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ON YI'S EXTENSION PROPERTY FOR TOTALLY PREORDERED TOPOLOGICAL SPACES

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ABSTRACT. The objective of this paper is to show further results concerning the problem of extending total preorders from a subset of a topological space to the entire space using the approach introduced by Gyoseob Yi.

1. Introduction

In the present paper we address the problem of extending a continuous total preorder defined on a subset of a topological space, to the whole space, keeping the continuity.

A motivation to analyze this problem is the search for interdisciplinary applications, mainly related with Economic Theory. Thus, in several contexts of utility theory some properties of extension appear in a natural way. It may happen, for instance, that we have local utility functions that act on a small part of a space on which a preference has been defined, and that we are interested in building a global utility function that may "glue" the local ones extending them to the whole space. (see e.g., Section 4.1 in Candeal et al. [7]). It may also happen that a preference (not necessarily representable through a utility function) has been defined on some subset of a given set, and we ask ourselves if it is possible to extend such preference to the entire set, satisfying some desirable properties (e.g., the extension should be a continuous preorder if the given preference was a continuous preorder on the subset). Despite

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both problems have an obvious relationship, they are, actually, of different nature: The extension of total preorders or "preferences" (studied in Yi [25] or Candeal et al. [6]) is quite different from the extension of order-preserving real-valued maps or "utility functions" (also known as the "lifting problem", and considered, e.g., in Herden [15]) as we shall discuss later.

Among the different families of extension properties that have been considered in the literature, in the framework of preferences and utility representations, we shall deal in the present paper with the approach introduced by the Korean mathematical economist Gyoseob Yi (see Yi [25]). Yi considered the problem of extending preferences from closed subsets of a commodity space to the entire space.

It is a plain corollary of the well-known Tietze extension theorem that continuous preferences admit an extension if they are defined on closed subsets of a normal topological space and, in addition, they are representable by continuous utility functions.

This context was generalized in Nachbin [20] through the concept of a normally preordered space and the so-called Nachbin's extension theorem. (See, e.g., Ch. 5 in Bridges and Mehta [7]).

But even if a topological space is normal, it could happen, however, that there exist continuous preferences that do not admit a utility representation. Thus, for instance, a continuous total preorder defined on a path-connected space is representable by a continuous utility function if and only if it is countably bounded. (See Monteiro [19]). Also, in a non-separable metric space it is always possible to define a continuous total preorder without a continuous utility representation. (See Estévez and Hervés [14]). Obviously, in those cases the problem of extending utility functions becomes different from the problem of extending preferences.

The problem of extension of preferences, and the problem of representability of preferences through continuous utility functions also have a close relationship. Indeed in the case of metric spaces that are path-connected, they are actually equivalent, as proved in Yi [25] and generalized in Candeal et al. [6] to the separably connected case. However, in the general case, both questions are no longer equivalent. Despite the problem of representability of preferences through continuous utility functions has been solved since long (see e.g. Ch. 3 in Bridges and Mehta [7]), the general problem of characterizing Yi's extension property for preferences has not been solved, or at least no such characterization has been published yet. (However, we must point out that a characterization of the topologies that satisfy the Yi's extension property has been announced to us by Prof. G. Herden.) In the present paper we furnish some partial results related to such question. We introduce some necessary and sufficient conditions for a topological space to have Yi's extension property.

Then we analyze the relationship between Yi's extension property and some properties that involve continuous representability of preferences.

In the last section, we analyze the analogous of Yi's extension property for the semicontinuous case, proving that semicontinuous extensions always exist, even without asking the subsets considered to be closed (as required in the "continuous" Yi's extension property). This means, in particular, that the original (continuous) Yi's extension property is more restrictive, because not every topological space satisfies the continuous Yi's extension property, but all satisfy its semicontinuous analogue.

2. Previous concepts and results

Let X be a nonempty set. For a preference $\precsim$ on the set X we will understand a total preorder (i.e., a reflexive, transitive and complete binary relation) defined on X. (If $\precsim$ is also antisymmetric, it is said to be a total order). We denote $x \prec y$ instead of $\neg(y \precsim x)$. Also $x \sim y$ will stand for $(x \precsim y) \wedge (y \precsim x)$ for every $x, y \in X$.

The total preorder $\precsim$ is said to be representable if there exists a real-valued order-preserving isotony (also called utility function) $f : X \longrightarrow \mathbb{R}$. Thus $x \precsim y \iff f(x) \leq f(y)$ $(x, y \in X)$. This fact is characterized (see e.g. Bridges and Mehta [5], p.23) by equivalent conditions of "order-separability" that the preorder $\precsim$ must satisfy. Thus, the total preorder $\precsim$ is said to be order-separable in the sense of Debreu if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists an element $d \in D$ such that $x \precsim d \precsim y$. Such subset D is said to be order-dense in $(X, \precsim)$.

If X is endowed with a topology τ, the total preorder $\precsim$ is said to be *continuously representable* if there exists a utility function f that is continuous with respect to the topology τ on X and the usual topology on the real line $\mathbb{R}$. The total preorder $\precsim$ is said to be τ-continuous if the sets $U(x) = \{y \in X \ , \ x \prec y\}$ and $L(x) = \{y \in X \ , \ y \prec x\}$ are τ-open, for every $x \in X$. In this case, the topology τ is said to be natural or compatible with the preorder $\precsim$. (see Bridges and Mehta [5], p.19). The coarsest natural topology is the order topology θ whose subbasis is the collection $\{L(x) \ : x \in X\} \cup \{U(x) \ : x \in X\}$.

Following Monteiro [19], a nonempty subset $Y \subseteq X$ is said to bound the total preorder $\precsim$ if for every $x \in X$ there exist $a, b \in Y$ such that $a \precsim x \precsim b$.

The total preorder $\precsim$ is said to be countably bounded if there exists a countable subset $D \subseteq X$ such that for every $x \in X$ there exist $a, b \in D$ such that $a \precsim x \precsim b$.

A powerful tool to obtain continuous representations of an order-separable totally preordered set $(X, \precsim)$ endowed with a natural topology τ is the Debreu's open gap lemma. (see Debreu [10], or Ch. 3 in Bridges and Mehta [5]). To this extent, let T be a subset of the real line $\mathbb{R}$. A lacuna L corresponding to T is a nondegenerate interval of $\mathbb{R}$ that has both a lower bound and upper bound in T and that has no points in common with T. A maximal lacuna is said to be a Debreu gap. Debreu's open gap lemma states that if S is a subset of the extended real line $\bar{\mathbb{R}}$, then there exists a strictly increasing function $g : S \longrightarrow \mathbb{R}$ such that all the Debreu gaps of $g(S)$ are open. Using Debreu's open gap lemma, the classical process to get a continuous real-valued isotony goes as follows: First, one can easily construct a (non necessarily continuous!) isotony f representing $(X, \precsim)$ when $\precsim$ is order-separable (see e.g. Birkhoff [3], Theorem 24 on p.200, or else Bridges and Mehta [5], Theorem 1.4.8 on p.14). Once we have an isotony f, Debreu's open gap is applied to find a strictly increasing function $g : f(X) \longrightarrow \mathbb{R}$ such that all the Debreu gaps of $g(f(X))$ are open. Consequently, the composition $F = g \circ f : X \longrightarrow \mathbb{R}$ is also a utility function representing $(X, \precsim)$, but now F is continuous with respect to any given natural topology τ on X.

A topological space (X, τ) is said to be separable if there exists a countable subset $D \subseteq X$ that meets every nonempty τ-open subset of X. (D is said to be τ-dense.) Also, (X, τ) is said to be second countable if the topology τ has a countable basis.

A topological space (X, τ) is said to be path-connected if for every $a, b \in X$ there exists a continuous map (a path) $f_{a,b} : [0, 1] \longrightarrow X$ such that $f(0) = a$, $f(1) = b$. Also, it is said to be separably connected if for every $a, b \in X$ there exists a connected and separable subset $C(a, b) \subseteq X$ such that $a, b \in C(a, b)$. Path connected implies separably connected, and this implies connected, but the converses are not true in general. (see Candeal et al. [6]).

A topological space (X, τ) is said to have Yi's extension property (Yi [25]) if an arbitrary continuous total preorder defined on an arbitrary τ-closed subset of X has a continuous extension to the whole X.

Similarly, (X, τ) is said to satisfy the continuous representability property (CRP) (see e.g. Candeal et al. [8]) if every continuous total preorder on X admits a continuous utility representation. The first studies on the CRP property are due to Herden, who used a different terminology. In Herden [16], if a topological space (X, τ) satisfies CRP then its topology τ is said to be a useful topology. The structure of such topologies was analyzed and characterized in Herden and Pallack [17].

The ordered sum of two disjoint totally preordered sets $(Z_1, \precsim_1)$ and $(Z_2, \precsim_2)$, that we shall denote $(Z_1, \precsim_1) + (Z_2, \precsim_2)$ is defined to be $(Z_1 \bigcup Z_2, \preceq)$, where $x \preceq y$ if and only if $x, y \in Z_1$ and $x \precsim_1 y$, or $x \in Z_1$ and $y \in Z_2$, or $x, y \in Z_2$ and $x \precsim_2 y$.

REMARK 2.1 From the definition of Yi's extension property the following results follow:

(a) If (X, τ) is a topological space that satisfies Yi's extension property, and $Y \subseteq X$ is a nonempty closed subset of X, then (Y, τ_Y) also satisfies Yi's extension property, where τ_Y is the topology that τ induces on Y. Indeed observe that a closed subset $Z \subseteq Y$ in τ_Y is closed in X, so that a τ_Y continuous total preorder $\precsim_Z$ defined on Z is also τ-continuous, so that it can be continuously extended to X, so that in particular it has a continuous extension to Y.

(b) Let (X_1, τ_{X_1}) and (X_2, τ_{X_2}) be two nonempty Hausdorff topological spaces. Let X be the product space $X_1 \times X_2$. Let τ_π be the product topology on X. If (X, τ_π) has Yi's extension property, then each factor (X_i, τ_{X_i}) $(i = 1, 2)$ also satisfies Yi's extension property. This is a consequence of the above Remark 2.1 (a), since for each elements $x_1 \in X_1$, $x_2 \in X_2$ it holds that X_1 is isomorphic to $X_1 \times \{x_2\}$, a closed subset of (X, τ_π). Similarly X_2 is isomorphic to $\{x_1\} \times X_2$.

(c) Suppose that $(X, \precsim)$ is the ordered sum of two disjoint totally preordered sets $(X_1, \precsim_1)$ and $(X_2, \precsim_2)$. Let X, X_1 and X_2 be endowed with the corresponding order topologies, that we shall denote, respectively, θ, θ_1 and θ_2. If (X, θ) satisfies Yi's extension property, then each (X_i, θ_i) $(i = 1, 2)$ also satisfies Yi's extension property. Observe that each (X_i, θ_i) $(i = 1, 2)$ is obviously isomorphic to a closed subset of (X, θ). Then apply Remark 2.1(a).

3. Yi's extension property for continuous preferences

Let (X, τ) be a Hausdorff topological space. (X, τ) is said to be normal (or T_4) if for each pair of disjoint τ-closed subsets $A, B \subseteq X$

there exists a pair of disjoint τ-open subsets $A^*, B^* \subseteq X$ such that $A \subseteq A^*$, $B \subseteq B^*$. (For basic topological definitions see Dugundji [11]).

It is well-known that this property of being normal is equivalent to an extension property for continuous real-valued functions. This is the "Tietze's extension theorem".

THEOREM 3.1. (Tietze's extension Theorem) *Let (X, τ) be a Hausdorff topological space. The following two properties are equivalent:*

(i) *(X, τ) is normal,*

(ii) *For every τ-closed subset $A \subseteq X$, each continuous map $f : A \to \mathbb{R}$ admits a continuous extension $F : X \to \mathbb{R}$.*

Moreover, if $f(X) \subseteq [-a, a]$ for some $a > 0 \in \mathbb{R}$, then F can be chosen so that $F(X) \subseteq [-a, a]$.

Now suppose that (X, τ) is a normal topological space. An immediate corollary of Tietze's extension theorem states that continuous and representable preferences defined on closed subsets of X can be continuously extended to the entire set X.

COROLLARY 3.2. *Let (X, τ) be a normal topological space. Let $S \subseteq X$ be a τ-closed subset of X. Let $\precsim_S$ be a continuous total preorder defined on S. Then if $\precsim_S$ is representable through a continuous utility function $u_S : S \to \mathbb{R}$, it can also be extended to a continuous total preorder $\precsim_X$ defined on the whole X.*

Proof. Just observe that, by Tietze's theorem, the utility function u_S admits a continuous extension to a map $u_X : X \to \mathbb{R}$. Then define $\precsim_X$ on X as $x \precsim_X y \iff u_X(x) \leq u_X(y) \quad (x, y \in X)$. □

Yi's extension property was initially understood as an strengthtening of Tietze's extension property, in a direction in which we are not interested in extending utility functions, but only preferences. Observe that Yi's and Tietze's extension properties are not equivalent in the general case. This is because preferences could fail to be representable.

In the case of separably connected metric spaces, Yi's extension property is equivalent to topological separability:

THEOREM 3.3. *Let (X, d) be a separably connected metric space, where d stands for the distance function. Consider on X the metric topology τ_d. The following properties are equivalent:*

(i) *(X, τ_d) satisfies Yi's extension property,*

(ii) *(X, τ_d) satisfies the continuous representability property (CRP),*

(iii) (X, τ_d) *is separable,*
(iv) (X, τ_d) *is second countable,*
(v) *Every* τ_d*-continuous total preorder defined on* X *is countably bounded.*

Proof. See Corollary 3 in Candeal et al. [6]. □

EXAMPLE 3.4. A metric topology is always normal. Thus, a separably connected, nonseparable metric space has Tietze's extension property, but not Yi's extension property.

Now we introduce some necessary conditions for a topological space (X, τ) to have Yi's extension property.

THEOREM 3.5. *Let* (X, τ) *be a topological space. Suppose that there exists a subset* $Y \subseteq X$ *endowed with a non-representable* τ_Y*-continuous total preorder* $\precsim_Y$ (where $\tau_Y$is the topology that τ induces on Y) *for which there exists a* τ*-connected and separable subset* $F \subseteq X$ *such that* $F \cap Y$ *bounds* $\precsim_Y$*. Then* $\precsim_Y$ *cannot be continuously extended to* X*. Thus, if* Y *is in addition closed, then* (X, τ) *does not satisfy Yi's extension property.*

Proof. Let us assume, by contradiction, that there is a continuous extension $\precsim_X$ to the entire set X of the total preorder $\precsim_Y$ defined on Y. The restriction of $\precsim_X$ to F, that we denote $\precsim_F$, is a continuous total preorder on F that will be continuously representable by the Eilenberg's representation theorem. (see Eilenberg [12], or Bridges and Mehta [5], Theorem 3.2.5 on p.46). Given an element $y \in Y$, let $A(y) = \{z \in F \ : \ z \precsim_X y\}$; $B(y) = \{t \in F \ : \ y \precsim_X t\}$. Since $\precsim_X$ is continuous, $A(y)$ and $B(y)$ are τ_F-closed, where τ_F is the restriction of the topology τ to the set F. Since $F \cap Y$ bounds $\precsim_Y$, it is clear that $A(y)$ and $B(y)$ are both nonempty. It follows then that there exists an element $y_F \in A(y) \cap B(y) \subseteq F$ since otherwise $\{A(y), B(y)\}$ will provoke a disconnection of F. Obviously, it follows by construction that $y_F \sim_X y$. Thus, a utility representation for the total preorder $\precsim_F$ on F will immediately furnish a utility function for the total preorder $\precsim_Y$ on Y, because each element $y \in Y$ has associated an element $y_F \in F$ such that $y_F \sim_X y$. But this leads to a contradiction, since $\precsim_Y$ was assumed to be non-representable. □

REMARKS 3.6.
(i) Theorem 3.5 can also be restated in the following way:

Let (X, τ) be a topological space. Let $Y \subseteq X$ a nonempty subset of X. Let τ_Y be the topology that τ induces on Y. Let $\precsim_Y$ be a τ_Y-continuous total preorder defined on Y. Let $F \subseteq X$ be a τ-connected and separable subset such that $F \cap Y$ bounds $\precsim_Y$. If $\precsim_Y$ has a continuous extension to the whole X, then $\precsim_Y$ is continuously representable.

As an immediate corollary, taking $X = Y$, we obtain Theorem 1 in Monteiro [19]:

Let (X, τ) be a topological space. Let $\precsim$ be a τ-continuous total preorder on X. If there exists a τ-connected and separable subset $F \subseteq X$ that bounds $\precsim$, then $\precsim$ is representable by means of a continuous utility function.

(ii) The above Theorem 3.5 has been inspired by Example 3.2 in Yi [25]. Such example corresponds to a topological space known as the "long rectangle", that is a typical example appearing under the conditions in the statement of Theorem 3.5. As a matter of fact, in Theorem 3.5 we have shown that the situation shown in that Example 3.2 in Yi [25] is, so to say, "general".

In the particular case of separably connected topological spaces, we can furnish more necessary conditions for the satisfaction of Yi's extension property. To do so, first we need to introduce some previous definition and lemma.

DEFINITION. Let (X, τ) be a topological space. A subset $Y \subseteq X$ is said to be τ-discrete if the topology τ_Y that τ induces on Y, is the discrete one. In other words, this means that for every element $y \in Y$ there exists a τ-open subset $O_y \subseteq X$ such that $O_y \cap Y = \{y\}$.

THEOREM 3.7. *Let (X, τ) be a separably connected topological space. If there exists an uncountable subset $Y \subseteq X$ that is τ-discrete and τ-closed, then (X, τ) does not satisfy Yi's extension property.*

Proof. See Theorem 3 in Candeal et al. [6]. (For the more restrictive case of path-connected topological spaces, a proof was given in Yi [25], Theorem 3.3.) □

EXAMPLE 3.8. A (sophisticated) example of the situation that appears under the statement of Theorem 3.7 is the following one, analyzed in Corson [9], pp.5–9: Consider the Banach space $C_0(X)$ of continuous complex valued functions which vanish at infinity, on a locally compact group X. If X is not metrizable then $C_0(X)$, in its weak topology ω, contains an uncountable ω-discrete closed subset. Since $(C_0(X), \omega)$ is a path-connected topological space, it is in particular separably connected.

LEMMA 3.9. *Let (X, τ) be a separably connected topological space. Let $\precsim$ be a total preorder on X. If $\precsim$ is countably bounded, then there exists a τ-connected and separable subset $F \subseteq X$ that bounds $\precsim$. Moreover, if $\precsim$ is τ-continuous then it is continuously representable.*

Proof. The existence of F is proved in Theorem 4 in Candeal et al. [6]. Once we have F, the continuous representability of $\precsim$ follows from the second part of Remark 3.6 (i). □

COROLLARY 3.10. *Let (X, τ) be a separably connected topological space. Let $\precsim$ be a τ-continuous total preorder defined on X. Then $\precsim$ is continuously representable if and only if it is countably bounded.*

Proof. (See also Corollary 2 in Candeal et al. [6]) It is plain that continuously representable implies countably bounded. Indeed if $u : X \to \mathbb{R}$ is a continuous utility function that represents $\precsim$, and that we take bounded without loss of generality, then the countable subset $u^{-1}(\mathbb{Q}) \cup u^{-1}(\{\inf u(X), \sup u(X)\})$ bounds $\precsim$. The converse has been stated in Lemma 3.9. □

REMARK 3.11. The result in Corollary 3.10 was already known for the path-connected case. (see Theorem 3 in Monteiro [19]).

THEOREM 3.12. *Let (X, τ) be a separably connected topological space. Let $Y \subseteq X$ be a nonempty subset of X. Let τ_Y be the topology that τ induces on Y. Let $\precsim_Y$ be a τ_Y-continuous total preorder defined on Y. If $\precsim_Y$ is countably bounded and non-representable, then $\precsim_Y$ cannot be continuously extended to X. Thus, if Y is in addition closed, then (X, τ) does not satisfy Yi's extension property.*

Proof. Let $D \subseteq Y$ be a countable subset that bounds $\precsim_Y$. Fix an element $d \in D$. For every other element $d^* \in D$ let $C_{\{d,d^*\}} \subseteq X$ be a connected and separable subset of X such that $d, d^* \in C_{\{d,d^*\}}$. It is clear that $F = \bigcup_{d^* \in D} C_{\{d,d^*\}}$ is connected and separable, and obviously $D \subseteq F \cap Y$, so that $F \cap Y$ also bounds Y. Therefore, by Theorem 3.5, $\precsim_Y$ cannot be continuously extended to X. □

REMARKS 3.13.

(i) Observe, as an application of Lemma 3.9 and Corollary 3.10, that the subset Y that appears in the statement of Theorem 3.12 cannot be separably connected. (In particular Y must be a proper subset of X, i.e., $Y \subsetneq X$).

(ii) Since a continuously representable total preorder is obviously countably bounded, Theorem 3.12 can also be restated in the following way, that is also a generalization of Corollary 3.10:

Let (X, τ) be a separably connected topological space. Let $Y \subseteq X$ a nonempty subset of X. Let τ_Y the topology that τ induces on Y. Let $\precsim_Y$ be a τ_Y-continuous total preorder defined on Y. Suppose also that $\precsim_Y$ can be continuously extended to the whole X. Then $\precsim_Y$ is countably bounded if and only if it is continuously representable.

(iii) As an immediate corollary we obtain the following result:

Let (X, τ) be a separably connected topological space. Let $Y \subseteq X$ a nonempty subset of X. Let τ_Y the topology that τ induces on Y. Let $\precsim_Y$ be a τ_Y-continuous total preorder defined on Y. Then if $\precsim_Y$ is countably bounded and can be continuously extended to the whole X, then $\precsim_Y$ is continuously representable.

(iv) If we take $Y = X$ in the statement of Theorem 3.12, since $\precsim_Y$ is plainly extended to X (because $Y = X$) it follows that either $\precsim_Y$ fails to be countably bounded or else it is representable. Equivalently, it is representable if and only if it is countably bounded. Thus we reobtain Corollary 3.10, now as a direct consequence of Theorem 3.12.

EXAMPLE 3.14. (Long rectangle) The following example was issued in Yi [25], Example 3.2 on p.550. It is noticeable that it appears under the conditions of the statements of Theorem 3.5, and Theorem 3.12: First, the long line L is constructed as follows (see e.g. Steen and Seebach [23], p.71 or Monteiro [19]): Let Ω be the first uncountable ordinal. Given any ordinal $\alpha \in [0, \Omega)$ insert the open interval $(0, 1)$ of real numbers just between α and its successor $\alpha + 1$. The resulting set is the long line L, ordered in the obvious natural way. On L we shall consider the order topology. The long rectangle $L_{\mathcal{R}}$ is defined as $L \times [0, 1]$, endowed with the product topology (where the closed real interval $[0, 1]$ is given the Euclidean topology).

The long rectangle is path-connected, so that it is in particular separably connected. The closed subset $Y = L \times \{0, 1\} \subsetneq L_{\mathcal{R}}$ can be given the non-representable continuous total preorder $\precsim_Y$ defined in Yi [25], p.550. Such total preorder is obviously countably bounded since it has a first point $(0, 0)$ and a last point $(0, 1)$. Therefore, this example corresponds to the statement of Theorem 3.12.

Now let $F = \{0\} \times [0,1] \subsetneq L_{\mathcal{R}}$. It is clear that F is connected and separable in $L_{\mathcal{R}}$. Moreover $F \cap Y$ bounds Y. Therefore, this example also corresponds to the statement of Theorem 3.5. Finally, observe that Y is not separably connected, because it is indeed disconnected.

We must also point out that in Yi [25] it is said that: "Using the intuition obtained in the above example, we provide a sufficient condition on the space under which continuous extension is not possible". (See Yi [25], p.551). As a matter of fact, the result provided by Yi is our Theorem 3.7 above (but stated only for the particular case of path-connected topological spaces). However, the example provided does not correspond to the statement of Theorem 3.7. This is because the subset $Y \subsetneq L_{\mathcal{R}}$, even being uncountable and closed, does not inherit the discrete topology. Actually, due to the presence of limit ordinals, it is straightforward to see that no uncountable closed subset of Y inherits the discrete topology. Consequently, Theorem 3.5 and Theorem 3.12 on the one hand, and Theorem 3.7 on the other hand are, so to say, "of a different nature".

4. Extensibility and continuous representability

There are several contexts in which the satisfaction of Yi's extension property leads to some property of continuous representability of preferences (as CRP or similars). In the present section we present further results in that direction, showing that Yi's extension property implies some representability properties. In general, such representability properties do not lead to the satisfaction of Yi's extension property, that is stronger.

To start with, remember that Theorem 3.3 stated that in a separably connected metric space, CRP and Yi's extension property are equivalent. However, in the general case CRP and Yi extension property are no longer equivalent, as the Example 4.5 below will show.

To introduce the example, we need some preparatory definition and results.

Let (L, θ_L) be the long line endowed with its usual total order. (See Example 3.14 above). Let $Y = L \times \{0,1\}$. Endow Y with the total preorder $\precsim_Y$ defined as follows:

i) $(a,0) \prec_Y (b,1)$ for every $a, b \in L$ such that either $a \neq 0$ or $b \neq 0$,
ii) $(0,0) \sim_Y (0,1)$,
iii) $(a,0) \precsim_Y (b,0) \iff b \precsim_L a$ for every $a, b \in L$,

iv) $(a,1) \precsim_Y (b,1) \iff a \precsim_L b$ for every $a, b \in L$. $(Y, \precsim_Y)$ is called the double long line. Separably connected topological spaces have a property of continuous ordinal representability in the double long line, as the next lemma states.

LEMMA 4.1. *Let (X, τ) be a separably connected topological space. Let $\precsim$ be a τ-continuous total preorder on X. Let θ_Y denote the order topology on the double long line Y. There exists a continuous map $F : (X, \tau) \longrightarrow (Y, \theta_Y)$ such that*

$$x \precsim y \iff F(x) \precsim_Y F(y) \quad (x, y \in X).$$

Proof. We can suppose without loss of generality that $\precsim$ is actually a *total order*, because since $\precsim$ is τ-continuous, the quotient space $(X/\sim) = \{[x] : x \in X\}$, where $[x] = \{y \in X : y \sim x\}$ is also separably connected with respect to the quotient topology induced by τ on $(X/\sim)$. Also, again because $\precsim$ is τ-continuous, it is enough to find a continuous isotony $F : (X, \theta_X) \to (Y, \theta_Y)$, where θ_X stands for the order topology on X. Since θ_X is coarser than τ, (X, θ_X) is also separably connected. Indeed, it is path-connected by Remark 2 (iv) in Candeal et al. [6]. Two cases may occur now: If $(X, \precsim)$ is representable, there exists a continuous real-valued isotony. But it is clear that the real line with its usual topology and order can be also continuously embedded in the double long line (Y, θ_Y). If otherwise $(X, \precsim)$ fails to be representable, Remark 3.2 (iii) in Beardon et al. [2] states that it can be also continuously embedded in (Y, θ_Y) through an isotony. (See also Theorem 5 in Monteiro [19], of which this result is a generalization). This finishes the proof. $\square$

DEFINITION. Let X be a nonempty set endowed with a topology τ. The topological space (X, τ) is said to satisfy the countable chain condition (ccc) if every family of pairwise disjoint τ-open subsets is countable.

LEMMA 4.2. *Let X be a Banach space endowed with its weak topology ω. Then (X, ω) satisfies the countable chain condition ccc.*

Proof. (See also Corson [9], p.8). Notice that a real Banach space in its weak topology is homeomorphic in a natural way to a dense subset of a product of copies of the real numbers. The real line $\mathbb{R}$ is second countable in its usual Euclidean topology, and a product of second countable spaces satisfies ccc in the product topology. (See Engelking [13], Corollary 2.3.18). Finally, a dense subset of a topological space that satisfies ccc also satisfies ccc in the induced topology. (See Szpilrajn-Marczewski [24], or Lemma 4 in Corson [9]). □

LEMMA 4.3. *Let (X, τ) be a separably connected topological space that satisfies the countable chain condition ccc. Then it also satisfies the continuous representability property CRP.*

Proof. By Lemma 4.1, if $\precsim$ is a τ-continuous total preorder on X, there exists a continuous map $F : (X, \tau) \longrightarrow (Y, \theta_Y)$ such that $x \precsim y \iff F(x) \precsim_Y F(y) \quad (x, y \in X)$ where $(Y, \precsim_Y)$ denotes the double long line. By continuity of F, the subset $F(X) \subseteq Y$ is connected.

Let us prove now that $F(X)$ is order-bounded in Y, that is, there exist elements $a, b \in Y$ such that $a \prec_Y x \prec_Y b$ for every $x \in X$: Suppose, by contradiction, that $F(X)$ is not order-bounded. For a given ordinal number $\alpha < \Omega$ let $\alpha + 1$ be its follower, and call $A_\alpha = \{z \in Y : (\alpha + 1, 0) \prec_Y z \prec_Y (\alpha, 0)\}$, $B_\alpha = \{z \in Y : (\alpha, 1) \prec_Y z \prec_Y (\alpha + 1, 1)\}$. It is clear that the family $\{F^{-1}(A_\alpha)\} \bigcup \{F^{-1}(B_\alpha)\} (\alpha < \Omega)$ is an uncountable family of nonempty and pairwise disjoint open subsets of X, which contradicts the fact of (X, τ) satisfying the countable chain condition ccc. Finally, observe that every order-bounded subset of the double long line is continuously isotonic to a subset of the real line. (See Monteiro [19] or Beardon et al. [2]). This finishes the proof. □

THEOREM 4.4. *Let X be a Banach space endowed with its weak topology ω. Then (X, ω) satisfies the continuous representability property CRP.*

Proof. We already know that any compatible topology in a topological vector space (in particular, the weak topology of a Banach space) is separably connected. Now, by Lemma 4.2 we have that (X, ω) satisfies the countable chain condition ccc. And finally, according to Lemma 4.3, (X, ω) must satisfy CRP. □

EXAMPLE 4.5. Consider again (see Example 3.8 above) the Banach space $C_0(X)$ of continuous complex valued functions which vanish at infinity, on a locally compact group X, where X is not metrizable. Endow $C_0(X)$ with the weak topology ω. It has been proved in Theorem 4.4. above that all Banach spaces satisfy CRP with respect to the weak topology. Thus $(C_0(X), \omega)$ always satisfies CRP. However, as commented in Example 3.8, if X is a non-metrizable locally compact group then

$(C_0(X), \omega)$ fails to satisfy Yi's extension property. Therefore the continuous representability property (CRP) does not imply Yi's extension property, in general.

Yi's extension property is in some sense more restrictive that the continuous representability property, as next result shows.

THEOREM 4.6. *Let (X, τ) be a separably connected topological space that has Yi's extension property. Let Y be a closed subset of X such that $X \setminus Y = \{x \in X : x \notin Y\}$ has at least two different elements. Let τ_Y the topology that τ induces on the subset Y. Then (Y, τ_Y) satisfies CRP.*

Proof. A first proof was given for the more restrictive case of path-connected spaces in Yi [25], Lemma 4.1 on p.554. The proof for the separably connected case goes as follows:

Let $Y' = Y \cup \{a, b\}$ where a and b are two different elements in $X \setminus Y$. Let $\tau_{Y'}$ the topology that τ induces on Y'. Let $\precsim_Y$ be a τ_Y-continuous total preorder on Y. We extend $\precsim_Y$ to the $\tau_{Y'}$-continuous total preorder $\precsim_{Y'}$ defined on Y' by declaring that $a \prec_{Y'} y \prec_{Y'} b$; $a \precsim_{Y'} b$, for every $y \in Y$. Since, being (X, τ) Hausdorff and Y a τ-closed subset, the subset Y' is also τ-closed, it follows by Yi's extension property that $\precsim_{Y'}$ has a continuous extension to the whole X. Let F be a τ-connected and separable subset to which both a and b belong. (Such F exists because (X, τ) is separably connected by hypothesis). It is plain that $F \cap Y'$ bounds $\precsim_{Y'}$, so that by Remark 3.6 (i), $\precsim_{Y'}$ has a continuous representation by means of a utility function $u_{Y'} : Y' \longrightarrow \mathbb{R}$. Considering now the restrictions to Y of the preorder $\precsim_{Y'}$ and the map $u_{Y'}$, it is then obvious that $\precsim_Y$ also has a continuous utility representation. Therefore (Y, τ_Y) satisfies CRP. □

COROLLARY 4.7. *Let (X, τ) be a separably connected topological space that has Yi's extension property. Suppose also that X can be decomposed as the product space $(X_1, \tau_{X_1}) \times (X_2, \tau_{X_2})$ of two nonempty Hausdorff topological spaces such that each X_i $(i = 1, 2)$ has at least two different points, and τ is the product topology on X. Then each factor (X_i, τ_{X_i}) $(i = 1, 2)$ has the continuous representability property CRP.*

Proof. It is plain that each factor (X_i, τ_{X_i}) $(i = 1, 2)$ is isomorphic to a closed subset Y_i of X such that $X \setminus Y_i$ $(i = 1, 2)$ contains at least two different points. The result follows now as a direct consequence of Theorem 4.6. □

COROLLARY 4.8. *Suppose that $(X, \precsim)$ is the ordered sum of two disjoint totally preordered sets $(X_1, \precsim_1)$ and $(X_2, \precsim_2)$ such that each of the sets X_1 and X_2 has at least two different points. Let X, X_1 and X_2 be endowed with the corresponding order topologies, respectively denoted θ, θ_1 and θ_2. If (X, θ) is separably connected and satisfies Yi's extension property, then each (X_i, θ_i) $(i = 1, 2)$ satisfies CRP.*

Proof. Now it is clear that each (X_i, θ_i) $(i = 1, 2)$ is isomorphic to a closed subset of (X, θ) such that $X \setminus X_i$ $(i = 1, 2)$ contains at least two different points. □

EXAMPLE 4.9. (Long line) The long line L (already considered in Example 3.14 above) does not satisfy Yi's extension property with respect to the order topology θ_L: First observe that (L, θ_L) is path-connected (see e.g. Steen and Seebach [23], pp.196–197), hence it is in particular separably connected. Moreover, $L \setminus \{[0, 1)\}$, endowed with the induced order topology, is plainly isomorphic to the entire long line L, and it is closed with respect to θ_L. Since the long line is not representable by means of a utility function, it does not satisfy CRP. Therefore $L \setminus \{[0, 1)\}$ does not satisfy CRP with respect to the induced order topology, either. Now we apply Theorem 4.6 to conclude that L fails to satisfy Yi's extension property.

EXAMPLE 4.10. (Replicant spaces) A topological space (X, τ) is said to be replicant if it can be decomposed as the product space $(X_1, \tau_{X_1}) \times (X_2, \tau_{X_2})$ of two topological spaces such that at least one of them (say, e.g., (X_1, τ_{X_1})) is homeomorphic to the entire space (X, τ).

This is a typical situation encountered in Banach space theory, where it is well-known that several classical Banach spaces are replicant. In particular, the Pelczyński decomposition theorem (see Pelczyński [21]) states that:

Let X be the Banach space C_0 or ℓ_p $(1 \leq p < \infty)$, endowed with the norm topology. Then every infinite-dimensional subspace $Y \subseteq X$ contains a subspace Z that is isomorphic to X and complemented in X (i.e., X is isomorphic to a product $Z \times Z'$ for some Z').

As an immediate consequence of Corollary 4.3, we have now that:

Suppose that (X, τ) is a replicant, separably connected Hausdorff topological space with at least two different points. If (X, τ) satisfies Yi's extension property, then it also satisfies the continuous representability property CRP.

The result given in Remark 3.13 (ii) says that if (X, τ) is a separably connected topological space and $Y \subseteq X$ is a nonempty subset of X endowed with the topology τ_Y that τ induces on Y, then if a τ_Y-continuous total preorder $\precsim_Y$ defined on Y can be continuously extended to the whole X, it holds that $\precsim_Y$ is countably bounded if and only if it is continuously representable. This fact carries some consequences about the relationship between Yi's extension property and the satisfaction of the continuous representability property CRP on closed subsets of X.

LEMMA 4.11. *Let (X, τ) be a separably connected topological space that has Yi's extension property. Then the following assertions are equivalent:*

(i) *Every τ-continuous total preorder $\precsim_X$ defined on X is countably bounded.*
(ii) *Every τ-closed subset Y of X satisfies CRP with respect to the topology τ_Y that τ induces on Y.*

Proof. (i) $\Longrightarrow$ (ii): Every τ_Y continuous total preorder $\precsim_Y$ defined on any τ-closed subset Y of X has a τ-continuous extension $\precsim_X$ to the whole X, because (X, τ) has Yi's extension property. By the hypothesis (i), $\precsim_X$ is countably bounded, so that it is also continuously representable by Remark 3. 13 (ii). It is then obvious that its restriction to Y, namely $\precsim_Y$, is also continuously representable. Therefore (Y, τ_Y) satisfies CRP.

(ii) $\Longrightarrow$ (i): Every τ-continuous total preorder $\precsim_X$ defined on X is by hypothesis continuously representable because (X, τ) satisfies CRP. Hence the preorder $\precsim_X$ is in particular countably bounded. □

THEOREM 4.12. *Let (X, τ) be a separably connected topological space. Let the topology τ be normal. Suppose also that every τ-continuous total preorder $\precsim_X$ defined on X is countably bounded. Then the following assertions are equivalent:*

(i) *(X, τ) has Yi's extension property.*
(ii) *Every τ-closed subset Y of X satisfies CRP with respect to the topology τ_Y that τ induces on Y.*

Proof. (i) $\Longrightarrow$ (ii): If (X, τ) has Yi's extension property, it follows that every τ_Y-continuous total preorder $\precsim_Y$ defined on any τ-closed subset Y of X has a τ-continuous extension $\precsim_X$ to X. This extension is, by hypothesis, countably bounded, hence continuously representable by Corollary 3.10. Obviously its restriction to Y, namely $\precsim_Y$, is also continuously representable. Therefore (Y, τ_Y) satisfies CRP. (Actually, once we know that the extension is countably bounded, a direct application of Lemma 4.11 also shows that (Y, τ_Y) satisfies CRP.)

(ii) $\Longrightarrow$ (i): Let Y be a τ-closed subset of X, and $\precsim_Y$ a τ_Y-continuous total preorder defined on Y. Since (Y, τ_Y) satisfies CRP, there exists a continuous utility function $u_Y : (Y, \tau_Y) \to \mathbb{R}$ that represents Y. Thus (X, τ) has Yi's extension property as a direct consequence of Corollary 3.2. □

THEOREM 4.13. *Let (X, τ) be a normal and separably connected topological space. Let the topology τ be Lindelöf (i.e., every τ-open covering of X has a countable subcovering). Then the following assertions are equivalent:*

(i) *(X, τ) has Yi's extension property.*
(ii) *Every τ-closed subset Y of X satisfies CRP with respect to the topology τ_Y that τ induces on Y.*

Proof. Just observe that Lindelöf property implies that every τ-continuous total preorder $\precsim_X$ defined on X is countably bounded: Actually, let us assume that $\precsim_X$ is not countably bounded. Suppose that X has neither a first nor a last element with respect to $\precsim_X$. (The arguments for the cases in which X has either a first or a last element are analogous). Being $x \in X$ let $a_x, b_x \in X$ be such that $a_x \prec_X x \prec_X b_x$. Then the family $\{(a_x, b_x) : x \in X\}$ is a τ-open covering of X that does not admit a countable subcovering because otherwise $\precsim_X$ would be countably bounded. But this contradicts the fact of the topology τ being Lindelöf. The result follows now as a direct consequence of Theorem 4.12. □

EXAMPLE 4.14. (Weak-star topology in dual Banach spaces) Let X be a Banach space. Consider its dual Banach space X^* endowed with the weak-star topology ω^*. The topological space (X^*, ω^*) is path-connected, hence separably connected. It is also Lindelöf and normal. (see e.g. Yi [25], p.550). It is also known that if X^* is separable with respect to the norm topology, then every ω^*-closed subset C of X^* satisfies CRP with respect to the restriction to C of the norm topology on X^*. (see Candeal et al. [6], Theorem 1). As a consequence of this result and Debreu's open gap lemma, it follows that C also satisfies CRP with respect to the restriction to C of the weak-star topology ω^* on X^*, because ω^* is weaker than the norm topology.

Therefore, by Theorem 4.13, it follows that if X^* is separable with respect to the norm topology, then (X^*, ω^*) has Yi's extension property.

Observe also that this result is not a consequence of Theorem 3.3, since weak-star topologies are non-metrizable in general. (Actually (X^*, ω^*) is metrizable if and only if X^* is finite-dimensional with respect to its norm topology. The proof is similar to that of Theorem 6.30 in Aliprantis and Border [1].)

5. Semicontinuous extension properties

Till this point we have analyzed questions related to the continuous Yi's extension property that corresponds to the situation when every continuous total preorder $\precsim_S$ defined on a closed subset S of a topological space (X, τ) has a τ-continuous extension $\precsim_X$ on the whole X.

We may still ask ourselves about what happens for the semicontinuous case, dealing with semicontinuous total preorders and extensions, in the sense of the next definitions.

DEFINITION. Given a topological space (X, τ) a total preorder $\precsim$ defined on X is said to be τ-lower semicontinuous if the sets $U(x) = \{y \in X \ , \ x \prec y\}$ are τ-open, for every $x \in X$. In a similar way, $\precsim$ is said to be τ-upper semicontinuous if the sets $L(x) = \{y \in X \ , \ y \prec x\}$ are τ-open, for every $x \in X$.

The topology τ is said to satisfy the semicontinuous representability property (SRP) if every τ-upper (respectively τ-lower) semicontinuous total preorder $\precsim$ defined on X admits a numerical representation by means of a τ-upper (respectively τ-lower) semicontinuous real-valued order-preserving utility function.

Topologies satisfying SRP are also known as "completely useful topologies" in the literature. (see e.g. Bosi and Herden [4]). In general SRP implies CRP, but the converse is not true, as proved in Proposition 4.4 in Bosi and Herden [4], where some characterizations of topologies satisfying the semicontinuous representability property (SRP) were achieved. A classical theorem by Rader [22] (see also Isler [18]) states that "every second countable topology satisfies SRP". As an immediate consequence of this result and the fact "SRP $\Longrightarrow$ CRP", we obtain that, for the particular case of metric spaces we can add "(X, δ) satisfies SRP" to the equivalent conditions in the statement of Theorem 3.3.

Next theorem shows that semicontinuous extensions of total preorders are always available.

THEOREM 5.1. *Let X be a nonempty set endowed with a topology τ.*

Let S be a nonempty subset of X, let τ_S be the topology that τ induces on S, and let $\precsim_S$ be a τ_S-upper(respectively, lower) *semicontinuous total preorder defined on S. Then there exists a τ-upper* (respectively, lower) *semicontinuous total preorder $\precsim_X$ defined on the whole X, that extends $\precsim_S$.*

Proof. We give the proof for the upper semicontinuous case. For the lower semicontinuous case, the arguments are analogous. Thus, let us assume that $\precsim_S$ is a τ_S-upper semicontinuous total preorder defined on S. By hypothesis, the subsets $L_S(a) = \{z \in S \ : \ z \prec_S a\}$ $(a \in S)$ are all τ_S-open. Given $a \in S$, define

$$V_a = \bigcup_{\{O \in \tau \ : \ O \cap S \subseteq L_S(a)\}} O.$$

Observe that, by definition, $V_a \in \tau$, for every $a \in S$. Moreover

$$V_a \cap S = L_S(a) \quad (a \in S).$$

Actually, for every $a \in S$, it happens that $a \notin V_a$, and V_a is the biggest τ-open subset $V \subseteq X$ such that $V \cap A = L_S(a)$.

Also, being $a, b \in S$ it is clear by construction that

$$a \precsim_S b \iff V_a \subseteq V_b.$$

Define now the total preorder $\precsim_X$ on X given by:

$$x \precsim_X y \iff \text{ for every } \ a \in S \text{ it holds that } \ y \in V_a \implies x \in V_a.$$

The preorder $\precsim_X$ is τ-upper semicontinuous since

$$L(x) = \bigcup_{a \in S \ , \ x \notin V_a} V_a.$$

To conclude, let us prove that $\precsim_X$ extends $\precsim_S$: Take $a, b \in S$ such that $a \precsim_S b$. Suppose also that $b \in V_c$, for some $c \in S$. This implies the existence of a τ-open subset O such that $b \in O$; $O \cap S \subseteq L_S(c)$. Thus, in particular, it holds that $b \in L_S(c)$ since $b \in S$, and also $a \in L_S(c)$ because $a \in S, a \precsim_S b$. Therefore $a \in V_c$ because $a \in S$ and $V_c \cap S = L_S(c)$. Since this happens for every $c \in S$, we conclude that $a \precsim_X b$. $\square$

COROLLARY 5.2. *The semicontinuous representability property* (SRP) *is hereditary. That is, if X is a nonempty set and τ is a topology on X satisfying SRP, then for any nonempty subset $S \subseteq X$, the topology τ_S that τ induces on S also satisfies the semicontinuous representability property SRP.*

Proof. Let $\precsim_S$ be a τ_S-upper semicontinuous total preorder on S. By Theorem 5.1, there exists a τ-upper semicontinuous total preorder, say $\precsim_X$, defined on the whole X, of the total preorder $\precsim_S$ given on S. Since (X, τ) satisfies SRP, there exists a τ-upper semicontinuous utility function u_X that represents $\precsim_X$. It is straightforward to see now that the restriction of u_X to S, that we denote U_S is a τ_S-upper semicontinuous utility function that represents $\precsim_S$. □

REMARK 5.3. We have already mentioned that SRP implies CRP, but the converse is not true, in general. (see Bosi and Herden [4]). Using Corollary 5.2, we find now an indirect argument to prove this last assertion: First observe that since SRP implies CRP, and SRP is hereditary by Corollary 5.2, we obtain that:

If X is a nonempty set and τ is a topology on X satisfying SRP, then for any nonempty subset $S \subseteq X$, the topology τ_S that τ induces on S also satisfies the continuous representability property CRP. (In other words, SRP implies "CRP hereditarily"). If it were true that the property CRP is equivalent to SRP, it would follow in particular that the property CRP would be always hereditary. However, CRP is not hereditary, in general, as next example shows.

EXAMPLE 5.4. (The topology of "point p included"). Let X be an uncountable set. Select a point $p \in X$. Consider on X the topology τ_p of "point p included", that is, given $A \subseteq X$, $A \in \tau_p \iff A = \emptyset$ or else $p \in A$. It is straightforward to see that the only continuous total preorder on X is the trivial one that declares indifferent all the elements belonging to X. Therefore (X, τ_p) trivially satisfies CRP.

However, $X \setminus \{p\}$ inherits the discrete topology, so that any total preorder on $X \setminus \{p\}$ is continuous. If we consider a well-ordering on $X \setminus \{p\}$, it is clear that such well-ordering fails to have a representation (even discontinuous!) in the real line. (See e.g. Beardon et al. [2] for a further account). Consequently, ($X \setminus \{p\}$, discrete topology) does not satisfy CRP, so that (X, τ_p) does not satisfy "CRP hereditarily". Neither it satisfies Yi's extension property because, by the previous discussion, a well-ordering defined on the τ_p-closed subset $X \setminus \{p\}$ cannot be continuously extended to X.

Observe finally that, as a consequence of Corollary 5.2, we obtain indirectly that (X, τ_p) does not satisfy SRP. Obviously, this fact could have been observed directly: For instance, a well-ordering on X such that the first element is p is τ_p-upper semicontinuous, but it fails to have a representation (even discontinuous).

REMARK 5.5. Analyzing a bit more the last example, we find more arguments to say that "Yi's extension property" and the "continuous representability property (CRP)" are of a different nature. As was pointed out in Remark 2.1 (a), if a topological space (X, τ) satisfies Yi's extension property, then every nonempty closed subset $Y \subseteq X$ also satisfies Yi's extension property with respect to the induced topology. The analogous situation for CRP does not hold: In the above Example 5.4 we see that (X, τ_p) satisfies the continuous representability property (CRP), but $X \setminus \{p\}$, that is a closed subset of X, does not satisfy CRP with respect to the induced topology.

REMARK 5.6. Observe also that in relation to representation properties continuity is weaker (less restrictive) than semicontinuity, because SRP implies CRP but the converse is not true. However, in what concerns extension properties, the restrictive one is Yi's extension property, that deals with continuity, whereas the analogous extension property for semicontinuity instead of continuity always holds, as proved in Theorem 5.1. This is a new argument to say again that, in general, extension properties and representability properties are of a different nature.

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