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THE BERGMAN KERNEL FUNCTION AND THE SZEGŐ KERNEL FUNCTION

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ABSTRACT. We compute the holomorphic derivative of the harmonic measure associated to a C^∞ bounded domain in the plane and show that the exact Bergman kernel function associated to a C^∞ bounded domain in the plane relates the derivatives of the Ahlfors map and the Szegő kernel in an *explicit* way. We find several formulas for the exact Bergman kernel and the Szegő kernel and the harmonic measure. Finally we survey some other properties of the holomorphic derivative of the harmonic measure.

1. Introduction

I showed in [7] that the exact Bergman kernel function associated to a C^∞ smoothly bounded domain in the plane can be expressed in terms of the derivative of the Ahlfors map and the harmonic measures. I also showed in [8] that the exact Bergman kernel function is expressed in terms of the derivative of the Ahlfors map and the Szegő kernel in the *first variable* explicitly. In this paper, by computing the derivative of the harmonic measure we shall show that the exact Bergman kernel is written as a sum of the derivative of the Ahlfors map and the Szegő kernel and the Garabedian kernel in it both variables explicitly. Furthermore, an explicit formula for a relationship between the exact Bergman kernel, the derivative of the Ahlfors map and the Szegő kernel will be presented when the domain is doubly connected. In the last section, we survey some other properties of the holomorphic derivative of the harmonic measure. In particular, we find another expression of the derivative of the harmonic measure from formulas introduced by Bell. The results

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of this paper are of practical importance because the Ahlfors map is a solution of an extremal problem in a multiply connected domain such as the Riemann mapping function in a simply connected domain. And Kerzman and Stein [11], Kerzman and Trummer [12], Trummer [15] and Bell [2] also showed that the Ahlfors map is highly computable object.

2. Preliminaries and notations

In this section, we review some preliminaries about the kernel functions and notations. To begin with, we shall assume that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary. Let $\gamma_j, j = 1, \ldots, n$, denote the n non-intersecting C^∞ simple closed curves defining the boundary $b\Omega$ of Ω. We assume that the boundary curve γ_j is parameterized in the standard sense by $z_j(t), 0 \le t \le 1$. For convenience, let γ_n denote the outer boundary curve of Ω. Let $T(z)$ be the C^∞ function defined on $b\Omega$ by the complex unit tangent vector in the direction of the standard orientation. For example, when $z = z_j(t) \in \gamma_j$, $T(z) = \dfrac{z_j'(t)}{|z_j'(t)|}$. We shall let $L^2(b\Omega)$ denote the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds and let $L^2(\Omega)$ denote the space of complex valued functions on Ω that are square integrable with respect to Lebesgue area measure dA. The Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω shall be written $H^2(b\Omega)$ and the Bergman space of holomorphic functions on Ω that are in $L^2(\Omega)$ shall be written $H^2(\Omega)$. The Bergman kernel $B(z,w)$ is the reproducing kernel for the Bergman projection which is the orthogonal projection B of $L^2(\Omega)$ onto $H^(\Omega)$.

The orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ with respect to the inner product

$$\langle u, v\rangle_{b\Omega} = \int_{b\Omega} u\, \bar{v}\, ds$$

is called the Szegő projection denoted by P. The Szegő kernel denoted by $S(z,w)$ is the kernel for P. It is well known that $S(z,w)$ extends to the boundary to be in $C^\infty\left((\bar{\Omega} \times \bar{\Omega}) \setminus \{(z,z) : z \in b\Omega\}\right)$. And it is a holomorphic function of z and an antiholomorphic function of w on $\Omega \times \Omega$. We note that $S(z,z)$ is real and positive for $z \in \Omega$ and $\overline{S(z,w)} = S(w,z)$. The Garabedian kernel $L(z,w)$ is the kernel for the orthogonal projection $P^\perp$ of $L^2(b\Omega)$ onto $H^2(b\Omega)^\perp$ defined by

$$(2.1) \qquad L(z,w) = i\,\overline{S(z,w)\,T(z)}, \quad \text{for } (z,w) \in b\Omega \times \Omega.$$

In fact, the Garabedian kernel satisfies the identity

$$(2.2) \qquad L(z,w) = \frac{1}{2\pi}\frac{1}{z-w} - iSP(\overline{C_w T})(z),$$

where

$$C_w(\zeta) = \overline{\frac{1}{2\pi i}\frac{T(\zeta)}{\zeta - w}}, \quad \zeta \in b\Omega,\ w \in \Omega$$

is the kernel for the Cauchy transform defining the Cauchy integral. For fixed $w \in \Omega$, $L(z,w)$ is a holomorphic function of z on $\Omega \setminus \{w\}$ with a simple pole at w with residue $1/2\pi$. Furthermore, $L(z,w)$ extends to be in $C^\infty\left((\bar{\Omega} \times \bar{\Omega}) \setminus \{(z,z) : z \in \bar{\Omega}\}\right)$. We also note that $L(w,z) = -L(z,w)$ and $L(z,w)$ is zero-free for all $(z,w) \in \bar{\Omega} \times \Omega$ with $z \neq w$. All of these properties can be found in Bell's book[4]. See also [6].

For fixed $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is an n-to-one proper holomorphic mapping of Ω onto the unit disc and extends C^∞ smoothly to the boundary of Ω. And it also maps each boundary curve γ_j one-to-one onto the unit disc. This Ahlfors map f_a is the unique solution to the extremal problem: among all holomorphic functions h mapping Ω into the unit disc, find the one taking $h'(a)$ real-positive valued and as large as possible. Hence it is very important to express classical kernel functions in terms of the *derivative* of the Ahlfors map. On the other hand, The Ahlfors map is given in terms of the Szegő kernel and Garabedian kernel (see [9]) by

$$f_a(z) = \frac{S(z,a)}{L(z,a)}.$$

Let $E^2(\Omega)$ denote the exact Bergman space of holomorphic functions in $H^2(\Omega)$ such that have single-valued indefinite integrals. It is clear that in a simply connected domain the exact Bergman space is equal to the Bergman space. Let $E(z,w)$ denote the exact Bergman kernel that is the kernel for the orthogonal projection of $L^2(\Omega)$ onto $E^2(\Omega)$. In the simply connected case, it is easy to see that the exact Bergman kernel function(and hence the Bergman kernel) is related (see [7]) via

$$E(z,w) = 2S(w,w)f_w'(z).$$

Furthermore, I proved in [7] that in multiply connected domains the exact Bergman kernel function can be written in terms of derivative of the Ahlfors map and the Szegő kernel. In the next section by using Bell's result [5] we shall find much more explicit form of formula than before.

3. Main results

The harmonic measure function $\omega_j, j = 1, \ldots, n$ associated to the boundary curves $\{\gamma_k\}$ of Ω is a harmonic function that solves the Dirichlet problem on Ω with boundary data equal to one on γ_j and zero on the other boundary curves. Then the function

$$F_j = 2\frac{\partial \omega_j}{\partial z}$$

is holomorphic in Ω and it is the derivative of the multivalued holomorphic function obtained by analytically continuing around Ω a germ of $\omega_j + iv$, where $v$is a local harmonic conjugate of ω_j.

It is a classical fact that the set of functions $\{F_j : j = 1, \ldots, n-1\}$ is linearly independent. In fact, the set $\{F_j : j = 1, \ldots, n-1\}$ is a basis for the space $H^2(\Omega) \setminus E^2(\Omega)$ of the complement of the exact Bergman space. From this, it is easy to see (see [6], [13]) that the exact Bergman kernel function $E(z, w)$ is related to the Szegő kernel $S(z, w)$ via the identity

$$E(z, w) = 4\pi S(z, w)^2 + \sum_{i,j=1}^{n-1} a_{ij} F_i(z) \overline{F_j(w)},$$

where a_{ij} are constants independent of the variables z and w.

Let $a \in \Omega$ be fixed. Since the Ahlfors map $f_a(z) = S(z, a)/L(z, a)$ is n-to-one, it has n zeroes. But $f_a(a) = 0, f_a'(a) = 2\pi S(a, a) \neq 0$. Thus the simple zero of f_a at a accounts for the simple pole of $L(z, a)$ at a. The other $n - 1$ zeroes of f_a come from exactly $n - 1$ zeroes of $S(z, a)$ in $\Omega \setminus \{a\}$. Let $a_1, a_2, \ldots, a_{n-1}$ denote these $n - 1$ zeroes counted with multiplicities. It was proved in [3] that for all but at most finitely many points $a \in \Omega$, the kernel $S(z, a)$ has $n - 1$ *distinct simple* zeroes in Ω as a function of z. We may thus assume without loss of generality that those $n - 1$ zeroes $a_1, a_2, \ldots, a_{n-1}$ of $S(z, a)$ are all distinct simple zeroes.

Schiffer [14] proved that the set of $n - 1$ functions $\{S(z, a_j)L(z, a) : j = 1, \ldots, n - 1\}$ and the set $\{F_j : j = 1, \ldots, n - 1\}$ span the same vector space of functions. Notice that since the pole of $L(z, a)$ at $z = a$ is cancelled out by the zero of $S(z, a_j)$ at $z = a$, the function $S(z, a_j)L(z, a)$ extends C^∞ smoothly to the boundary of Ω. It is also proved in [4] that the linear span of $\{S(z, a_j)L(z, a) : j = 1, \ldots, n - 1\}$ is the same as the linear span of the set $\{S(z, a)L(z, a_j) : j = 1, \ldots, n - 1\}$.

Hence we have obtained the following formula relating the exact Bergman kernel to the Szegő kernel.

THEOREM 3.1. *Ω is an n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let $a \in \Omega$ be fixed and let $a_1, a_2, \ldots, a_{n-1}$ be $n-1$ distinct simple zeroes of the Szegő kernel function $S(z,a)$. Then the exact Bergman kernel $E(z,w)$ is related to the Szegő kernel via the identity*

$$E(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \mu_{ij} S(z,a)\overline{S(w,a)} L(z,a_i)\overline{L(w,a_j)}$$

or

$$E(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \nu_{ij} S(z,a_i)\overline{S(w,a_j)} L(z,a)\overline{L(w,a)}.$$

Notice (see [5]) that the $(n-1)\times(n-1)$ matrix

$$A = [S(a_i, a_j)]$$

is nonsingular. On the other hand, Bell [5] also proved that the Szegő kernel is an easily computable object of one complex variable with the following identity:

$$S(z,w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(\frac{S(z,a)\overline{S(w,a)}}{S(a,a)} + \sum_{i,j=1}^{n-1} {A^{-1}}_{ij} S(z,a_i)\overline{S(w,a_j)} \right).$$

Using the above identity, we obtain a formula for the exact Bergman kernel expressed as a function of one variable very similar to the Bergman kernel function as follows.

THEOREM 3.2. *Ω is an n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let $a \in \Omega$ be fixed and let $a_1, a_2, \ldots, a_{n-1}$ be $n-1$ distinct simple zeroes of the Szegő kernel function $S(z,a)$. Then the exact Bergman kernel $E(z,w)$ is related to the Szegő kernel* (and Garabedian kernel) *via the identity*

$$E(z,w) = \left(\frac{L(z,a)\overline{L(w,a)}}{L(z,a)\overline{L(w,a)} - S(z,a)\overline{S(w,a)}} \right)^2 \times \left(\sum_{\substack{0\le i\le j\le n-1 \\ 0\le k\le m\le n-1}} \alpha_{ijkm} S(z,a_i)S(z,a_j)\overline{S(w,a_k)S(w,a_m)} \right)$$

$$+\sum_{i,j=1}^{n-1} S(z,a_i)\overline{S(w,a_j)}L(z,a)\overline{L(w,a)},$$

where $a_0 = a$.

This theorem says that the exact Bergman kernel of two complex variables is actually a function of one complex variable. In fact, all the elements of the exact Bergman kernel can be computed by means of one dimensional line integrals. See [2] for this matter.

By the definition of the exact Bergman kernel, $E(z,w)$ is the derivative of a holomorphic function on Ω and hence it is very important to find an indefinite integral of the kernel explicitly. I proved in [7] that the exact Bergman kernel is related to the derivative of the Ahlfors map via

$$(3.1)\quad E(z,\zeta) = 2S(\zeta,\zeta)f'_\zeta(z) - 2\sum_{j,k=1}^{n-1} \overline{\alpha^{jk}}\ \overline{F_j(\zeta)} \int_{\lambda_k} S(\zeta,w)\frac{\partial}{\partial z}\left(\frac{S(z,w)}{L(z,\zeta)}\right) d\overline{w},$$

where α^{jk} is the (j,k)-th entry of the *inverse* of the matrix $A = [\alpha_{lm}]$.

As mentioned before, the F_j is a linear combination of the functions $S(\zeta,a)L(\zeta,a_m), m = 1,2,\dots,n-1$. Thus we can write the exact Bergman kernel via the identity

$$E(z,\zeta) = 2S(\zeta,\zeta)f'_\zeta(z) - 2\sum_{j,k,m=1}^{n-1} \beta_{jkm}S(a,\zeta)\overline{L(\zeta,a_m)} \int_{\lambda_k} S(\zeta,w)\frac{\partial}{\partial z}\left(\frac{S(z,w)}{L(z,\zeta)}\right) d\overline{w}.$$

On the other hand, in (3.1) we would like to find the exact expression of the holomorphic function F_j in terms of the Szegő kernel and the Garabedian kernel so that the (exact) Bergman kernel function can be expressed in terms of only the Szegő kernel and the Garabedian kernel with *explicit coefficients.*

THEOREM 3.3. *Ω is an n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let $a \in \Omega$ be fixed and let $a_1, a_2, \dots, a_{n-1}$ be $n-1$ distinct simple zeroes of the Szegő kernel function $S(z,a)$. Then the holomorphic derivative F_k of the harmonic measure with respect to Ω has the identity*

$$(3.2)\qquad F_k(z) = \sum_{j=1}^{n-1} \frac{2\pi F_k(a_j)}{S'(a_j,a)} S(z,a)L(z,a_j),$$

where the $S'(a_j, a)$ means the derivative taken with respect to the holomorphic variable.

Proof. Suppose that $z(t)$ parameterizes a boundary curve of Ω. Since the harmonic measure ω_k is constant on the boundary of Ω, we see that the chain rule of complex functions gives

$$0 = \frac{d\omega_k \circ z}{dt}(t) = \frac{\partial \omega_k(z(t))}{\partial z}\frac{dz(t)}{dt} + \frac{\partial \omega_k(z(t))}{\partial \bar{z}}\overline{\left(\frac{dz(t)}{dt}\right)}$$

$$\stackrel{\cdot}{=} \frac{\partial \omega_k(z(t))}{\partial z}\frac{dz(t)}{dt} + \overline{\left(\frac{\partial \omega_k(z(t))}{\partial z}\right)}\;\overline{\left(\frac{dz(t)}{dt}\right)}.$$

Hence the function $F_k = 2\dfrac{\partial \omega_k}{\partial z}$ has the property on the boundary of Ω:

$$F_k T = -\overline{F_k T}. \tag{3.3}$$

Inserting (2.1) into the above formula implies

$$F_k(z) i \frac{\overline{L(z,a)}}{S(z,a)} = -\overline{F_k(z) T(z)}$$

or

$$\frac{F_k(z)}{S(z,a)} = \overline{\left(-i\frac{F_k(z)}{L(z,a)}\right)}\,\overline{T(z)}, \qquad z \in b\Omega \tag{3.4}$$

Notice that the Szegő kernel $S(z,a)$ has all simple zeroes $a_1, a_2, \ldots, a_{n-1}$. Thus the left hand side of the above is

$$\frac{F_k(z)}{S(z,a)} = \sum_{j=1}^{n-1} \frac{F_k(a_j)/S'(a_j,a)}{z - a_j} + H(z),$$

where $H(z)$ is a holomorphic function in $H^2(b\Omega)$. On the other hand, it is easy to see from Cauthy's theorem and identity $T(z)ds = dz$ that any function of the form $\overline{HT}$ where H is holomorphic in $H^2(b\Omega)$ is orthogonal to the Hardy space $H^2(b\Omega)$. (We remark here (see [1]) that the class of such forms is exactly the same as the orthogonal complement of $H^2(b\Omega)$ in $L^2(b\Omega)$.) Since $L(z,a)$ never vanishes in $\overline{\Omega} \times \Omega$, it follows that the functions in (3.4) is orthogonal to $H^2(b\Omega)$ and hence

$$\overline{\left(-i\frac{F_k(z)}{L(z,a)}\right)}\,\overline{T(z)} = P^{\perp}\left(\frac{F_k(z)}{S(z,a)}\right) = \sum_{j=1}^{n-1} \frac{F_k(a_j)}{S'(a_j,a)} P^{\perp}\left(\frac{1}{z-a_j}\right).$$

By (2.2), $P^{\perp}\left(\frac{1}{z-a_j}\right) = 2\pi L(z,a_j)$. Thus from (3.4), we have

$$\frac{F_k(z)}{S(z,a)} = \sum_{j=1}^{n-1} \frac{2\pi F_k(a_j)}{S'(a_j,a)} L(z,a_j),$$

which finishes the proof of the identity (3.2). □

Now from Theorem 3.3 obtain the following identity which relates the exact Bergman kernel to the Szegő kernel in an explicit way.

THEOREM 3.4. *Suppose Ω is an n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let γ_k be the boundary curves of Ω and let $F_k = 2\frac{\partial \omega_k}{\partial z}$ where ω_k is the harmonic measure relative to Ω. Let $a \in \Omega$ be fixed and let $a_1, a_2, \ldots, a_{n-1}$ be $n-1$ distinct simple zeroes of the Szegő kernel function $S(z,a)$. Let $f_\zeta(z) = \frac{S(z,\zeta)}{L(z,\zeta)}$ be the Ahlfors map associated to the point $\zeta \in \Omega$. Then the exact Bergman kernel function $E(z,\zeta)$ is related to the Szegő kernel function via the identity*

$$\begin{aligned} E(z,\zeta) &= 2S(\zeta,\zeta) f'_\zeta(z) \\ &\quad - 4\pi S(z,\zeta) \sum_{j,k,m=1}^{n-1} \overline{\alpha^{jk} \left(\frac{F_j(a_m)}{S'(a_m,a)}\right)}\, \overline{L(\zeta,a_m)} \\ &\quad \times \int_{\gamma_k} S(\zeta,w) \frac{\partial}{\partial z}\left(\frac{S(z,w)}{S(z,\zeta)}\right) d\overline{w}; \end{aligned}$$

where α^{jk} is the (j,k)-th entry of the inverse of the matrix $A = [\alpha_{lm}]$.

Since the Bergman kernel function is related similarly to the Sezgő kernel (see [8]), we have the following result.

THEOREM 3.5. *Suppose Ω is an n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let γ_k be* *the boundary curves of* Ω *and let* $F_k = 2\dfrac{\partial \omega_k}{\partial z}$ *where* ω_k *is the harmonic measure relative to* Ω. *Let* $a \in \Omega$ *be fixed and let* $a_1, a_2, \ldots, a_{n-1}$ *be* $n-1$ *distinct simple zeroes of the Szegő kernel function* $S(z,a)$. *Let* $f_\zeta(z) = \dfrac{S(z,\zeta)}{L(z,\zeta)}$ *be the Ahlfors map associated to the point* $\zeta \in \Omega$. *Then the Bergman kernel function* $B(z,\zeta)$ *is related to the Szegő kernel function via the identity*

$$B(z,\zeta) = 2S(\zeta,\zeta)f'_\zeta(z) + \sum_{j=1}^{n-1} \frac{2\pi}{S'(a_j,\zeta)}\Big(B(z_j,\zeta) - 2S(\zeta,\zeta)f'_\zeta(a_j) - H(a_j,\zeta)\Big)L(z,a_j)S(z,\zeta) + H(z,\zeta),$$

where

$$H(z,\zeta) = -4\pi S(z,\zeta)\sum_{j,k,m=1}^{n-1} \overline{\alpha^{jk}}\,\overline{\left(\frac{F_j(a_m)}{S'(a_m,a)}\right)}\,\overline{L(\zeta,a_m)} \times \int_{\gamma_k} S(\zeta,w)\frac{\partial}{\partial z}\left(\frac{S(z,w)}{S(z,\zeta)}\right)\,d\overline{w}.$$

Now we want to find a formula for the exact Bergman kernel and the Szegő kernel that does not contain computation of complex line integrals. Using the invertibility of $(n-1)\times(n-1)$ matrix

$$B = [S(a_j,a_k)^2].$$

I also found in [8] the identity

$$S(z,a)^2 = \frac{S(a,a)f'_a(z)}{2\pi} - \frac{S(a,a)}{2\pi}\sum_{j=1}^{n-1} c_j(a)S(z,a_j)^2, \tag{3.5}$$

where

$$\begin{bmatrix} c_1(a) \\ \vdots \\ c_{n-1}(a) \end{bmatrix} = B^{-1}\begin{bmatrix} \dfrac{S'(a_1,a)}{L(a_1,a)} \\ \vdots \\ \dfrac{S'(a_{n-1},a)}{L(a_{n-1},a)} \end{bmatrix}.$$

On the other hand, it follows from Theorem 3.1 that by letting $w = a$,

$$E(z,a) = 4\pi S(z,a)^2 + \sum_{i,j=1}^{n-1} \mu_{ij} S(z,a)S(a,a)L(z,a_i)\overline{L(a,a_j)}.$$

Hence by inserting (3.5) into the above identity we obtain the following useful formula between the exact Bergman kernel function and the derivative of the Ahlfors map.

THEOREM 3.6. *Ω is a n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let $a \in \Omega$ be fixed and let $a_1, a_2, \ldots, a_{n-1}$ be $n-1$ distinct simple zeroes of the Szegő kernel function $S(z,a)$. Then the exact Bergman kernel $E(z,a)$ is related to the derivative of the Ahlfors map via the identity*

$$\begin{aligned} E(z,a) &= 2S(a,a)f_a'(z) - \sum_{j=1}^{n-1} 2S(a,a)c_j(a)S(z,a_j)^2 \\ &\quad + \sum_{i,j=1}^{n-1} \mu_{ij} S(z,a)S(a,a)L(z,a_i)\overline{L(a,a_j)}, \end{aligned}$$

where

$$\begin{bmatrix} c_1(a) \\ \vdots \\ c_{n-1}(a) \end{bmatrix} = [S(a_j,a_k)^2]^{-1} \begin{bmatrix} \dfrac{S'(a_1,a)}{L(a_1,a)} \\ \vdots \\ \dfrac{S'(a_{n-1},a)}{L(a_{n-1},a)} \end{bmatrix}.$$

In particular, when $n = 2$, i.e., Ω is doubly connected, since $c_1(a) = \dfrac{1}{S(a_1,a_1)^2}\dfrac{S'(a_1,a)}{L(a_1,a)}$ and $f_a'(a) = 2\pi S(a,a)$, it follows from Theorem 3.6 that

$$E(a,a) = 4\pi S(a,a)^2 + \mu_{11} S(a,a)^2 |L(a,a_1)|^2.$$

Notice that $S(a,a_1) = 0$. Thus we have

$$\begin{aligned} E(z,a) &= 2S(a,a)f_a'(z) \\ &\quad + \frac{E(a,a) - 4\pi S(a,a)^2}{S(a,a)^2|L(a,a_1)|^2} S(a,a)\overline{L(a,a_1)}L(z,a_1)S(z,a) \\ &\quad + \frac{2S(a,a)S'(a_1,a)}{S(a_1,a_1)^2 L(a,a_1)} S(z,a_1)^2. \end{aligned}$$

THEOREM 3.7. *Ω is a doubly connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let $a \in \Omega$ be fixed and let a_1 be the unique simple zero of the Szegő kernel function $S(z,a)$. Then the exact Bergman kernel $E(z,a)$ is related to the derivative of the*

Ahlfors map via the identity

$$\begin{aligned} E(z,a) &= 2S(a,a)f_a'(z) + \frac{E(a,a) - 4\pi S(a,a)^2}{S(a,a)L(a,a_1)} L(z,a_1)S(z,a) \\ &+ \frac{2S(a,a)S'(a_1,a)}{S(a_1,a_1)^2 L(a,a_1)} S(z,a_1)^2. \end{aligned}$$

Theorem 3.7 shows that once the value of $E(a,a)$ on the diagonal point $\{(a,a)\}$ is computed, the value of $E(z,a)$ is easily obtained using the values of the Szegő kernel.

4. Further properties of F_k

It is known (see [6]) that the holomorphic derivative F_k of the harmonic measure ω_k satisfies

$$\overline{F_m(w)} = -i \int_{\gamma_m} K(z,w)dz,$$

where $K(z,w)$ is the Bergman kernel. Since the Bergman kernel function is related to the F_k's via

$$k(z,w) = 4\pi S(z,w)^2 + \sum_{j,k=1}^{n-1} C_{jk} F_j(z) \overline{F_k(w)},$$

it follows that

$$\begin{aligned} \overline{F_m(w)} &= -4\pi i \int_{\gamma_m} S(z,w)^2 dz + \sum_{j,k=1}^{n-1} C_{jk} A_{mj} \overline{F_k(w)} \\ &= -4\pi i \int_{\gamma_m} S(z,w)^2 dz + \sum_{k=1}^{n-1} (AC)_{mk} \overline{F_k(w)}, \end{aligned}$$

where $A_{mj} = \int_{\gamma_m} F_j(z)dz$ is the period of F_j with respect to γ_m, $A = [A_{mj}]$ and $C = [C_{jk}]$ are $(n-1) \times (n-1)$ matrices. Hence we obtain the following.

THEOREM 4.1. *Suppose Ω is a n-connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let γ_k be the boundary curves of Ω and let $F_k = 2\frac{\partial \omega_k}{\partial z}$, where ω_k is the harmonic* *measure relative to* Ω. *Then there exists a real positive definite symmetric matrix* $C = [C_{jk}]$ *for which the Szegő kernel function is related to* F_k *via the identity*

$$\begin{bmatrix} \int_{\gamma_1} S(z,w)^2 dz \\ \vdots \\ \int_{\gamma_{n-1}} S(z,w)^2 dz \end{bmatrix} = \frac{i}{4\pi}(I - AC) \begin{bmatrix} \overline{F_1(w)} \\ \vdots \\ \overline{F_{n-1}(w)} \end{bmatrix},$$

where $A = [A_{mj}] = [\int_{\gamma_m} F_j(z)dz]$.

REMARK 4.2. S. Saitho proved in [13] that the class of functions $S(z,w)^2, w \in \Omega$ is complete in $H^2(b\Omega)$ using the fact that the set of functions

$$\int_{\gamma_m} S(z,w)^2 dz, m = 1, 2, \ldots, n-1$$

is linearly independent. The constant matrix C in the proof of Theorem 4.1 is real positive definite symmetric (see [10]).

From now on, we for simplicity assume that Ω is doubly connected. Thus there is exactly one inner boundary γ of Ω. We introduce two methods solving the Dirichlet problem on Ω due to Bell ([1], [3]) to relate the Szegő kernel to the function $F(z) = 2\frac{\partial \omega}{\partial z}$ where ω is the harmonic measure associated to the boundary curve γ. As before, let a be fixed in Ω and let a_1 be the only simple zero of $S(z,a)$. Bell showed in [1] that given any fixed point p of the bounded connected component of $\mathbb{C} \setminus \overline{\Omega}$ and given the boundary data $\varphi \in C^\infty(b\Omega)$ for the Dirichlet problem, the harmonic extension Φ of φ to Ω is

(4.1)

$$\begin{aligned}\Phi(z) &= \frac{P(\varphi(\zeta)S(\zeta,a))(a_1)}{P(S(\zeta,a)\log|\zeta - p|)(a_1)} \log|z-p| \\ &+ \frac{1}{S(z,a)} P\left(\left(\varphi(\zeta) - \frac{P(\varphi(w)S(w,a))(a_1)}{P((S(w,a)\log|w-p|))(a_1)} \log|\zeta - p|\right) S(\zeta,a)\right)(z) \\ &+ \overline{\frac{1}{L(z,a)} P\left(\left(\varphi(\zeta) - \frac{P(\varphi(w)S(w,a))(a_1)}{P((S(w,a)\log|w-p|))(a_1)} \log|\zeta - p|\right) L(\zeta,a)\right)(z)}\end{aligned}$$

Bell also proved in [3] that the harmonic extension Φ of φ to Ω is given by

(4.2)

$$\Phi(z) = \frac{iP(S(\zeta,a)\varphi(\zeta))(a_1)}{\int_\gamma L(\zeta,a_1)S(\zeta,a)d\zeta}\,\omega(z) + \frac{1}{S(z,a)}P\left(\left(\varphi(\zeta) - \frac{iP(S(\zeta,a)\varphi(\zeta))(a_1)}{\int_\gamma L(\zeta,a_1)S(\zeta,a)d\zeta}\omega(\zeta)\right)S(\zeta,a)\right)(z) + \frac{1}{\overline{L(z,a)}}\overline{P\left(\left(\overline{\varphi(\zeta) - \frac{iP(S(\zeta,a)\varphi(\zeta))(a_1)}{\int_\gamma L(\zeta,a_1)S(\zeta,a)d\zeta}\omega(\zeta)}\right)L(\zeta,a)\right)(z)}$$

Differentiating both sides of (4.1) and (4.2), since $F(z) = 2\dfrac{\partial\omega}{\partial z}$, we get

$$\frac{P(\varphi(\zeta)S(\zeta,a))(a_1)}{P(S(\zeta,a)\log|\zeta-p|)(a_1)}\frac{1}{z-p} - 2\frac{P(\varphi(\zeta)S(\zeta,a))(a_1)}{P(S(\zeta,a)\log|\zeta-p|)(a_1)}\frac{\partial}{\partial z}\left(\frac{P(\log|\zeta-p|S(\zeta,a))(z)}{S(z,a)}\right)$$

$$= \frac{iP(S(\zeta,a)\varphi(\zeta))(a_1)}{\int_\gamma L(\zeta,a_1)S(\zeta,a)d\zeta}F(z) - 2\frac{iP(S(\zeta,a)\varphi(\zeta))(a_1)}{\int_\gamma L(\zeta,a_1)S(\zeta,a)d\zeta}\frac{\partial}{\partial z}\left(\frac{P(\omega(\zeta)S(\zeta,a))(z)}{S(z,a)}\right)$$

Hence using the definition of the Szegő projection and the harmonic measure, we can represent the $F(z)$ in terms of the Szegő kernel as follows.

THEOREM 4.3. *Suppose Ω is a doubly connected domain in the plane bounded by the non-intersecting C^∞ simple closed curves. Let γ be the inner boundary curve of Ω and let $F = 2\dfrac{\partial\omega}{\partial z}$, where ω is the harmonic measure relative to γ. Let a be fixed in Ω and let a_1 be the only simple zero of $S(z,a)$. Suppose that*

$$\int_{b\Omega} S(a_1,\zeta)S(\zeta,a)\varphi(\zeta)ds \neq 0.$$

Then $F(z)$ can be written in terms of the Szegő kernel explicitly via the identity

$$F(z) = \frac{\int_\gamma S(\zeta,a)L(\zeta,a_1)d\zeta}{\int_{b\Omega} S(a_1,\zeta)S(\zeta,a)\log|\zeta-p|ds}\,\frac{1}{z-p}$$
$$+ 2i\,\frac{\partial}{\partial z}\left(\frac{\int_\gamma S(z,\zeta)S(\zeta,a)ds_\zeta}{S(z,a)}\right)$$
$$- 2\frac{\int_\gamma S(\zeta,a)L(\zeta,a_1)d\zeta}{\int_{b\Omega} S(a_1,\zeta)S(\zeta,a)\log|\zeta-p|ds}$$
$$\times\,\frac{\partial}{\partial z}\left(\frac{\int_\gamma S(\zeta,a)S(z,\zeta)\log|\zeta-p|ds_\zeta}{S(z,a)}\right).$$

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