2 of [5]. Moreover, if  $f \in b^2(D^n)$ , then it is not hard to see  $g, h \in A^2(D^n)$ . Here,  $A^2(D^n)$  denotes the well-known holomorphic Bergman space which consists of all holomorphic functions in  $L^2(D^n)$ . As a result of this observation we see the following simple relation:

$$b^2(D^n) = A^2(D^n) + \overline{A^2(D^n)}.$$

This yields

$$(1.2) \quad R_z = K_z + \overline{K_z} - 1,$$

where  $K_z$  denotes the well-known holomorphic Bergman kernel whose explicit formula is given by

$$K_z(w) = \prod_{j=1}^n \frac{1}{(1 - \bar{z}_j w_j)^2}, \quad z, w \in D^n.$$

By (1.1) and (1.2), the orthogonal projection  $Q$  admits the integral representation

$$(1.3) \quad Q(\varphi)(z) = \int_{D^n} \varphi(w) (\overline{K_z(w)} + K_z(w) - 1) dV_n(w), \quad z \in D^n$$

for functions  $\varphi \in L^2(D^n)$ .

For  $u \in L^2(D^n)$ , the Toeplitz operator  $T_u$  with symbol  $u$  is defined by

$$T_u f = Q(uf)$$

for  $f \in b^2(D^n)$ . The operator  $T_u$  is densely defined. In fact, we have  $Q(uf) \in b^2(D^n)$  for any bounded holomorphic function  $f$  on  $D^n$ .

Here, we are concerned with the characterizing problem of symbols of commuting Toeplitz operators. This problem has been studied by several authors in case of the holomorphic Bergman spaces. In [1], Axler and

Čučković first obtained a complete description of harmonic symbols of commuting Toeplitz operators on the Bergman Space  $A^2(D)$ . Later, this result has been extended to various domains such as the annulus ([4]), the ball ([9]) and the polydisk ([2], [8]).

In case of the pluriharmonic Bergman space, the problem is more subtle and less is known. The case of pluriharmonic Bergman space was first studied on the unit disk ([3]) and then on the ball ([6]). Here, we study the same on the polydisk. Our results are parallel to those of [6]. Our first result is the characterization of holomorphic symbols for which associated Toeplitz operators are commuting.

**THEOREM 1.1.** *Let  $f, g \in A^2(D^n)$ . Then  $T_f T_g = T_g T_f$  on  $b^2(D^n)$  if and only if  $f, g$  and  $1$  are linearly dependent.*

Recall that (a densely defined) linear operator on a Hilbert space is said to be normal if it commutes with its adjoint operator. We have the following characterization of normal Toeplitz operators with pluriharmonic symbols.

**THEOREM 1.2.** *Let  $u \in b^2(D^n)$ . Then  $T_u$  is normal on  $b^2(D^n)$  if and only if  $u(D^n)$  is a part of a line in  $\mathbb{C}$ . In particular, for  $f \in A^2(D^n)$ ,  $T_f$  is normal if and only if  $f$  is constant.*

Section 2 is devoted to the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, some results for special types of semi-commutators are included.

## 2. Proofs

We first introduce some notations. Let  $z = (z_1, \dots, z_k, \dots, z_n) \in D^n$  be an arbitrary point. For  $1 \leq k \leq n$ , we denote for simplicity

$$\hat{z}_k = (z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n).$$

For  $\lambda \in D$ , we let

$$(\lambda; \hat{z}_k) = (z_1, \dots, z_{k-1}, \lambda, z_{k+1}, \dots, z_n).$$

Also, we use the conventional multi-index notations. That is, for an ordered  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$  of nonnegative integers, we let

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}.$$

In what follows,  $A^1(D^n)$  denotes the space consisting of all holomorphic functions in  $L^1(D^n)$ . The notation  $dA$  means the area measure on  $D$  normalized to have total mass 1.

LEMMA 2.1. Let  $f \in A^1(D^n)$ . Then,  $f(a, \cdot) \in A^1(D^{n-1})$  for every  $a \in D$ .

*Proof.* Let  $a \in D$ . Then, for any  $w \in D^{n-1}$ , we have by subharmonicity

$$\begin{aligned} |f(a, w)| &\leq \frac{1}{(1 - |a|)^2} \int_{|\lambda - a| < 1 - |a|} |f(\lambda, w)| dA(\lambda) \\ &\leq \frac{1}{(1 - |a|)^2} \int_D |f(\lambda, w)| dA(\lambda). \end{aligned}$$

Now, integrating with respect to  $dV_{n-1}(w)$ , we conclude  $f(a, \cdot) \in A^1(D^{n-1})$ . The proof is complete.  $\square$

The following lemma is proved for  $p = 2$  in Lemma 2 of [3] and the same proof works for  $p = 1$ .

LEMMA 2.2. Let  $\psi \in A^1(D)$ . Then, we have

$$\int_D \frac{\bar{\lambda}\psi(\lambda)}{(1 - a\bar{\lambda})^2} dA(\lambda) = \frac{1}{a}\psi(a) - \frac{1}{a^2} \int_0^a \psi(\zeta) d\zeta$$

for each  $a \in D$ .

In what follows,  $P$  denotes the Bergman projection, which is the Hilbert space orthogonal projection from  $L^2(D^n)$  onto  $A^2(D^n)$ . As is well known, the projection  $P$  is represented by an integral operator

$$(2.1) \quad P(\varphi)(z) = \int_{D^n} \varphi(w) \overline{K_z(w)} dV_n(w), \quad z \in D^n$$

for  $\varphi \in L^2(D^n)$ .

Note that, via the integral representation (2.1),  $P$  extends to an integral operator from  $L^1(D^n)$  into the space of holomorphic functions on  $D^n$ . Moreover, it is well known that

$$(2.2) \quad P(f) = f \quad \text{and} \quad P(\bar{f}) = \bar{f}(0)$$

for  $f \in A^1(D^n)$ . Similarly, via the integral representation (1.3),  $Q$  extends to an integral operator from  $L^1(D^n)$  into the space of pluriharmonic functions on  $D^n$ . Note that  $Q$  can be written in terms of  $P$  by (1.3) and (2.1):

$$(2.3) \quad Q(\varphi) = P(\varphi) + \overline{P(\bar{\varphi})} - P(\varphi)(0)$$

for  $\varphi \in L^1(D^n)$ . In addition, we see from (2.2) and (2.3) that

$$(2.4) \quad Q(f) = f$$

for  $f \in A^1(D^n)$ . The following property of Bergman integrals are useful for our purpose.

LEMMA 2.3. *Let  $f \in A^1(D^n)$ . Then, the following hold.*

(a) *For  $z \in D^n$  and  $1 \leq k \leq n$ , we have*

$$P(\overline{w_k}f)(z) = \frac{1}{z_k}f(z) - \frac{1}{z_k^2} \int_0^{z_k} f(\zeta; \hat{z}_k) d\zeta.$$

(b) *If  $f(0) = 0$ , then*

$$\overline{P(w_k \overline{f})} = P(\overline{w_k}f)(0) = \frac{1}{2} \partial_k f(0).$$

*Proof.* Fix  $k$  and let  $z \in D^n$ . Then, for each  $w_k \in D$ , the function  $\hat{z}_k \mapsto f(w_k; \hat{z}_k)$  belongs to  $A^1(D^{n-1})$  by Lemma 2.1. Thus, we have

$$f(w_k; \hat{z}_k) = \int_{D^{n-1}} f(w_k; \hat{w}_k) \prod_{j \neq k} \frac{1}{(1 - z_j \overline{w_j})^2} dV_{n-1}(\hat{w}_k), \quad w_k \in D.$$

Now, multiply  $\overline{w_k}(1 - z_k \overline{w_k})^{-2}$  on both sides of the above and then integrate with respect to the measure  $dA(w_k)$ . The result is

$$\begin{aligned} P(\overline{w_k}f)(z) &= \int_D \frac{\overline{w_k}f(w_k; \hat{z}_k)}{(1 - z_k \overline{w_k})^2} dA(w_k) \\ &= \frac{1}{z_k}f(z) - \frac{1}{z_k^2} \int_0^{z_k} f(\zeta; \hat{z}_k) d\zeta, \end{aligned}$$

where the second equality holds by Lemma 2.2. This proves (a).

Next, assume  $f(0) = 0$  and let

$$f(w) = \sum_{j=1}^n a_j w_j + \sum_{|\alpha| \geq 2} a_\alpha w^\alpha$$

be the power series expansion of  $f$  at the origin. Then we have

$$(2.5) \quad P(w_k \overline{f}) = \sum_{j=1}^n \overline{a_j} P(w_k \overline{w_j}) + \sum_{|\alpha| \geq 2} \overline{a_\alpha} P(w_k \overline{w^\alpha}).$$

For the first sum of the above, note that we have by (a)

$$(2.6) \quad P(w_k \overline{w_j}) = \begin{cases} 0 & \text{for } j \neq k, \\ \frac{1}{2} & \text{for } j = k. \end{cases}$$

For the second sum, we claim

$$(2.7) \quad P(w_k \overline{w^\alpha}) = 0 \quad \text{for } |\alpha| \geq 2.$$

To see this, assume  $|\alpha| \geq 2$ . In case  $\alpha_k \leq 1$ , pick  $j \neq k$  such that  $\alpha_j \geq 1$ . Then we have by the mean value property

$$\int_D \frac{\overline{w}_j^{\alpha_j}}{(1 - z_j \overline{w}_j)^2} dA(w_j) = 0$$

for all  $z_j \in D$ . In case  $\alpha_k \geq 2$ , again by the mean value property, we have

$$\begin{aligned} \int_D \frac{w_k \overline{w}_k^{\alpha_k}}{(1 - z_k \overline{w}_k)^2} dA(w_k) &= \int_D \frac{\overline{w}_k^{\alpha_k - 1}}{(1 - z_k \overline{w}_k)^2} |w_k|^2 dA(w_k) \\ &= \frac{1}{\pi} \int_0^1 r^{\alpha_k + 2} \int_0^{2\pi} \frac{e^{-(\alpha_k - 1)i\theta}}{(1 - rz_k e^{-i\theta})^2} d\theta dr \\ &= 0 \end{aligned}$$

for all  $z_k \in D$ . So, in either case, we see that

$$P(w_k \overline{w}^\alpha)(z) = \int_D \frac{w_k \overline{w}_k^{\alpha_k}}{(1 - z_k \overline{w}_k)^2} dA(w_k) \times \prod_{j \neq k} \int_D \frac{\overline{w}_j^{\alpha_j}}{(1 - z_j \overline{w}_j)^2} dA(w_j) = 0$$

for  $z \in D^n$ . Thus (2.7) holds. Now, we conclude from (2.5), (2.6), and (2.7)

$$\overline{P(w_k f)} = \frac{a_k}{2} = \frac{1}{2} \partial_k f(0).$$

On the other hand, since

$$f(z) = \int_{D^n} f(w) \prod_{j=1}^n \frac{1}{(1 - z_j \overline{w}_j)^2} dV_n(w),$$

differentiation with respect to  $z_k$  under the integral sign yields

$$\partial_k f(0) = 2 \int_{D^n} f(w) \overline{w}_k dV_n(w) = 2P(\overline{w}_k f)(0).$$

This proves (b). The proof is complete. □

As a consequence of Lemma 2.3, we have the following.

**LEMMA 2.4.** *Let  $f \in A^1(D^n)$  and  $1 \leq k \leq n$ . Then, we have  $\partial_k f = 0$  if and only if  $P(\overline{w}_k f) = 0$ .*

*Proof.* First, suppose  $\partial_k f = 0$ . Since  $f$  is independent of  $z_k$ , we have

$$\int_0^{z_k} f(\zeta; \hat{z}_k) d\zeta = z_k f(z), \quad z \in D^n$$

and therefore  $P(\overline{w}_k f) = 0$  by (a) of Lemma 2.3.

Conversely, suppose  $P(\bar{w}_k f) = 0$ . By Lemma 2.3, we have

$$z_k f(z) = \int_0^{z_k} f(\zeta; \hat{z}_k) d\zeta.$$

Thus, differentiating with respect to  $z_k$ , we obtain  $\partial_k f = 0$ . The proof is complete.  $\square$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* The sufficiency is trivial. We now prove the necessity. So, suppose  $T_f T_g = T_g T_f$  on  $b^2(D^n)$ . Without loss of generality, we may assume  $f$  and  $g$  are both nonconstant. We may further assume that  $f(0) = g(0) = 0$ . With these assumptions, we need to show  $f = \alpha g$  for some constant  $\alpha$ .

Let  $1 \leq k \leq n$ . By Lemma 2.3 and (2.3), we have

$$\begin{aligned} T_f(\bar{w}_k) &= Q(\bar{w}_k f) \\ &= P(\bar{w}_k f) + \overline{P(w_k \bar{f})} - P(\bar{w}_k f)(0) = P(\bar{w}_k f), \end{aligned}$$

so that

$$T_g T_f(\bar{w}_k) = T_g(P(\bar{w}_k f)) = gP(\bar{w}_k f),$$

where the second equality holds by (2.4). Now, since  $T_f T_g = T_g T_f$ , we have by symmetry

$$(2.8) \quad fP(\bar{w}_k g) = gP(\bar{w}_k f), \quad k = 1, \dots, n.$$

By Lemma 2.4 and (2.8), we see that  $\partial_k f = 0$  if and only if  $\partial_k g = 0$ . Thus, by changing the coordinate system if necessary, we may write

$$f(z) = f(z_1, \dots, z_r), \quad g(z) = g(z_1, \dots, z_r),$$

where  $r \leq n$  is chosen so that  $\partial_j g \neq 0$  for  $j \leq r$  and  $\partial_j f = \partial_j g = 0$  for  $j > r$ . Let  $1 \leq k \leq r$ . Since  $fP(\bar{w}_k g) = gP(\bar{w}_k f)$ , a little manipulation using Lemma 2.3 yields

$$f(z)G_k(z) = g(z)F_k(z), \quad z \in D^n,$$

where

$$F_k(z) = \int_0^{z_k} f(\zeta; \hat{z}_k) d\zeta, \quad G_k(z) = \int_0^{z_k} g(\zeta; \hat{z}_k) d\zeta.$$

Note  $\partial_k F_k = f$  and  $\partial_k G_k = g$ . Thus, we have

$$(2.9) \quad (\partial_k F_k)G_k = (\partial_k G_k)F_k.$$

Now consider

$$V_k = \{\hat{z}_k \in D^{n-1} \mid G_k(\cdot; \hat{z}_k) \neq 0\}.$$

Then, it follows from (2.9) that

$$\partial_k \left( \frac{F_k}{G_k} \right) (\cdot; \hat{z}_k) = 0 \quad \text{on } D$$

for each fixed  $\hat{z}_k \in V_k$ . That is,

$$(2.10) \quad F_k(z) = c(\hat{z}_k)G_k(z), \quad z_k \in D, \quad \hat{z}_k \in V_k,$$

where the coefficient  $c(\hat{z}_k)$  is independent of  $z_k$ . Let  $E_k = \{z \in D^n \mid z_k \in D, \hat{z}_k \in V_k\}$ . Then, differentiating both sides of (2.10) with respect to  $z_k$ , we have

$$f(z) = c(\hat{z}_k)g(z), \quad z \in E_k.$$

Since  $k$  is arbitrary, it follows that

$$f(z) = \alpha g(z), \quad z \in \bigcap_{k=1}^r E_k$$

for some constant  $\alpha$ .

Now, it remains to show that  $\bigcap_{k=1}^r E_k$  is dense in  $D^n$ . Note that each  $V_k$  is open in  $D^{n-1}$  by continuity. Also, it is not hard to verify that each  $V_k$  is dense in  $D^{n-1}$ . In fact, if some  $V_k$  is not dense in  $D^{n-1}$ , then there exists some open subset  $U$  of  $D^{n-1}$  such that

$$G_k(z_k; \hat{z}_k) = 0, \quad z_k \in D, \quad \hat{z}_k \in U.$$

Hence,  $G_k$  vanishes on some open subset of  $D^n$  and thus on all of  $D^n$ , which is a contradiction. It follows that each  $E_k$  is both open and dense in  $D^n$ . Thus, by Baire's theorem, we conclude that  $\bigcap_{k=1}^r E_k$  is dense in  $D^n$ , as desired. The proof is complete.  $\square$

It is easy to see that the adjoint operator of  $T_u$  is  $T_{\bar{u}}$ . Thus, considering normal Toeplitz operators with holomorphic symbols  $f \in A^2(D^n)$ , and considering the conjugates of the coordinate functions as in the proof of Theorem 1.1, one is led to the following system of integral equations:

$$(2.11) \quad P(f\overline{P(\bar{w}_k f)}) = P(w_k |f|^2)$$

for  $k = 1, 2, \dots, n$ . It turns out that just one of the above equations is enough to conclude Theorem 1.2.

**PROPOSITION 2.5.** *Suppose  $f \in A^2(D^n)$  and (2.11) holds for some  $k$ . Then  $f = 0$ .*

*Proof.* Without loss of generality, let  $k = 1$ . From (2.1),  $P$  is the operator of integration against the Bergman kernel. Differentiating both



sides of the equation (2.11) with respect to the first variable under the integral sign and evaluating at the origin, we get

$$(2.12) \quad \int_{D^n} f(z) \overline{P(\overline{w_1 f})(z)} \bar{z}_1 dV_n(z) = \int_{D^n} |z_1|^2 |f(z)|^2 dV_n(z).$$

Let  $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$  be the power series expansion of  $f$  at the origin. A straightforward calculation using Lemma 2.3 yields

$$\overline{P(\overline{w_1 f})(z)} \bar{z}_1 = \sum_{\alpha} \bar{a}_{\alpha} \frac{\alpha_1}{1 + \alpha_1} \bar{z}^{\alpha},$$

so that

$$\begin{aligned} & \int_{D^n} f(z) \overline{P(\overline{w_1 f})(z)} \bar{z}_1 dV_n(z) \\ &= \int_{D^n} f(z) \left( \sum_{\alpha} \bar{a}_{\alpha} \frac{\alpha_1}{1 + \alpha_1} \bar{z}^{\alpha} \right) dV_n(z) \\ &= \sum_{\alpha} |a_{\alpha}|^2 \left( \frac{\alpha_1}{1 + \alpha_1} \right) \int_{D^n} |z^{\alpha}|^2 dV_n(z) \\ &= \sum_{\alpha} |a_{\alpha}|^2 \left( \frac{\alpha_1}{1 + \alpha_1} \right) \prod_{j=1}^n \frac{1}{1 + \alpha_j}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{D^n} |z_1|^2 |f(z)|^2 dV_n(z) &= \sum_{\alpha} |a_{\alpha}|^2 \int_{D^n} |z_1 z^{\alpha}|^2 dV_n(z) \\ &= \sum_{\alpha} |a_{\alpha}|^2 \left( \frac{1 + \alpha_1}{2 + \alpha_1} \right) \prod_{j=1}^n \frac{1}{1 + \alpha_j}. \end{aligned}$$

It follows from (2.12) that

$$\begin{aligned} 0 &= \sum_{\alpha} |a_{\alpha}|^2 \left( \frac{1 + \alpha_1}{2 + \alpha_1} - \frac{\alpha_1}{1 + \alpha_1} \right) \prod_{j=1}^n \frac{1}{1 + \alpha_j} \\ &= \sum_{\alpha} |a_{\alpha}|^2 \frac{1}{(2 + \alpha_1)(1 + \alpha_1)} \prod_{j=1}^n \frac{1}{1 + \alpha_j} \end{aligned}$$

and thus  $a_{\alpha} = 0$  for all  $\alpha$ . Consequently,  $f$  is identically 0. The proof is complete.  $\square$

Having Proposition 2.5, we now turn to the proof of Theorem 1.2.

*Proof of Theorem 1.2.* The theorem follows from exactly the same argument as in the proof of Theorem 13 of [6].  $\square$

### 3. Semi-commutators

In this section we consider the semi-commuting property of Toeplitz operators. All the arguments are parallel to those of [6]. The key step in [6] is the ball version of the following proposition.

PROPOSITION 3.1. *Suppose  $f, g \in A^2(D^n)$  and*

$$(3.1) \quad P(\bar{w}_k f g) = f P(\bar{w}_k g), \quad 1 \leq k \leq n.$$

*Then, either  $f$  is constant or  $g = 0$ .*

*Proof.* Assume (3.1) and  $g \neq 0$ . We need to show that  $f$  is constant. By (3.1) and Lemma 2.3 we have

$$(3.2) \quad H_k(z) = f(z)G_k(z), \quad z \in D^n, \quad 1 \leq k \leq n,$$

where

$$H_k(z) = \int_0^{z_k} f(\zeta; \hat{z}_k)g(\zeta; \hat{z}_k) d\zeta, \quad G_k(z) = \int_0^{z_k} g(\zeta; \hat{z}_k) d\zeta.$$

Differentiating both sides of (3.2), we have  $(\partial_k f)G_k = 0$  for each  $k$ . Since  $g \neq 0$ , we have  $G_k \neq 0$  for each  $k$ . It follows that  $\partial_k f = 0$  for each  $k$ . Thus  $f$  is constant. The proof is complete.  $\square$

Now, all the results below can be proved by repeating exactly the same argument as in the proofs of Lemma 14, Theorem 15, Corollary 16, and Corollary 18 of [6]. Note that the class of symbols of Toeplitz operators is naturally extended to  $L^1(D^n)$  by (2.3).

PROPOSITION 3.2. *Let  $u, v \in b^2(D^n)$ . Assume  $u = f + \bar{g}$ ,  $v = h + \bar{k}$  for holomorphic functions  $f, g, h$  and  $k$ . If  $T_{uv} = T_u T_v$  on  $b^2(D^n)$ , then at least one of  $f$  and  $h$  is constant, and at least one of  $g$  and  $k$  is constant.*

THEOREM 3.3. *Let  $f, g \in A^2(D^n)$ . Then  $T_{fg} = T_f T_g$  on  $b^2(D^n)$  if and only if either  $f$  or  $g$  is constant.*

THEOREM 3.4. *Let  $u \in b^2(D^n)$ . Then  $T_{u^2} = T_u T_u$  on  $b^2(D^n)$  if and only if  $u$  is constant.*

THEOREM 3.5. *Let  $u \in b^2(D^n)$ . Then  $T_{|u|^2} = T_u T_{\bar{u}}$  on  $b^2(D^n)$  if and only if  $u$  is constant;  $T_{|u|^2} = T_{\bar{u}} T_u$  on  $b^2(D^n)$  if and only if  $u$  is constant.*

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