J. Korean Math. Soc. **43** (2006), No. 1, pp. 183–198

q-EXTENSIONS OF GENOCCHI NUMBERS

MEHMET CENKCI, MÜMÜN CAN, AND VELI KURT

ABSTRACT. In this paper q-extensions of Genocchi numbers are defined and several properties of these numbers are presented. Properties of q-Genocchi numbers and polynomials are used to construct q-extensions of p-adic measures which yield to obtain p-adic interpolation functions for q-Genocchi numbers. As an application, general systems of congruences, including Kummer-type congruences for q-Genocchi numbers are proved.

1. Introduction

The Genocchi numbers G_n may be defined by the generating function

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty} G_n\frac{t^n}{n!}, \quad (|t|<\pi). \tag{1}$$

It satisfies $G_1=1, G_3=G_5=G_7=\cdots=0$, and even coefficients are given by

$$G_n=2\left(1-2^n\right)B_n=2nE_{2n-1}(0), \tag{2}$$

where B_n are Bernoulli numbers and $E_n(x)$ are Euler polynomials.

The Bernoulli numbers are defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, (|t|<2\pi),$$

which can be written symbolically as $e^{Bt}=\frac{t}{e^t-1}$, interpreted to mean that B^n must be replaced by B_n when we expand on the left. This

Received February 15, 2005.
2000 Mathematics Subject Classification: Primary 11B68, 43A05, 11S80, 11A07.
Key words and phrases: q-Genocchi numbers, p-adic measures, p-adic integral, Kummer congruences.
This work was supported by Akdeniz University Scientific Research Project Unit.

relation can also be written $e^{(B+1)t} - e^{Bt} = 1$, or, if we equate powers of t,

$$B_0 = 1, (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

where again we must first expand and then replace B^i by B_i. The Bernoulli polynomials are then

$$B_n(x) = (B+x)^n = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}.$$

The multiplication theorem for Bernoulli polynomials can be stated as follows: If n and m are positive integers, with $m > 1$, then

$$m^{1-n} B_n(mx) = \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right).$$

One of the most important theorem relating to Bernoulli numbers is the Staudt-Clausen Theorem:

THEOREM 1. ([3]) *For* $m \geq 1$,

$$B_{2m} = A_{2m} - \sum_{(p-1)|2m} \frac{1}{p},$$

where A_{2m} *is an integer and the summation is over all primes* p *such that* $(p-1)|2m$.

What Theorem 1 tells us is equivalent to if $m \geq 1$, then the denominator of B_{2m} (in lowest terms) is exactly the product of those primes p for which $p-1$ divides $2m$.

It follows from (2) and the Staudt-Clausen Theorem that the Genocchi numbers are integers.

The Euler polynomials $E_n(x)$ may be defined by the generating function

$$(3) \qquad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi).$$

From (3) and (1) we deduce that

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}.$$

For real x, the Genocchi polynomials $G_n(x)$ can be defined as follows:

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad (|t|<\pi). \tag{4}$$

Note that $G_n(0)=G_n$, and

$$G_n(x)=\sum_{k=0}^{n}\binom{n}{k}G_k x^{n-k}.$$

For an odd positive integer m, the multiplication theorem for the Genocchi polynomials can be stated as

$$m^{n-1}\sum_{j=0}^{m-1}(-1)^j G_n\left(\frac{x+j}{m}\right)=G_n(x), \tag{5}$$

which follows from (4).

In [1] and [2] Carlitz defined a set of numbers $\eta_n=\eta_n(q)$ inductively by

$$\eta_0=1,(q\eta+1)^n-\eta_n=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1,\end{cases}$$

with the usual convention about replacing η^i by η_i. These numbers are q-extensions of the ordinary Bernoulli numbers B_n, but they do not remain finite when $q=1$. So he modified the definition as

$$\beta_0=1,q(q\beta+1)^n-\beta_n=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases}$$

These numbers were called the q-Bernoulli numbers, which reduce to B_n when $q=1$. Defining q-Bernoulli polynomials, he also proved properties generalizing those satisfied by B_n and $B_n(x)$. In [19], Koblitz used these properties, especially the distribution relation for q-Bernoulli polynomials, to construct q-extension of p-adic L-functions which interpolate the q-Bernoulli numbers. By Koblitz's suggestion, q-analogues of the Dirichlet L-series were constructed by Satoh [20]. The series were essentially defined as a sum of two q-series, what causes difficulty in studying these series. In [10], Kim gave explicit formulas of p-adic q-L-functions which interpolate generalized q-Bernoulli numbers attached to a primitive Dirichlet character χ.

The remarkable relation between Bernoulli and Genocchi numbers (2) represents a method to define q-Genocchi numbers in connection with q-Bernoulli numbers. In [17], Kim et al. defined q-Genocchi numbers and q-zeta functions which interpolated q-Genocchi numbers at non-positive integers, with the help of this relation. Han and Zeng treated a q-analogue of the median Genocchi numbers and discussed their relations to some polynomials and ordinary Genocchi numbers, including some continued fraction expansions, in [5]. In [6] Han et al. gave a new q-analogue of Euler numbers, and unlike the generating functions of the previous q-analogues of these numbers (e.g., G. E. Andrews, I. Gessel, Proc. Amer. Math. Soc. 68(1978), no. 3, 380–384; and G. E. Andrews, D. Foata, European J. Combin. 1(1980), no. 4, 183–287), the generating functions for these new analogues had elegant continued fraction expansions. They also gave combinatorial interpretations of their q-Euler numbers and explained the relation to ordinary Genocchi numbers.

In this paper we give another construction of q-Genocchi numbers using the methods appear in Kim's recent papers [11], [12], [14] and [15]. We prove several properties for q-Genocchi numbers, and using these properties we define q-extensions of p-adic measures which enables us to obtain p-adic interpolation function for q-Genocchi numbers. Furthermore we give some applications of this p-adic interpolation function, in particular, we obtain general systems of congruences, including Kummer-type congruences for q-Genocchi numbers, following the approach in Young's papers [21] and [22].

2. Construction of q-extensions of Genocchi numbers

Throughout this paper p will denote an odd prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{C}$ the field of complex numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$, as usual. If K is a finite extension of $\mathbb{Q}_p$, then $\mathbb{D}_K$ will denote its ring of integers and $\mathbb{D}_K^\times$ will denote the multiplicative group of units in $\mathbb{D}_K$. When talking about q-extensions, q can be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, it will be assumed that $|1-q|_p < p^{-1/(p-1)}$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$, where $\nu_p(p)$ be the normalized exponential valuation of $\mathbb{C}_p$. Thus for $|x|_p \leq 1$, we have $q^x = \exp(\log_p q)$, where $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$ is the Iwasawa p-adic logarithm, the unique function which is given by the usual series $\sum(-1)^{n+1}(x-1)^n/n$ when $|x-1|_p < 1$; satisfies $\log_p(xy) = \log_p x + \log_p y$ and normalized by the condition $\log_p p = 0$ (see [8]).

We use the notation

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Thus

$$\lim_{q\to 1}[x:q]=x,$$

for any x in the complex case and any x with $|x|_p\le 1$ in the p-adic case.

The Teichmüller character w on $\mathbb{Z}_p^\times$ is defined by setting $w(x)$ be the unique $(p-1)$th root of unity congruent to x modulo $p\mathbb{Z}_p$.

In the complex case, we denote the generating function of q-Genocchi numbers $G_k(q)$ by $F_q^{(G)}(t)$ and define by

$$F_q^{(G)}(t)=\sum_{k=0}^{\infty}G_k(q)\frac{t^k}{k!}=q(1+q)t\sum_{n=0}^{\infty}(-1)^n q^n e^{[n]t}, \tag{6}$$

where q is a complex number with $|q|<1$. The remarkable point is that the series on the right of (6) is uniformly convergent in the wider sense. Hence, expanding e^t and comparing the powers of t, we have

$$G_k(q)=kq(1+q)\sum_{n=0}^{\infty}(-1)^n q^n[n]^{k-1}. \tag{7}$$

$F_q^{(G)}(t)$ is uniquely determined as a solution of the following q-difference equation:

$$F_q^{(G)}(t)=-e^tF_q^{(G)}(qt)+q(1+q)t.$$

From (6), it is easy to see that

$$G_k(q)=\frac{q(1+q)}{(1-q)^{k-1}}\sum_{m=0}^{k}\binom{k}{m}\frac{m(-1)^{m+1}}{1+q^m}. \tag{8}$$

We also have

$$\begin{aligned}\sum_{k=0}^{\infty}G_k(q)\frac{t^k}{k!}&=2q(1+q)t\sum_{n=0}^{\infty}q^{2n}e^{[n:q^2][2]t}-q(1+q)t\sum_{n=0}^{\infty}q^ne^{[n]t}\\&=\frac{q^2-1}{2\log q}e^{\frac{[2]t}{1-q^2}}-\frac{2q(1+q)}{[2]}\sum_{k=0}^{\infty}\beta_k(q^2)[2]^k\frac{t^k}{k!}\\&\quad-\frac{q-1}{\log q}e^{\frac{t}{1-q}}+q(1+q)\sum_{k=0}^{\infty}\beta_k(q)\frac{t^k}{k!},\end{aligned}$$

where $\beta_k(q)$ are q-analogues of Bernoulli numbers defined by the generating function

$$F_q(t)=\frac{q-1}{\log q}e^{\frac{t}{1-q}}-t\sum_{n=0}^{\infty}q^ne^{[n]t},$$

for $|t| < 1$ (for additional information about generating functions of q-Bernoulli numbers see, for example, [4], [13], [18], [20]). Equating powers of t we obtain

$$G_k(q) = q(1+q)\left(\beta_k(q) - 2[2]^{k-1}\beta_k(q^2)\right) + \frac{(q-1)^2}{2\log q}\frac{1}{(1-q)^k},$$

which is the q-analogue of (2).

Using (8) we can determine q-Genocchi numbers explicitly. For example, the first few q-Genocchi numbers are

$$G_0(q) = 0, G_1(q) = q, G_2(q) = -\frac{2q^2}{1+q^2},\ G_3(q) = -\frac{3q^2(1-q^2)}{(1+q^2)(1+q^3)}$$

For the limiting case $q = 1$, we obtain the ordinary Genocchi numbers G_k.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define

$$\zeta_q^{(G)}(s) = q(1+q)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^n}{[n]^s}. \tag{9}$$

Note that $\zeta_q^{(G)}(s)$ is a memoporphic function on $\mathbb{C}$ with only one simple pole at $s = 1$. The values of $\zeta_q^{(G)}(s)$ at non-positive integers are obtained by the following theorem:

THEOREM 2. *For any positive integer k, we have*

$$\zeta_q^{(G)}(1-k) = -\frac{G_k(q)}{k}.$$

Proof. It is clear by (7). □

REMARK 3. The main motivation of this paper originates from the limiting case $q = 1$ in (9). This yields the formula

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{2t^{s-1}}{e^t+1}dt,$$

from which the ordinary Genocchi numbers appear as residue of the integral. Thus defining

$$\zeta_G(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s},$$

we have

$$\zeta_G(1-k) = \begin{cases} -1 & \text{if } k = 1 \\ -\frac{G_k}{k} & \text{if } k > 1 \end{cases}$$

(see [9]). Therefore q-Genocchi numbers are the q-analogues of the values of $\zeta_G(s)$ at non-positive integers.

For positive integer k, we define q-Genocchi polynomials $G_k(q, x)$ as

$$(10) \qquad G_k(x,q) = (q^x G(q) + [x])^k = \sum_{m=0}^{k} \binom{k}{m} G_m(q) q^{mx} [x]^{k-m}.$$

The generating function of q-Genocchi polynomials is then

$$(11) \qquad F_q^{(G)}(x,t) = \sum_{k=0}^{\infty} G_k(x,q) \frac{t^k}{k!} = F_q^{(G)}(q^x t) e^{[x]t}.$$

From (11) it follows that

$$F_q^{(G)}(x,t) = q(1+q)t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]t},$$

and

$$G_k(x,q) = \frac{q(1+q)}{(1-q)^{k-1}} \sum_{m=0}^{k} \binom{k}{m} \frac{m(-1)^{m+1} q^{mx}}{1+q^m}.$$

For a primitive Dirichlet character χ of conductor an odd natural number f, we define the generalized q-Genocchi numbers attached to χ as

$$(12) \qquad G_{k,\chi}(q) = [f]^{k-1} \sum_{a=1}^{f} \chi(a)(-1)^a G_k\left(\frac{a}{f}, q^f\right)$$

We conclude this section with the following lemma which is important for the construction of the p-adic q-Genocchi measures.

LEMMA 4. (q-distribution relation) *For any positive odd integer m, we have*

$$(13) \qquad [m]^{k-1} \sum_{a=0}^{m-1} \frac{(-1)^a}{q^m(1+q^m)} G_k\left(\frac{x+a}{m}, q^m\right) = \frac{G_k(x,q)}{q(1+q)},$$

for all $k > 0$.

Proof.

$$\sum_{k=0}^{\infty}[m]^{k-1}\sum_{a=0}^{m-1}\frac{(-1)^a}{q^m(1+q^m)}G_k\left(\frac{x+a}{m},q^m\right)\frac{t^k}{k!}$$
$$=\sum_{a=0}^{m-1}\frac{(-1)^a}{q^m(1+q^m)}\frac{1}{[m]}\sum_{k=0}^{\infty}G_k\left(\frac{x+a}{m},q^m\right)\frac{([m]t)^k}{k!}$$
$$=\sum_{a=0}^{m-1}\frac{(-1)^a}{q^m(1+q^m)}\frac{[m]t}{[m]}q^m(1+q^m)\sum_{n=0}^{\infty}(-1)^n(q^m)^{n+\frac{x+a}{m}}e^{[n+\frac{x+a}{m}:q^m][m]t}$$
$$=t\sum_{a=0}^{m-1}(-1)^a\sum_{n=0}^{\infty}(-1)^n q^{mn+x+a}e^{[mn+x+a]t}$$
$$=t\sum_{a=0}^{m-1}\sum_{n=0}^{\infty}(-1)^{mn+a}q^{mn+x+a}e^{[mn+x+a]t}$$
$$=t\sum_{j=0}^{\infty}(-1)^j q^{j+x}e^{[j+x]t}$$
$$=\frac{1}{q(1+q)}\sum_{k=0}^{\infty}G_k(x,q)\frac{t^k}{k!}.$$

Comparing the coefficients of $t^k/k!$ yields the stated result. □

REMARK 5. Writing $x=0$ in (13), we get

$$[m]^{k-1}\sum_{a=0}^{m-1}\frac{(-1)^a}{q^m(1+q^m)}G_k\left(\frac{a}{m},q^m\right)=\frac{G_k(q)}{q(1+q)}.$$

Then using (10), we obtain

$$\begin{aligned}&\frac{[m]}{q(1+q)}G_k(q)-\frac{[m]^k}{q^m(1+q^m)}G_k(q^m)\\ &=\sum_{j=0}^{k-1}\binom{k}{j}\frac{G_j\left(q^m\right)}{q^m(1+q^m)}[m]^j\sum_{a=0}^{m-1}(-1)^a q^a[a]^{k-j}.\end{aligned}\tag{14}$$

(14) is the q-analogue of the recurrence formula for ordinary Genocchi numbers presented by Howard [7].

3. q-Genocchi measures

For a positive odd integer f, let

$$\mathbb{X} = \varprojlim_{n} \mathbb{Z}/fp^n\mathbb{Z},$$
$$a + fp^n\mathbb{Z}_p = \{x \in \mathbb{X} : x \equiv a (\mathrm{mod} fp^n)\},$$
$$\mathbb{X}^* = \bigcup_{\substack{0<a<fp^n \\ (a,p)=1}} (a + fp^n\mathbb{Z}_p).$$

The natural map $\mathbb{Z}/fp^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ induces $\pi : \mathbb{X} \longrightarrow \mathbb{Z}_p$. If f is a function on $\mathbb{Z}_p$, we also use f to denote the function $f \circ \pi$ on $\mathbb{X}$.

DEFINITION 6. A $\mathbb{C}_p$-valued measure μ on $\mathbb{X}$ is a bounded finitely additive map from the set of compact open $U \subset \mathbb{X}$ to $\mathbb{C}_p$.

A bounded function μ on compact open sets of the form $a + fp^n\mathbb{Z}_p$ extends to a measure if and only if additivity is checked for the disjoint unions $a + fp^n\mathbb{Z}_p = \bigcup (b + fp^{n+1}\mathbb{Z}_p)$ with the union taken over the p values of b, $0 \le b < fp^{n+1}$, for which $b \equiv a (\mathrm{mod}\, fp^n)$. A measure μ extends to continuous functions:

$$\int f d\mu = \lim_{n\to\infty} \sum_{a=0}^{fp^n-1} f(a)\mu(a + fp^n\mathbb{Z}_p). \tag{15}$$

We now define q-Genocchi measures.

DEFINITION 7. For $k \ge 0$ and $n \ge 0$,

$$g_q^k(a + fp^n\mathbb{Z}_p) = [fp^n]^{k-1} \frac{(-1)^a}{q^{fp^n}\left(1 + q^{fp^n}\right)} G_k\left(\frac{a}{fp^n}, q^{fp^n}\right).$$

Note that if $k = 1$ then

$$g_q^1(a + fp^n\mathbb{Z}_p) = g_q(a + fp^n\mathbb{Z}_p) = \frac{(-1)^a}{q^{fp^n}\left(1 + q^{fp^n}\right)} q^a q^{fp^n} = \frac{(-1)^a q^a}{1 + q^{fp^n}}$$

is the q-Genocchi distribution.

LEMMA 8. *g_q^k is a measure on $\mathbb{X}$ for all $k \ge 0$.*

Proof. It suffices to check that

$$\sum_{i=0}^{p-1} g_q^k(a + ifp^n + fp^{n+1}\mathbb{Z}_p) = g_q^k(a + fp^n\mathbb{Z}_p).$$

$$\begin{aligned}
&\sum_{i=0}^{p-1} g_q^k(a+ifp^n+fp^{n+1}\mathbb{Z}_p) \\
&= \sum_{i=0}^{p-1} [fp^{n+1}]^{k-1} \frac{(-1)^{a+ifp^n}}{q^{fp^{n+1}}\left(1+q^{fp^{n+1}}\right)} G_k\left(\frac{a+ifp^n}{fp^{n+1}}, q^{fp^{n+1}}\right) \\
&= [fp^{n+1}]^{k-1}(-1)^a \sum_{i=0}^{p-1} \frac{\left((-1)^{fp^n}\right)^i}{(q^{fp^n})^p\left(1+(q^{fp^n})^p\right)} G_k\left(\frac{\frac{a}{fp^n}+i}{p}, \left(q^{fp^n}\right)^p\right).
\end{aligned}$$

Using Lemma 4 and the relation $[fp^{n+1}] = [fp^n]\left[p : q^{fp^n}\right]$, the lemma follows, since f is odd. □

We can express the q-Genocchi numbers as an integral over $\mathbb{X}$, by using the measure g_q^k.

LEMMA 9. *For any $k \geq 0$, we have*

$$\int_{\mathbb{X}} \chi(x)dg_q^k(x) = \begin{cases} \frac{G_{k,\chi}(q)}{q^f(1+q^f)} & \text{if } \chi \neq 1, \\ \frac{G_k(q)}{q(1+q)} & \text{if } \chi = 1. \end{cases}$$

Proof.

$$\begin{aligned}
&\int_{\mathbb{X}} \chi(x)dg_q^k(x) \\
&= \lim_{n\to\infty} \sum_{a=0}^{fp^n-1} [fp^n]^{k-1} \frac{\chi(a)(-1)^a}{q^{fp^n}(1+q^{fp^n})} G_k\left(\frac{a}{fp^n}, q^{fp^n}\right) \\
&= \lim_{n\to\infty} [f]^{k-1} \sum_{a=0}^{f-1} \chi(a)(-1)^a \left[p^n : q^f\right]^{k-1} \\
&\quad \times \sum_{i=0}^{p^n-1} \frac{(-1)^{i+f}}{(q^f)^{p^n}\left(1+(q^f)^{p^n}\right)} G_k\left(\frac{\frac{a}{f}+i}{p^n}, \left(q^f\right)^{p^n}\right) \\
&= \lim_{n\to\infty} [f]^{k-1} \sum_{a=0}^{f-1} \frac{\chi(a)(-1)^a}{q^f(1+q^f)} G_k\left(\frac{a}{f}, q^f\right) \\
&= \begin{cases} \frac{G_{k,\chi}(q)}{q^f(1+q^f)} & \text{if } \chi \neq 1, \\ \frac{G_k(q)}{q(1+q)} & \text{if } \chi = 1. \end{cases}
\end{aligned}$$

□

Finally we give a relation between g_q^k and g_q.

LEMMA 10. *For any $k \geq 0$ we have*

$$dg_q^k(x) = k[x]^{k-1} dg_q(x).$$

Proof. By (10) we have

$$\begin{aligned}
&g_q^k\left(a + fp^n\mathbb{Z}_p\right)\\
&= [fp^n]^{k-1} \frac{(-1)^a}{q^{fp^n}\left(1+q^{fp^n}\right)} G_k\left(\frac{a}{fp^n}, q^{fp^n}\right)\\
&= [fp^n]^{k-1} \frac{(-1)^a}{q^{fp^n}\left(1+q^{fp^n}\right)} \sum_{i=0}^{k} \binom{k}{i} G_i\left(q^{fp^n}\right) q^a \left[\frac{a}{fp^n} : q^{fp^n}\right]^{k-i}\\
&= \frac{(-1)^a q^a}{q^{fp^n}\left(1+q^{fp^n}\right)} \sum_{i=0}^{k} \binom{k}{i} G_i\left(q^{fp^n}\right) [a]^{k-i} [fp^n]^{i-1}\\
&= \frac{(-1)^a q^a}{q^{fp^n}\left(1+q^{fp^n}\right)} k q^{fp^n} [a]^{k-1} + [fp^n] \times (p\text{-integral}).
\end{aligned}$$

Therefore we obtain

$$dg_q^k(x) = k[x]^{k-1} dg_q(x). \qquad \square$$

4. Interpolation function and congruences for q-Genocchi numbers

In this section, using the integral representation of q-Genocchi numbers in the foregoing section, we define a q-l-series which interpolates q-Genocchi numbers at non-positive integers. As an application of this representation we prove general systems of congruences for q-Genocchi numbers, including Kummer-type congruences.

Let w be the Teichmüller character mod$p$. For $x \in \mathbb{X}^*$, we set $\langle x : q \rangle = [x]/w(x)$. Note that since $|\langle x : q \rangle - 1|_p < p^{-1/(p-1)}$, $\langle x : q \rangle^s$ is defined by $\exp(s\log_p \langle x : q \rangle)$, for $|s|_p \leq 1$ and $\langle x : q \rangle^{p^n} \equiv 1 \,(\text{mod} p^n)$.

Fix an embedding of the algebraic closure of $\mathbb{Q}$, $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$. We may then consider the values of Dirichlet character χ as lying in $\mathbb{C}_p$. For $n \in \mathbb{Z}$ we define the product $\chi_n = \chi w^{-n}$ in the sense of the product of characters. This implies that $f_{(\chi_n)} | f_{(\chi)} p$. However, since we can write $\chi = \chi_n w^n$, we also have $f_{(\chi)} | f_{(\chi_n)} p$. Thus $f_{(\chi)}$ and $f_{(\chi_n)}$ differ by a factor that is a power of p. In fact, either $f_{(\chi_n)}/f_{(\chi)} \in \mathbb{Z}$ and divides p, or $f_{(\chi)}/f_{(\chi_n)} \in \mathbb{Z}$ and divides p.

We define an interpolation function for q-Genocchi numbers as follows:

Definition 11. For $s \in \mathbb{Z}_p$

$$l_{p,q}^{(G)}(s,\chi) = \int_{\mathbb{X}^*} (1-s)\langle x:q\rangle^{-s} \chi(x) dg_q(x).$$

The values of this function at non-positive integers are given by:

Theorem 12. *For any $k \geq 1$, we have*

$$l_{p,q}^{(G)}\left(1-k,\chi w^{k-1}\right) = \begin{cases} \frac{G_{k,\chi}(q)}{q^f(1+q^f)} - \frac{[p]^{k-1}\chi(p)}{q^{pf}(1+q^{pf})} G_{k,\chi}(q^p) & \text{if } \chi \neq 1, \\ \frac{G_k(q)}{q(1+q)} - \frac{[p]^{k-1}}{q^p(1+q^p)} G_{k,\chi}(q^p) & \text{if } \chi = 1. \end{cases}$$

Proof.

$$\begin{aligned} & l_{p,q}^{(G)}\left(1-k,\chi w^{k-1}\right) \\ &= \int_{\mathbb{X}^*} k\langle x:q\rangle^{k-1} \chi w^{k-1}(x) dg_q(x) \\ &= \int_{\mathbb{X}^*} k[x]^{k-1}\chi(x) dg_q(x) \\ &= k\int_{\mathbb{X}} [x]^{k-1}\chi(x) dg_q(x) - k\int_{p\mathbb{X}} [x]^{k-1}\chi(x) dg_q(x) \\ &= k\int_{\mathbb{X}} [x]^{k-1}\chi(x) dg_q(x) - k\int_{\mathbb{X}} [px]^{k-1}\chi(px) dg_q(px) \\ &= k\int_{\mathbb{X}} [x]^{k-1}\chi(x) dg_q(x) \\ &\quad - k[p]^{k-1}\chi(p)\int_{\mathbb{X}} [x:q^p]^{k-1}\chi(x) dg_{q^p}(x) \\ &= \int_{\mathbb{X}} \chi(x) dg_q^k(x) - [p]^{k-1}\chi(p)\int_{\mathbb{X}} \chi(x) dg_{q^p}^k(x) \\ &= \begin{cases} \frac{G_{k,\chi}(q)}{q^f(1+q^f)} - \frac{[p]^{k-1}\chi(p)}{q^{pf}(1+q^{pf})} G_{k,\chi}(q^p) & \text{if } \chi \neq 1, \\ \frac{G_k(q)}{q(1+q)} - \frac{[p]^{k-1}}{q^p(1+q^p)} G_{k,\chi}(q^p) & \text{if } \chi = 1, \end{cases} \end{aligned}$$

where we use Lemma 10,

$$g_q\left(px + fp^{n+1}\mathbb{Z}_p\right) = \frac{(-1)^{px} q^{px}}{1+q^{fp^{n+1}}} = \frac{(-1)^x (q^p)^x}{1+(q^p)^{fp^n}} = g_{q^p}\left(x + fp^n\mathbb{Z}_p\right)$$

and Lemma 9. □

We now give general systems of congruences for q-Genocchi numbers.

Let $K_q = \mathbb{Q}_p(q)$. For $i \in \mathbb{Z}$, we consider $w^{i-1}g_q$ as a $\mathbb{D}_{K_q}$-valued measure on $\mathbb{D}_{K_q}$. Let $q \in \mathbb{D}_{K_q}$ with $|1-q|_p < p^{-1/(p-1)}$. Then $q \equiv 1(\mathrm{mod} p\mathbb{D}_{K_q})$. If $x \in \mathbb{D}_{K_q}^\times$ then $(x,p)=1$ and

$$[x] = \frac{1-q^x}{1-q} = 1 + q + \cdots + q^{x-1} \equiv x \left(\mathrm{mod}\ p\mathbb{D}_{K_q}\right).$$

Thus we have $\langle x : q\rangle \equiv 1 \left(\mathrm{mod}\ p\mathbb{D}_{K_q}\right)$.

If c is a nonnegative integer, the difference operator Δ_c operates on the sequence $\{\alpha_m\}$ by

$$\Delta_c \alpha_m = \alpha_{m+c} - \alpha_m.$$

The powers Δ_c^l of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^l = \Delta_c \circ \Delta_c^{l-1}$ for positive integers l, so that

$$\Delta_c^l \alpha_m = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \alpha_{m+jc}. \tag{16}$$

THEOREM 13. *Let $c \equiv 0 \left(\mathrm{mod}(p-1)p^A\right)$ with $A \geq 0$. Then*

$$\Delta_c^l \frac{G_m(q)}{m} \equiv 0 \left(\mathrm{mod} p^{A'}\mathbb{D}_{K_q}\right)$$

for all $m \geq 1$ and $l \geq 0$, where $A' = \min\{m-1, l(A+1)\}$.

Proof. From Lemma 9 for a primitive character χ we have

$$\int_{\mathbb{D}_{K_q}} dg_q^k(x) = \int_{\mathbb{D}_{K_q}} k[x]^{k-1} dg_q(x) = \frac{G_k(q)}{q(1+q)}. \tag{17}$$

The function $T_{g_q}(s,i)$ defined for $s \in \mathbb{Z}_p$ by

$$T_{g_q}(s,i) = \int_{\mathbb{D}_{K_q}^\times} \langle x : q\rangle^s w^{i-1}(x) dg_q(x)$$

is the p-adic q-Γ-transform of the measure $w^{i-1}g_q$. Furthermore when n is a nonnegative integer, $n \geq 1$, with $n \equiv i(\mathrm{mod}(p-1))$, we have

$$T_{g_q}(n-1,i) = \int_{\mathbb{D}_{K_q}^\times} [x]^{n-1} dg_q(x). \tag{18}$$

It follows from (16) and (18) that for $c \equiv 0 \left(\bmod (p-1)p^{A}\right)$, we have

$$\begin{aligned}\Delta_c^l T_{g_q}(m-1,i) &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}T_{g_q}(m-1+jc,i)\\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\int\limits_{\mathbb{D}_{K_q}^{\times}}[x]^{m-1+jc}dg_q(x)\\ &= \int\limits_{\mathbb{D}_{K_q}^{\times}}[x]^{m-1}\left([x]^c-1\right)^l dg_q(x),\end{aligned}$$

when $m \equiv i(\bmod(p-1))$. Since $([x]^c-1)^l \equiv 0\left(\bmod\, p^{lA'}\mathbb{D}_{K_q}\right)$ for all $x \in \mathbb{D}_{K_q}^{\times}$ (where $A' = A+1$), and g_q is an $\mathbb{D}_{K_q}$-valued measure, this implies

$$\Delta_c^l T_{g_q}(m-1,i) \equiv 0\left(\bmod\, p^{lA'}\mathbb{D}_{K_q}\right).$$

On the other hand, $[x]^{m-1} \equiv 0\left(\bmod\, p^{m-1}\mathbb{D}_{K_q}\right)$ for all $x \in p\mathbb{D}_{K_q}$, so from (17), (18) we obtain

$$\begin{aligned}T_{g_q}(m-1,i) &= \int\limits_{\mathbb{D}_{K_q}^{\times}}[x]^{m-1}dg_q(x)\\ &\equiv \frac{1}{m}\int\limits_{\mathbb{D}_{K_q}} m[x]^{m-1}dg_q(x)\\ &= \frac{G_m(q)}{m}\left(\bmod\, p^{m-1}\mathbb{D}_{K_q}\right).\end{aligned}$$

Therefore

$$\Delta_c^l T_{g_q}(m-1,i) \equiv \Delta_c^l\frac{G_m(q)}{m}\left(\bmod\, p^{m-1}\mathbb{D}_{K_q}\right),$$

which yields the stated result. □

5. Final remarks

Professor Taekyun Kim has pointed out the following connections between q-Genocchi numbers and q-Volkenborn integral:

In [11] he defined the p-adic q-integrals as

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x)\, q^x,$$

that is, $\mu_q(x)$ defined by

$$\mu_q\left(x + p^N \mathbb{Z}_p\right) = \frac{q^x}{[p^N]}.$$

The q-Genocchi numbers then can be defined as

$$G_k(q) = q \int_{\mathbb{Z}_p} [x]^{k-1}\, d\mu_{(-q)}(x).$$

In [16] he considered q-numbers by using q-Volkenborn integral as follows:

$$\int_{\mathbb{Z}_p} [x]^k\, d\mu_{(-q)}(x) = K_{k,q}$$

for positive integer k. From this it can be noted that

$$K_{k,q} = [2] \left(\frac{1}{1-q}\right)^k \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}},$$

which is similar to (8).

ACKNOWLEDGMENTS. The authors wish to express their sincere gratitude to the referees for their valuable suggestions and comments and to Professor Taekyun Kim for his kind help and collaboration.

References

[1] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.

[2] ———, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc. **76** (1954), 332–350.

[3] ———, *The Staudt-Clausen theorem*, Math. Mag. **34** (1961), 131–146.

[4] M. Cenkci, M. Can, and V. Kurt, *p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers*, Advan. Stud. Contem. Math. **9** (2004), no. 2, 203–216.

[5] G. -N. Han and J. Zeng, *On a sequence that generalizes the median Genocchi numbers*, Ann. Sci. Math. Quebec **23** (1999), no. 1, 63–72.

[6] G. -N. Han, A. Randrianarivony, and J. Zeng, *Un autre q-analogue des nombres d'Euler*, Seminaire Lotharingien de Combinatorie **42** Art. B42e, (1999), 22pp. (electronic).

[7] F. T. Howard, *Applications of a recurrence formula for the Bernoulli numbers*, J. Number Theory **52** (1995), no. 1, 157–172.
[8] K. Iwasawa, *Lectures on p-adic L-functions*, Ann. of Math. Studies Vol: 74, Princeton Univ. Press, Princeton, N. J., 1972.
[9] L. -C. Jang, T. Kim, D. -H. Lee, and D. -W. Park, *An application of polylogarithms in the analogs of Genocchi numbers*, Notes Number Theory Discrete Math. **7** (2001), no. 3, 65–69.
[10] T. Kim, *On explicit formulas of p-adic q-L-functions*, Kyushu J. Math. **48** (1994), no. 1, 73–86.
[11] ———, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory **76** (1999), no. 2, 320–329.
[12] ———, *q-Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), no. 3, 288–299.
[13] ———, *On p-adic q-L-functions and sums of powers*, Discrete Math. **252** (2002), no. 1–3, 179–187.
[14] ———, *Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russ. J. Math. Phys. **10** (2003), no. 1, 91–98.
[15] ———, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys. **10** (2003), no. 3, 261–267.
[16] ———, *A note on q-Volkenborn integration*,(English. English summary) Proc. Jangjeon Math. Soc. **8** (2005), no. 1, 13–17.
[17] T. Kim, L. -C. Jang, and H. K. Pak, *A note on q-Euler and Genocchi numbers*, Proc. Japan Acad. Ser. A Math. Sci. **77** (2001), no. 8, 139–141.
[18] T. Kim, Y. Simsek, and H. M. Srivastava, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12** (2005), no. 2, 201–228.
[19] N. Koblitz, *On Carlitz's q-Bernoulli numbers*, J. Number Theory **14** (1982), no. 3, 332–339.
[20] J. Satoh, *q-analogue of Riemann's ζ-function and q-Euler numbers*, J. Number Theory **31** (1989), no. 3, 346–362.
[21] P. T. Young, *Congruences for Bernoulli, Euler and Stirling numbers*, J. Number Theory **78** (1999), no. 2, 204–227.
[22] ———, *On the behaviour of some two-variable p-adic L-functions*, J. Number Theory **98** (2003), no. 1, 67–88.

Mehmet Cenkci, Mümün Can, and Veli Kurt
Akdeniz University
Department of Mathematics
07058-Antalya, Turkey
E-mail: cenkci@akdeniz.edu.tr
mcan@akdeniz.edu.tr
vkurt@akdeniz.edu.tr