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EINSTEIN SPACES AND CONFORMAL VECTOR FIELDS

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ABSTRACT. We study Riemannian or pseudo-Riemannian manifolds which admit a closed conformal vector field. Subject to the condition that at each point $p \in M^n$ the set of conformal gradient vector fields spans a non-degenerate subspace of T_pM, using a warped product structure theorem we give a complete description of the space of conformal vector fields on arbitrary non-Ricci flat Einstein spaces.

1. Introduction

In 1925, Brinkmann studied conformal mappings between Riemannian or pseudo-Riemannian Einstein spaces ([1]). Later conformal vector fields, or infinitesimal conformal mappings on Einstein spaces were reduced to the case of gradient vector fields, leading to a very fruitful theory of conformal gradient vector fields in general. Brinkmann's work has attracted renewed interest, especially in the context of general relativity ([2, 3, 4, 5, 9, 14, 17]), and the following local structure theorem has been shown:

PROPOSITION 1.1. (Kerckhove [9]) *Let (M^n, g) be an n-dimensional Einstein but not Ricci flat pseudo-Riemannian manifold with* Ric $= (n-1)kg(k \neq 0)$, *which carries a conformal vector field. Here we denote by* Ric *the Ricci tensor of (M^n, g). Suppose that the subspace $\Delta(p)$ spanned by the set of conformal gradient vector fields at $p \in M^n$ satisfies the following:*

(a) *$\Delta(p)$ is a non-degenerate subspace of T_pM for each $p \in M$,*

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(b) *the dimension m of $\Delta(p)$ is independent of the choice of the point p.*
Then (M^n, g) is locally isometric to a warped product $B^m(k)\times_f F$, where $B^m(k)$ is an m-dimensional space of constant sectional curvature k and the fibre (F, g_F) is an Einstein space with $\mathrm{Ric}_F = (n-m-1)\alpha g_F$ for some constant α.

Generalizing Kerckhove's results, the first two authors established a local structure theorem as follows:

PROPOSITION 1.2. ([11]) *Let (M^n, g) be an n-dimensional connected pseudo-Riemannian manifold. Suppose that there exists a nonzero constant $k \in \mathbb{R}$ such that*

(a) $\dim A_k(M^n, g) = m \geq 1$,
(b) *each subspace $\Delta(p)$ is a nondegenerate subspace of T_pM.*

Then, for a fixed $p_o \in M^n$ the following hold:

(1) *if $\dim \Delta(p_o) < m$, then (M^n, g) is locally isometric to a space form $B^n(k)$,*
(2) *if $\dim \Delta(p_o) = m$, then (M^n, g) is locally isometric to a warped product space $B^m(k) \times_{\sigma_T} F$, where σ_T is the height function for a vector T in the pseudo-Euclidean space of $B^m(k)$ and the fiber (F^{n-m}, g_F) is a pseudo-Riemannian manifold.*

Furthermore, F satisfies the following:

(i) *if $\langle T, T\rangle \neq 0$, then $A_\alpha(F, g_F) = \{0\}$, where $\alpha = \langle T, T\rangle$,*
(ii) *if $\langle T, T\rangle = 0$, then F carries no nontrivial homothetic gradient vector fields.*

In either case, we have $A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S, T\rangle = 0\}$, where $\tilde{\sigma}_S$ denotes the lifting of σ_S.

For the definitions of $A_k(M^n, g)$, $\Delta(p)$ and σ_T, see Section 2. Note that if (F, g_F) has constant sectional curvature $\alpha = \langle T, T\rangle$, then the warped product $M^n = B^m(k) \times_{\sigma_T} F$ has constant sectional curvature k. This shows that the space of closed conformal vector fields on $M^n = B^m(k) \times_{\sigma_T} F$ is of dimension $n+1$, which is greater than the dimension m of $\{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S, T\rangle = 0\}$. This shows that not all closed conformal vector fields on the warped product need to be lifted from the base.

On the other hand, the first two authors prove as follows that the necessary condition on the fibre F in (2) of Proposition 1.2 is sufficient for any closed conformal vector fields on the warped product $M^n = B^m(k) \times_{\sigma_T} F$ to be lifted from the base.

PROPOSITION 1.3. ([11]) *Let* (M^n, g) *be a warped product space* $B^m(k) \times_{\sigma_T} F$, *where* T *is a vector in the ambient pseudo-Euclidean space of* $B^m(k)$. *Suppose that the fibre* (F, g_F) *satisfies the following:*

(1) *if* $\langle T, T\rangle \neq 0$, *then* $A_\alpha(F, g_F) = \{0\}$, *where* $\alpha = \langle T, T\rangle$,
(2) *if* $\langle T, T\rangle = 0$, *then* F *carries no nontrivial homothetic gradient vector fields.*

Then (M^n, g) *satisfies the following:*

$$A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S, T\rangle = 0\}.$$

In particular, each subspace $\Delta(p)$ *is nondegenerate and of dimension* m.

In this paper, we study Riemannian or pseudo-Riemannian manifolds which admit a closed conformal vector field. Subject to the condition that at each point $p \in M^n$ the set of conformal gradient vector fields spans a non-degenerate subspace of T_pM, using Proposition 1.2 and Proposition 1.3 we give a complete description of the space of conformal vector fields on arbitrary non-Ricci flat Einstein spaces (Theorem 3.3).

2. Closed conformal vector fields

We consider an n-dimensional connected pseudo-Riemannian manifold (M^n, g) carrying a closed conformal vector fields V. Hence there is a smooth function ϕ on M^n such that

$$\nabla_X V = \phi X \tag{2.1}$$

for all $X \in TM$, where ∇ denotes the Levi-Civita connection on M^n. Then for every point $p \in M^n$ one can find a neighborhood U and a function f such that $V = \nabla f$, the gradient of f. Hence the Hessian $\nabla^2 f$ and the Laplacian Δf of f are given respectively by

$$\nabla^2 f = \phi g, \quad \Delta f = \mathrm{div} V = n\phi. \tag{2.2}$$

We denote by $CC(M^n, g)$ the vector space of closed conformal vector fields. First of all, we state some useful lemmas for later use.

LEMMA 2.1. *Let* V *be a non-trivial closed conformal vector field.*

(1) *If* $\gamma : [0, \ell) \to M^n$ *is a geodesic with* $V(\gamma(0)) = a\gamma'(0)$ *for some* $a \in \mathbb{R}$, *then we have*

$$V(\gamma(t)) = (a + \int_0^t \phi(\gamma(s))ds)\gamma'(t). \tag{2.3}$$

(2) *If* $V(p) = 0$, *then* $\mathrm{div}V(p) = n\phi(p) \neq 0$, *in particular, all zeros of* V *are isolated.*

Proof. See Propositions 2.1 and 2.3 in [16]. □

LEMMA 2.2. *Let (M^n, g) be an n-dimensional connected pseudo-Riemannian manifold. Then the following hold:*

(1) $\dim CC(M^n, g) \leq n+1$,
(2) *if* $\dim CC(M^n, g) \geq 2$, *there exists a constant $k \in \mathbb{R}$ such that for all $V \in CC(M^n, g)$, $\nabla \phi = -kV$, where $n\phi$ is the divergence of V.*

Proof. See Proposition 2.3 in [16] and Proposition 4 in [7]. □

The model spaces $B^n(\epsilon a^2)(\epsilon = \pm 1)$ are the hyperquadrics $S^n_\nu(a^2)$, $H^n_\nu(-a^2)$, which are given by

$$S^n_\nu(a^2) = \{x \in \mathbb{R}^{n+1}_\nu | \langle x, x\rangle = 1/a^2\},$$

$$H^n_\nu(-a^2) = \{x \in \mathbb{R}^{n+1}_{\nu+1} | \langle x, x\rangle = -1/a^2\}.$$

For a fixed vector T in $\mathbb{R}^{n+1}_\nu$ or $\mathbb{R}^{n+1}_{\nu+1}$, let σ_T be the height function in the direction of T defined by $\sigma_T(x) = \langle T, x\rangle$. Then one can easily show that on $B^n(k)$,

$$\nabla \sigma_T(x) = T - k\sigma_T(x)x, \tag{2.4}$$

$$\nabla_X \nabla \sigma_T = -k\sigma_T X \tag{2.5}$$

for all $X \in TB^n(k)$([9]). (2.5) implies that for any constant vector T in $\mathbb{R}^{n+1}_\nu$ or $\mathbb{R}^{n+1}_{\nu+1}$, $\nabla\sigma_T$ is a closed conformal vector field on the hyperquadric $B^n(k), k = \epsilon a^2$. Furthermore, by counting dimensions, we see that $\nabla \sigma_T$ represents every element of $CC(B^n(k))$.

For the flat space form $\mathbb{R}^n_\nu$ with index ν, the vector field V defined by $V(x) = bx + c(b \in \mathbb{R}, c \in \mathbb{R}^n_\nu)$ is a closed conformal vector field. Obviously, by counting dimensions, we have

$$CC(\mathbb{R}^n_\nu) = \{bx + c | b \in \mathbb{R}, \quad c \in \mathbb{R}^n_\nu\}.$$

For the space of conformal vector fields of pseudo-Riemannian space forms, the authors et al. gave a complete description in [12].

Now we introduce a function space $A_k(M^n, g)(k \neq 0)$ and a symmetric bilinear form Φ_k on the space as follows:

$$A_k(M^n, g) = \{f \in C^\infty(M) | \nabla_X \nabla f = -kfX, \quad X \in TM\}, \tag{2.6}$$

$$\Phi_k(f,h) = \langle \nabla f, \nabla h \rangle + kfh, \quad f,h \in A_k(M^n, g). \tag{2.7}$$

For the non-flat space form $B^n(k)(k = \epsilon a^2)$ (2.4) shows that

$$\Phi_k(\sigma_T, \sigma_S) = \langle \nabla \sigma_T, \nabla \sigma_S \rangle + k\sigma_T\sigma_S = \langle T, S \rangle. \tag{2.8}$$

This implies that the symmetric bilinear form Φ_k is just the usual scalar product on the ambient pseudo-Euclidean space.

Recall that $\mathfrak{L}_V g$ denotes the Lie derivative of g with respect to V.

LEMMA 2.3. ([21]) *Let (M^n, g) be a totally umbilic submanifold of a pseudo-Riemannian space $(\bar{M}, \bar{g})$. If V is a conformal vector field on $\bar{M}$ with $\mathfrak{L}_V g = 2\sigma g$, then the tangential part V^T of V on M^n is a conformal vector field on M^n with*

$$\mathfrak{L}_{V^T} g = 2\{\sigma + \bar{g}(V, H)\}g,$$

where H denotes the mean curvature vector field of M^n in $\bar{M}$.

In [12], the authors et al. prove a converse of Lemma 2.3 for hypersurfaces of a pseudo-Riemannian space form.

For a point $p \in M^n$, let $\Delta(p)$ denote the span of the set of closed conformal vector fields at p, that is,

$$\Delta(p) = \{V(p) \in T_p(M) | V \in CC(M^n, g)\}.$$

Suppose that $CC(M^n, g)$ is of dimension $m \geq 2$. Then (2.1) and Lemma 2.2 imply that there exits a constant $k \in \mathbb{R}$ such that for all $V \in CC(M^n, g)$ with $\phi = (1/n)\mathrm{div}V$

$$\nabla\phi = -kV, \tag{2.9}$$

so that we have

$$\nabla_X \nabla\phi = -k\phi X, \quad X \in TM. \tag{2.10}$$

Hence, if k is nonzero, then $CC(M^n, g)$ and $\Delta(p)$ may be identified with $A_k(M^n, g)$ and $\{\nabla f(p) | f \in A_k(M^n, g)\}$, respectively.

3. Einstein spaces admitting conformal vector fields

Let (M^n, g) denote a non-Ricci flat connected Einstein space with Ric$= k(n-1)g(k \neq 0)$ which admits conformal vector fields. Recall that by definition the Lie derivative of the metric g is given by $\mathfrak{L}_V g(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ for arbitrary tangent vectors X, Y, where ∇ denotes the Levi-Civita connection.

We denote by $C(M^n, g)$ the Lie algebra of all conformal vector fields on (M^n, g), and by $I(M^n, g)$ the subalgebra of isometric vector fields. It is well known ([8, 22]) that if V is a conformal vector field on M^n with $\mathfrak{L}_V g = 2fg$, then

$$\nabla_X \nabla f = -kfX \tag{3.1}$$

for all $X \in TM$. Hence we see that $kV + \nabla f$ is an isometric vector field. Since $k \neq 0$, this together with (3.1) shows that $C(M^n, g)$ may be identified with $I(M^n, g) \oplus A_k(M^n, g)$ by the correspondence $V \to (kV + \nabla f, f)$.

If V is a closed conformal vector field on M^n with $\nabla_X V = fX$, $X \in TM$, then (3.1) shows that $w = kV + \nabla f$ is parallel on M^n. Since (M^n, g) is Einstein with nonzero scalar curvature k, w must be trivial, that is, $V = -\frac{1}{k}\nabla f$. Hence $CC(M^n, g)$ can be identified with $A_k(M^n, g)$ by the correspondence $V \to f$. Thus $\Delta(p)$ is the subspace of T_pM spanned by $\nabla f(p)$, $f \in A_k(M^n, g)$.

Using Proposition 1.2, we can improve the Kerkhove's results (Proposition 1.1) as follows:

PROPOSITION 3.1. *Let (M^n, g) be a non-Ricci flat connected Einstein space with* Ric $= (n-1)kg(k \neq 0)$ *which admits a nonisometric conformal vector field. We denote by m the dimension of the space $A_k(M^n, g)$. Suppose that each subspace $\Delta(p)$ is a nondegenerate subspace of T_pM. Then for a fixed point $p_o \in M^n$ one of the following holds:*

(1) *if $\dim \Delta(p_o) < m$, then (M^n, g) is locally isometric to $B^n(k)$.*
(2) *if $\dim \Delta(p_o) = m$, then (M^n, g) is locally isometric to a warped product space $B^m(k) \times_{\sigma_T} F$, where $\sigma_T \in A_k(B^m(k))$ and the fibre (F, g_F) is an $(n-m)$-dimensional Einstein space with $Ric_F = (n-m-1)\langle T, T\rangle g_F$.*

Furthermore, F satisfies the following:

(i) *if $\langle T, T\rangle \neq 0$, then F admits no nonisometric conformal vector fields.*

(ii) *if* $\langle T, T\rangle = 0$, *then* F *admits no nontrivial homothetic gradient vector fields.*

In either case, $A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S, T\rangle = 0\}$, *where* $\tilde{\sigma}_S$ *denotes the lifting of* σ_S.

Proof. Since (M^n, g) admits a nonisometric conformal vector field, the space $A_k(M^n, g)$ is of dimension $m \geq 1$. Hence the proof follows from Proposition 1.2 except the condition on the fibre F with $\langle T, T\rangle \neq 0$. Furthermore, the fibre (F, g_F) is an $(n-m)$-dimensional Einstein space with $\mathrm{Ric}_F = (n-m-1)\alpha g_F$, where α denotes $\langle T, T\rangle$. Since $\alpha \neq 0$, (3.1) shows that $A_\alpha(F, g_F) = \{0\}$ is equivalent to the condition that F carries no nonisometric conformal vector fields. ☐

REMARK. Suppose that $n-m-1 \leq 0$ holds in Proposition 3.1, that is, the dimension m of $CC(M^n, g)$ is greater than or equal to $n-1$. Then the proof of Proposition 2.3 in [11] shows that (M^n, g) has constant sectional curvature k. Thus (M^n, g) is locally isometric to $B^n(k)$. In particular, we have $m = n+1$. This shows that if (M^n, g) is not locally isometric to $B^n(k)$, then we have $m < n-1$, that is, $n-m-1 \geq 1$.

Suppose that the fibre F in Proposition 3.1 satisfies $\langle T, T\rangle = 0$. Then F carries no nontrivial homothetic gradient vector fields. For any conformal vector field V on F with $\mathfrak{L}_V g_F = 2f g_F$, (3.1) shows that ∇f is isometric, hence it is a homothetic gradient vector field. Thus ∇f vanishes identically, that is, V is a homothetic vector field. Therefore the fiber space F carries no nonhomothetic conformal vector fields.

The following lemma is useful for the proof of our theorem.

LEMMA 3.2. *Suppose that* (M^n, g) *is a non-Ricci flat Einstein space with* $\mathrm{Ric} = k(n-1)g (k \neq 0)$. *Then, for any* $w \in I(M^n, g)$ *and* $\sigma \in A_k(M^n, g)$, *we have* $\langle w, \nabla\sigma\rangle \in A_k(M^n, g)$ *and* $\nabla\langle w, \nabla\sigma\rangle = [w, \nabla\sigma]$.

Proof. For the proof, see p.18 in [8]. ☐

Now let's describe the space $C(M^n, g)$ of a non-Ricci flat Einstein warped product space $(M^n, g) = B^m(k) \times_{\sigma_T} F (k \neq 0)$ in Proposition 3.1. Since the space $C(M^n, g)$ can be identified with $I(M^n, g) \oplus A_k(M^n, g)$ and the function space $A_k(M^n, g)$ is given by Proposition 1.3, it suffices to describe the space $I(M^n, g)$ of isometric vector fields.

THEOREM 3.3. *Let* (M^n, g) *be a non-Ricci flat Einstein warped product space* $B^m(k) \times_{\sigma_T} F (k \neq 0)$, *where* (F, g_F) *is an* $(n-m)$*-dimensional*

Einstein space with $\mathrm{Ric}_F = (n-m-1)\langle T,T\rangle g_F$. *Suppose that the fiber* F *satisfies the following:*

(i) *if* $\langle T,T\rangle \neq 0$, *then* F *admits no nonisometric conformal vector fields,*

(ii) *if* $\langle T,T\rangle = 0$, *then* F *admits no nontrivial homothetic gradient vector fields.*

Then the following hold:

(1) *if* $\langle T,T\rangle = 0$ *and* F *admits a nonisometric homothetic vector field, then* M^n *admits an isometric vector field* w_o *with* $\langle w_o, \nabla\sigma_T\rangle = \sigma_T$ *and we have*

$$\begin{aligned} I(M^n,g) = \mathbb{R}w_o &\oplus \{\tilde{w}_1 | w_1 \in I(B^m(k)), \langle w_1, \nabla\sigma_T\rangle = 0\} \\ &\oplus \{\tilde{w}_2 | w_2 \in I(F, g_F)\}, \end{aligned}$$

(2) *otherwise, we have*

$$\begin{aligned} I(M^n,g) =&\{\tilde{w}_1 | w_1 \in I(B^m(k)),\ \langle w_1, \nabla\sigma_T\rangle = 0\} \\ &\oplus \{\tilde{w}_2 | w_2 \in I(F, g_F)\}, \end{aligned}$$

where for each $i = 1,2$, $\tilde{w}_i$ *denotes the lifting of* w_i.

Proof. First, note that the condition (ii) on F implies that F admits no nonhomothetic conformal vector fields. Furthermore, it follows from Proposition 1.3 that (M^n, g) satisfies the following:

$$A_k(M^n,g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S,T\rangle = 0\}, \tag{3.2}$$

where $\tilde{\sigma}_S$ denotes the lifting of σ_S.

For $\tilde{\sigma}_S \in A_k(M^n,g)$ with $S \neq 0$, (2.4) and $\Phi_k(\sigma_S, \sigma_T) = \langle S,T\rangle = 0$ imply that $\nabla\sigma_S(p) \neq 0$ for any $p \in B^m(k)$. This means that for each $p \in B^m(k)$, $\{\nabla\sigma_S(p) | \langle S,T\rangle = 0\}$ spans $T_pB^m(k)$.

Now we give a proof of Theorem 3.3 step by step in the following procedures (a)–(d).

(a) *For every* $w \in I(M^n,g)$ *we have* $\langle w, \nabla\sigma_T\rangle = c\sigma_T$ *for some constant* c *in case* (1) *and we have* $\langle w, \nabla\sigma_T\rangle = 0$ *in case* (2).

For each $p \in B^m(k)$, consider the tangential part w^F of w on the fiber $\{p\}\times F$. Since $\{p\}\times F$ is a totally umbilic submanifold of M^n with mean curvature vector field $-\frac{1}{\sigma_T}\nabla\sigma_T$, we obtain from Lemma 2.3

$$\mathfrak{L}_{w^F} g_p = -\frac{2}{\sigma_T}\langle w, \nabla\sigma_T\rangle g_p, \tag{3.3}$$

where g_p denotes the restriction of g to the fibre $\{p\}\times F$. Because $\{p\}\times F$ is homothetic to F and F admits no nonhomothetic conformal vector fields, (3.3) shows that $\langle w, \nabla\sigma_T\rangle = \sigma_T h$ for some function h on $B^m(k)$. Hence in case (2), it follows again from (3.3) that h must be trivial. In case (1), using Lemma 3.2, we can prove that h is constant. In fact, since $\langle T, T\rangle = 0$, we have $\langle w, \nabla\sigma_T\rangle = \sigma_{\bar{T}}$ for a vector $\bar{T}$ with $\langle T, \bar{T}\rangle = 0$. Let S be an arbitrary vector with $\langle S, T\rangle = 0$. Then we also have $\langle w, \nabla\sigma_S\rangle = \sigma_{\bar{S}}$ for a vector $\bar{S}$ with $\langle \bar{S}, T\rangle = 0$. Because w is isometric and $\sigma_S, \sigma_{\bar{S}}, \sigma_T$ belongs to $A_k(M^n, g)$, we obtain

$$\begin{aligned}
\nabla\sigma_S(\sigma_{\bar{T}}) &= \nabla\sigma_S\langle w, \nabla\sigma_T\rangle \\
&= \langle \nabla_{\nabla\sigma_S} w, \nabla\sigma_T\rangle + \langle w, \nabla_{\nabla\sigma_S}\nabla\sigma_T\rangle \\
&= -\langle \nabla\sigma_S, \nabla_{\nabla\sigma_T} w\rangle - k\sigma_T\sigma_{\bar{S}} \\
&= -\nabla\sigma_T\langle \nabla\sigma_S, w\rangle + \langle \nabla_{\nabla\sigma_T}\nabla\sigma_S, w\rangle - k\sigma_T\sigma_{\bar{S}} \\
&= -\nabla\sigma_T(\sigma_{\bar{S}}) - k\sigma_S\langle w, \nabla\sigma_T\rangle - k\sigma_T\sigma_{\bar{S}} \\
&= -k\sigma_S\sigma_{\bar{T}},
\end{aligned} \tag{3.4}$$

where the last equality follows from $\langle \bar{S}, T\rangle = 0$. It follows from (3.4) that $\langle S, \bar{T}\rangle = 0$. This together with $\langle S, T\rangle = 0$ implies

$$\nabla\sigma_S(h) = \nabla\sigma_S\left(\frac{\sigma_{\bar{T}}}{\sigma_T}\right) = 0.$$

This completes the proof of (a) because $\nabla\sigma_S(p)$ with $\langle S, T\rangle = 0$ generates the tangent space of $B^m(k)$ at each point $p \in B^m(k)$.

(b) *For a linear map Ψ from $I(B^m(k))$ into $A_k(B^m(k))$ defined by $\Psi(w) = \langle w, \nabla\sigma_T\rangle$, the image of Ψ is the subspace $\{\sigma_S | \langle S, T\rangle = 0\}$.*

First of all, we show that the image of Ψ is contained in the subspace $\{\sigma_S | \langle S, T\rangle = 0\}$. It is easily seen that Ψ is well-defined by Lemma 3.2. Since $\langle w, \nabla\sigma_T\rangle$ belongs to $A_k(B^m(k))$, we see that $\langle w, \nabla\sigma_T\rangle = \sigma_S$ for some S in the ambient pseudo-Euclidean space. Furthermore, we have $\nabla\sigma_S = [w, \nabla\sigma_T]$ by Lemma 3.2. Since $\sigma_T \in A_k(B^m(k))$ we have $\nabla_w\nabla\sigma_T = -k\sigma_T w$. On the other hand, $\nabla_w\nabla\sigma_T = \nabla_{\nabla\sigma_T} w + [w, \nabla\sigma_T] = \nabla_{\nabla\sigma_T} w + \nabla\sigma_S$, so that we obtain

$$\nabla_{\nabla\sigma_T} w + \nabla\sigma_S + k\sigma_T w = 0. \tag{3.5}$$

Since w is isometric, taking the scalar products of the both sides of (3.5) with $\nabla\sigma_T$, we see that

$$\Phi_k(\sigma_S, \sigma_T) = \langle \nabla\sigma_S, \nabla\sigma_T\rangle + k\sigma_S\sigma_T = 0.$$

This shows that the image of Ψ is contained in the m-dimensional subspace $\{\sigma_S | \langle S, T\rangle = 0\}$.

Let $w \in \mathrm{Ker}\Psi$, that is, $\langle w, \nabla\sigma_T\rangle = 0$. For a fixed point $p \in B^m(k)$, (3.5) implies that

$$(3.6) \qquad \langle w(p), \nabla\sigma_T(p)\rangle = 0, \nabla_{\nabla\sigma_T(p)} w = -k\sigma_T(p)w(p).$$

Conversely, if w satisfies (3.6), then (3.5) together with (3.6) shows that the function $\sigma_S = \langle w, \nabla\sigma_T\rangle$ satisfies $\sigma_S(p) = 0$ and $\nabla\sigma_S(p) = 0$. Since $\sigma_S \in A_k(B^m(k))$, Lemma 2.1 yields that $\sigma_S = \langle w, \nabla\sigma_T\rangle$ vanishes identically. This means that $w \in \mathrm{Ker}\Psi$ is characterized by (3.6). Hence the proof of Lemma 28 ([18], p.253) gives $\dim\mathrm{Ker}\Psi \leq m(m-1)/2$. Since $\dim I(B^m(k)) = m(m+1)/2$, we see that $\mathrm{Im}\Psi$ is of at least m-dimensional. This completes the proof of (b).

(c) *If $\langle T, T\rangle = 0$ and F carries a nonisometric homothetic vector field u, then there exists an isometric vector field $w_o \in I(M^n, g)$ which satisfies $\langle w_o, \nabla\sigma_T\rangle = \sigma_T$. Furthermore, we have*

$$I(M^n, g) = \mathbb{R}w_o \oplus \{w \in I(M^n, g) | \langle w, \nabla\sigma_T\rangle = 0\}.$$

Since $\langle T, T\rangle = 0$, (b) shows that there exists an isometric vector field w_1 on $B^m(k)$ such that $\langle w_1, \nabla\sigma_T\rangle = \sigma_T$ on $B^m(k)$. We may assume that the nonisometric homothetic vector field u satisfies $\mathfrak{L}_u g_F = 2g_F$. We put $w_o = \tilde{w}_1 - \tilde{u}$, where $\tilde{w}_1$ and $\tilde{u}$ denote the liftings of w_1 and u, respectively. Then it is easy to prove that w_o is an isometric vector field on M^n which satisfies $\langle w_o, \nabla\sigma_T\rangle = \sigma_T$ on M^n. For any $w \in I(M^n, g)$, $\langle w, \nabla\sigma_T\rangle = c\sigma_T$ for some constant c. It gives $\langle w - cw_o, \nabla\sigma_T\rangle = 0$. This completes the proof of (c).

(d) *For $w \in I(M^n, g)$, we consider the decomposition $w = w^B + w^F$, where w^B and w^F are tangent to the leaves and to the fibers of M^n, respectively. Suppose that w satisfies $\langle w, \nabla\sigma_T\rangle = 0$. Then w^B and w^F are isometric vector fields on M^n. Furthermore, w^B and w^F are liftings of $w_1 \in I(B^m(k))$ and $w_2 \in I(F, g_F)$, respectively. Obviously, w_1 satisfies $\langle w_1, \nabla\sigma_T\rangle = 0$.*

For any liftings $\tilde{X}, \tilde{Y}$ and $\tilde{U}, \tilde{V}$ of $X, Y \in TB^m(k)$ and $U, V \in TF$, respectively, one can see that $\mathfrak{L}_{w^F} g(\tilde{X}, \tilde{Y}) = 0$. $\langle w, \nabla\sigma_T\rangle = 0$ gives $\mathfrak{L}_{w^B} g(\tilde{U}, \tilde{V}) = 0$. Hence we have $\mathfrak{L}_{w^B} g(\tilde{X}, \tilde{Y}) = 0$ and $\mathfrak{L}_{w^F} g(\tilde{U}, \tilde{V}) = 0$ because w is isometric. We need to prove $\mathfrak{L}_{w^B} g(\tilde{X}, \tilde{U}) = \tilde{U}\langle \tilde{X}, w\rangle$ to show that $\mathfrak{L}_{w^B} g(\tilde{X}, \tilde{U}) = 0$ holds. Since $\nabla\sigma_S (\sigma_S \in A_k(M^n, g))$ generates the tangent space of the base $B^m(k)$, it suffices to show that $\tilde{U}\langle\nabla\sigma_S, w\rangle = 0$ for all $\sigma_S \in A_k(M^n, g)$. Lemma 3.2 implies that $\langle\nabla\sigma_S, w\rangle$ belongs to $A_k(M^n, g)$, and hence (3.2) shows that it is a function on the base $B^m(k)$. This shows that w^B(and hence w^F) is isometric on M^n.

Now we fix a point $q_o \in F$. Let w_1 be the restriction $w^B|_{B\times\{q_o\}}$ of w^B. Since $\langle w_1, \nabla\sigma_T\rangle = 0$, we see that the lifting $\tilde{w}_1$ is an isometric vector field on M^n. Let η_1 be the isometric vector field $w^B - \tilde{w}_1$ on M^n. From the following:

$$\mathfrak{L}_{w^B} g(\tilde{X}, \tilde{U}) = \tilde{U}\langle\tilde{X}, w^B\rangle = 0,$$
$$\mathfrak{L}_{\tilde{w}_1} g(\tilde{X}, \tilde{U}) = \tilde{U}\langle\tilde{X}, \tilde{w}_1\rangle = 0,$$

we see that $\tilde{U}\langle\tilde{X}, \eta_1\rangle = 0$ for any $X \in TB^m(k)$ and $U \in TF$. Since $\langle\tilde{X}, \eta_1\rangle = 0$ on $B \times \{q_o\}$, $\langle\tilde{X}, \eta_1\rangle$ is identically zero on M^n. Hence $\eta_1 = w^B - \tilde{w}_1$ vanishes identically on M^n.

For a fixed point $p_o \in B^m(k)$, let w_2 be the restriction $w^F|_{\{p_o\}\times F}$ of w^F. Then the lifting $\tilde{w}_2$ is isometric on M^n. Let η_2 denote the isometric vector field $w^F - \tilde{w}_2$. For any $\sigma_S \in A_k(M^n, g)$ it is easy to show the following:

$$\mathfrak{L}_{w^F} g(\nabla\sigma_S, \tilde{U}) = \nabla\sigma_S\langle w^F, \tilde{U}\rangle - 2\frac{\langle\nabla\sigma_S, \nabla\sigma_T\rangle}{\sigma_T}\langle w^F, \tilde{U}\rangle = 0,$$
$$\mathfrak{L}_{\tilde{w}_2} g(\nabla\sigma_S, \tilde{U}) = \nabla\sigma_S\langle\tilde{w}_2, \tilde{U}\rangle - 2\frac{\langle\nabla\sigma_S, \nabla\sigma_T\rangle}{\sigma_T}\langle\tilde{w}_2, \tilde{U}\rangle = 0.$$

Hence, from $\Phi_k(\sigma_S, \sigma_T) = 0$ we obtain for all $\sigma_S \in A_k(M^n, g)$

$$\nabla\sigma_S\langle\eta_2, \tilde{U}\rangle + 2k\sigma_S\langle\eta_2, \tilde{U}\rangle = 0. \tag{3.7}$$

Since $\nabla\sigma_S$, $\sigma_S \in A_k(M^n, g)$ generates the tangent spaces of leaves and $\langle\eta_2, \tilde{U}\rangle = 0$ on $\{p_o\} \times F$, (3.7) implies that $\langle\eta_2, \tilde{U}\rangle = 0$ for any $U \in TF$, and hence $\eta_2 = w^F - \tilde{w}_2$ vanishes identically on M^n. In fact, let $\gamma(t)$ be an integral curve of $\nabla\sigma_S$ on M^n with an initial point on $\{p_o\} \times F$ and h denote the function $\langle\eta_2, \tilde{U}\rangle$. Then (3.7) shows that the function $y(t) = h(\gamma(t))$ satisfies the first order differential equation $y'(t) + 2kf(t)y(t) = 0$, where $f(t)$ denotes $\sigma_S(\gamma(t))$. Because of the initial condition $y(0) = 0$, the function $y(t)$ vanishes identically. This completes the proof of (d).

Combining from (a) to (d), the proof of our theorem is completed. □

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