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p-ADIC q-HIGHER-ORDER HARDY-TYPE SUMS

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ABSTRACT. The goal of this paper is to define p-adic Hardy sums and p-adic q-higher-order Hardy-type sums. By using these sums and p-adic q-higher-order Dedekind sums, we construct p-adic continuous functions for an odd prime. These functions contain p-adic q-analogue of higher-order Hardy-type sums. By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we give fundamental properties of these sums. We also establish relations between p-adic Hardy sums, Bernoulli functions, trigonometric functions and Lambert series.

1. Introduction, definitions, and notations

Recall that the Dedekind eta-function, which was introduced by Dedekind in 1877, is defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

for $z \in \mathbb{H}$, upper-half plane. In many applications of elliptic modular functions to Number Theory Dedekind eta-function plays a central role. The classical Dedekind sum appears in the transformation formulae of this function. If h and k are coprime integers with $k > 0$, the classical Dedekind sum is defined by

$$s(h,k) = \sum_{\mu \bmod k} ((\frac{\mu}{k}))((\frac{h\mu}{k})),$$

where the function $((x))$ is defined by

$$((x)) = \begin{cases} x - [x]_G - \frac{1}{2}, & x \text{ is not an integer} \\ 0, & \text{otherwise,} \end{cases}$$

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where $[x]_G$ is the largest integer $\leq x$ ([1], [17]).

Generalized Dedekind sums $s(h,k:p)$ are defined by [1]

$$s(h,k:p)=\sum_{a \bmod k}\frac{a}{k}\overline{B}_p(\frac{ha}{k}), \tag{1.1}$$

where h and k are coprime positive integers and $\overline{B}_p(x)$ is the p-th Bernoulli function, which is defined by

$$\overline{B}_p(x)=B_p(x-[x]_G)=-p!\,(2\pi i)^{-p}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} m^{-p}e^{2\pi imx}, \tag{1.2}$$

where $B_p(x)$ is the usual p-th Bernoulli polynomial.

Observe that when $p=1$, the sum $s(h,k:1)$ is known as the classical Dedekind sums, $s(h,k)$. Also the sums $s(h,k:p)$ are related to the Lambert series, $G_p(x)$, which are defined as follows:

$$G_p(x)=\sum_{n=1}^{\infty} n^{-p}\frac{x^n}{1-x^n},$$

where $p\geq 1$.

Using a Mellin transform technique developed by Rademacher, Apostol [1] obtained transformation formulae relating $G_p(e^{2\pi iz})$ to $G_p(e^{2\pi iAz})$ for odd p, where $Az=\frac{az+b}{cz+d}$ is a modular substitution. The sum $s(h,k:p)$ appears in these transformation formulae. The sum $s(h,k:p)$ is expressible as infinite series related to certain Lambert series. A representation of $s(h,k:p)$ as infinite series was given by Apostol [1]:

THEOREM 1. ([1]) *Let $(h,k)=1$. For odd $p\geq 1$, we have*

$$s(h,k:p)=\frac{p!}{(2\pi i)^p}\sum_{\substack{n=1\\ n\not\equiv 0(k)}}^{\infty} n^{-p}\left(\frac{e^{\frac{2\pi inh}{k}}}{1-e^{\frac{2\pi inh}{k}}}-\frac{e^{-\frac{2\pi inh}{k}}}{1-e^{-\frac{2\pi inh}{k}}}\right). \tag{1.3}$$

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{N}$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the set of natural numbers. Let v_p be the normalized exponential valuation of $\mathbb{Q}_p$ with $|p|_p=p^{-v_p(p)}=p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate. Let $\mathbb{C}_p$ be denoted by the completion of algebraic closure of $\mathbb{Q}_p$. If $q\in\mathbb{C}_p$, we assume that $|q-1|_p<p^{-\frac{1}{p-1}}$, so that $q^x=\exp(x\log q)$ for $x\in\mathbb{Z}_p$ (for detail see [6], [8]-[12]).

Rosen and Snyder [16] defined p-adic Dedekind sums. They commenced this sums by interpolating certain m-th Bernoulli functions. Let

p be an odd prime. Let a and N be positive integers such that $(p,a)=1$ and $p|N$. We wish to extend $N^{m-1}B_m(\frac{a}{N})$. The function $F(s;a,N)$ is defined by

$$F(s;a,N)=w^{-1}(a)\frac{\langle a\rangle^s}{N}\sum_{j=0}^{\infty}\binom{s}{j}\left(\frac{N}{a}\right)^j B_j, \tag{1.4}$$

where $s\in\mathbb{Z}_p$, $\langle a\rangle$ denotes the principal unit associated with a, i.e., $\langle a\rangle$ is the unique unit given via the decomposition $a=\langle a\rangle w(a)$ where $w(a)=\lim_{n\to\infty}a^{p^n}$. Since $\left|\binom{s}{j}\right|_p\le 1$, $\left|\frac{N}{a}\right|_p<1$ and $|B_j|_p\le p$ for all $j\ge 0$ (see [16]), we know that $\sum_{j=0}^{\infty}\binom{s}{j}(\frac{N}{a})^j B_j$ converges to a continuous p-adic function of s on $\mathbb{Z}_p$. Let $s=m$, where m is a positive integer. We set

$$F(m;a,N)=w^{-m-1}N^{m-1}B_m\left(\frac{a}{N}\right).$$

In particular, if $m+1\equiv 0\,(\mathrm{mod}\,p-1)$, then

$$F(m;a,N)=N^{m-1}B_m\left(\frac{a}{N}\right). \tag{1.5}$$

Therefore $F(s;a,N)$ is a continuous p-adic extension of $N^{m-1}B_m\left(\frac{a}{N}\right)$ ([16]).

This function interpolates $k^m s(h,k:m)$ where $p|k$ and $(h\mu,p)=1$ for each $\mu=1,2,\ldots,k-1$ ([16]):

$$k^m s(h,k:m)=\sum_{\mu=1}^{k-1}\mu F\left(m;(h\mu)_k,k\right), \tag{1.6}$$

for all $m+1\equiv 0\,(\mathrm{mod}\,p-1)$, where $(\alpha)_n$ denotes the integer x such that $0\le x<n$ and $x\equiv\alpha\,(\mathrm{mod}\,n)$, (for detail see [16]).

p-adic Dedekind sums in the Rosen and Snyder [16] sense are given as follows:

DEFINITION 1. ([16]) Let p be an odd prime. Let h and k relatively prime positive integers such that $p\nmid k$. Let $u\in\mathbb{Z}_p$. The p-adic Dedekind sum is defined to be

$$s(u;h,k:p)=\sum_{\mu=1}^{k-1}\mu F(u;h\mu,k). \tag{1.7}$$

If $p\equiv 1\,(\mathrm{mod}\,k)$ (p odd, as always), then we obtain immediately that

$$s(m;h,k:p)=\left(1-p^{m-1}\right)k^m s(h,k:m), \tag{1.8}$$

for all m such that $m+1 \equiv 0 \,(\mathrm{mod}\, p-1)$.

In [14], Kudo studied on p-adic Dedekind sums different from Rosen and Snyder [16]. By using p-adic measure and p-adic continuous function which interpolates of Euler numbers, Kudo [14] defined p-adic Dedekind sums.

In this paper, we use the notation

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Hence,

$$\lim_{q \to 1} [x] = x$$

for any x with $|x| \leq 1$ in the present p-adic case.

In [9]-[12], Kim defined the q-Volkenborn integration. By using q-Volkenborn integration, he evaluated q-Bernoulli polynomials and numbers. By using non-Archimedean q-integration, he introduced multiple Changhee q-Bernoulli polynomials which form a q-extension of Barnes' multiple Bernoulli polynomials. He also construct the Changhee q-zeta functions which give q-analogs of Barnes' multiple zeta function. He gave the relationships between the Changhee q-zeta function and Daehee q-zeta function. He constructed a new measure. In [6], Kim gave definition of a q-analogue of p-adic Haar distribution as follows: For any positive integer N,

$$\mu_q(a + p^N \mathbb{Z}_p) = \frac{(1-q)q^a}{1-q^{p^N}} = \frac{q^a}{[p^N]}$$

for $0 \leq a < p^N - 1$ and this can be extended to a distribution on $\mathbb{Z}_p$. This distribution yields an invariant p-adic q-integral for each nonnegative integer m and the m-th Carlitz's q-Bernoulli number β_m^* can be represented by this p-adic q-integral as follows:

$$\beta_m^* = \int_{\mathbb{Z}_p} [a]^m d\mu_q(a) = \lim_{N \to \infty} \sum_{a=0}^{p^N-1} \frac{q^a [a]^m}{[p^N]},$$

where the limit is convergent (see [6]-[8]).

In [13], T. Kim and H. S. Kim defined q-Bernoulli number $\beta_m (= \beta_m(q) \in \mathbb{C}_p$, by making use of this integral, as follows

$$\int_{\mathbb{Z}_p} q^{-a} [a]^m d\mu_q(a) = \beta_m.$$

Note that $\lim_{q\to 1}\beta_m(q) = B_m$, where B_m is the m-th Bernoulli number. They also defined q-Bernoulli polynomials by

$$\beta_m(x:q) = \int_{\mathbb{Z}_p} q^{-t}[x+t]^m d\mu_q(t) \tag{1.9}$$

for $x \in \mathbb{Z}_p$, $m \in \mathbb{N}$.

These numbers $\beta_m(x:q)$ can be represented as

$$\beta_m(x:q) = \sum_{k=0}^{m} \binom{m}{k} \beta_k(q)[x]^{m-k} \quad \text{(see [13])}.$$

In [8], by using an invariant $p-$adic q-integrals on $\mathbb{Z}_p$, Kim constructed a p-adic continuous function for an odd prime to contain a p-adic q-analogue of higher-order Dedekind sums $k^m s(h,k:m+1)$.

Theorem 2. ([8]) *Assume that h, k are fixed integers with $(h,k) = (p,k) = 1$. Let*

$$s_q(h,k:m;q^l) = \sum_{M=0}^{k-1} \frac{[M]}{[k]} \int_{\mathbb{Z}_p} q^{-lx}\left[x + \left\{\frac{hM}{k}\right\} : q^l\right]^m d\mu_{q^l}(x),$$

where $\{x\}$ denotes the fractional part of x. Then, there exists a continuous function $s_q(s;h,k:p;q^k)$ on $\mathbb{Z}_p$ which satisfies

$$\begin{aligned} s_q(m+1;h,k:p;q^k) &= [k]^m s_q(h,k:m;q^k) \\ &\quad - [k]^m[p:q^k]^{m-1} s_q((p^{-1}h)_k, k:m;q^{pk}), \end{aligned} \tag{1.10}$$

for all $m+1 \equiv 0 \,(\mathrm{mod}\, p-1)$, where $\left(p^{-1}a\right)_N$ denotes the integer x such that $0 \le x < N$ and $px \equiv \alpha \,(\mathrm{mod}\, N)$.

Here we express Hardy sums explicitly in terms of $s(h,k)$. In stating Hardy sums we will use the notation of Berndt [3] and Sitaramachandrarao [21].

Let h and k are integers with $k > 0$, the Hardy sums are defined as follows:

$$S(h,k) = \sum_{j=1}^{k-1}(-1)^{j+1+\left[\frac{hj}{k}\right]_G}, \; s_1(h,k) = \sum_{j=1}^{k}(-1)^{\left[\frac{hj}{k}\right]_G}\left(\left(\frac{j}{k}\right)\right),$$

$$s_2(h,k) = \sum_{j=1}^{k}(-1)^j\left(\left(\frac{hj}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right), \; s_3(h,k) = \sum_{j=1}^{k}(-1)^j\left(\left(\frac{hj}{k}\right)\right),$$

$$s_4(h,k) = \sum_{j=1}^{k-1}(-1)^{\left[\frac{hj}{k}\right]_G}, \; s_5(h,k) = \sum_{j=1}^{k}(-1)^{j+\left[\frac{hj}{k}\right]_G}\left(\left(\frac{j}{k}\right)\right).$$

The relations between Hardy sums and Dedekind sums are given as follows:

THEOREM 3. ([21]) *Let* $(h,k)=1$. *Then if* $h+k$ *is odd,*

$$S(h,k)=8s(h,2k)+8s(2h,k)-20s(h,k);$$

if h *is even,*

$$s_1(h,k)=2s(h,k)-4s(h,2k);$$

if k *is even,*

$$s_2(h,k)=-s(h,k)+2s(2h,k); \tag{1.11}$$

if k *is odd,*

$$s_3(h,k)=2s(h,k)-4s(2h,k);$$

if h *is odd,*

$$s_4(h,k)=-4s(h,k)+8s(h,2k);$$

if $h+k$ *is even,*

$$s_5(h,k)=-10s(h,k)+4s(2h,k)+4s(h,2k)$$

and each one of $S(h,k)$ $(h+k$ *even*), $s_1(h,k)$, (h *odd*), $s_2(h,k)$ (k *odd*), $s_3(h,k)$ (k *even*), $s_4(h,k)$ (h *even*) *and* $s_5(h,k)$ ($h+k$ *odd*) *is zero.*

The proof of this theorem was given by Sitaramachandrarao [21]. Sitaramachandrarao [21], Berndt [3] and Dieter [5] and the author ([17]-[20]) gave fundamental properties of the Hardy Sums.

A brief summary of the paper follows:

In Section 2, we define p-adic Hardy sums. We give relations between these sums, Bernoulli polynomials and trigonometric functions. In Section 3, we shall establish new relations connection between the sums $s(h,k:p)$ and the Lambert series $G_p(e^{2\pi ih/k})$. In Section 4, we define p-adic q-higher-order Hardy-type sums. By using these sums and the sum $s_q(h,k:m;q^l)$, we construct a p-adic continuous function for an odd prime. This functions contain p-adic q-analogue of higher-order Hardy-type sums. By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we give fundamental properties of q-analogue of p-adic Hardy type sums. We give relations between p-adic q-analogue of higher-order Hardy-type sums and p-adic q-analogue of higher-order Dedekind sums as well.

2. p-adic Hardy sums

Our aim in this section is to define p-adic Hardy sums. By using (1.7) and Theorem 3, we define p-adic Hardy sums. By applying (1.5) we prove the following theorems. p-adic Hardy sums are defined as follows [20]:

DEFINITION 2. Let p be an odd prime. Let h and k be relatively prime positive integers such that $p \nmid k$. Let $s \in \mathbb{Z}_p$. Let $h+k$ be odd,

$$S(s,h,k:p) = 4 \sum_{\mu (\operatorname{mod} k)} \mu(2F(s;h\mu,2k) + 2F(s;2h\mu,k) - 5F(s;h\mu,k)),$$

let h be even,

$$s_1(s,h,k:p) = 2 \sum_{\mu (\operatorname{mod} k)} \mu(F(s;h\mu,k) - 2F(s;h\mu,2k)),$$

let k be even,

$$s_2(s,h,k:p) = - \sum_{\mu (\operatorname{mod} k)} \mu(F(s;h\mu,k) - 2F(s;2h\mu,k)), \tag{2.1}$$

let k be odd,

$$s_3(s,h,k:p) = 2 \sum_{\mu (\operatorname{mod} k)} \mu(F(s;h\mu,k) - 2F(s;2h\mu,k)),$$

let h be odd,

$$s_4(s,h,k:p) = 4 \sum_{\mu (\operatorname{mod} k)} \mu(-F(s;h\mu,k) + 2F(s;h\mu,2k)),$$

let $h+k$ be even,

$$s_5(s,h,k:p) = 2 \sum_{\mu (\operatorname{mod} k)} \mu(-5F(s;h\mu,k) + 2F(s;2h\mu,k) + 2F(s;h\mu,2k)).$$

THEOREM 4. *Let p be an odd prime. Let h and k be relatively prime positive integers such that $p \nmid k$. Let $s \in \mathbb{Z}$. If k is even,*

$$s_2(s,h,k:p) = -k^{s-1} \sum_{c=0}^{s} \binom{s}{c} B_c \left(\frac{h}{k}\right)^{s-c} \left(\frac{1-2^{s-c+1}}{s-c+2}\right) (B_{s-c+2}(k) - B_{s-c+2}); \tag{2.2}$$

if k is odd,

(2.3)

$$s_3(s,h,k:p) = 2k^{s-1}\sum_{c=0}^{s}\binom{s}{c}B_c\left(\frac{h}{k}\right)^{s-c}\left(\frac{1-2^{s-c+1}}{s-c+2}\right)\left(B_{s-c+2}(k)-B_{s-c+2}\right).$$

Proof. We need the following well-known relation:

$$\sum_{v=1}^{n-1} v^p = \frac{B_{p+1}(n)-B_{p+1}}{p+1} \tag{2.4}$$

By using (1.5) and (2.4) in (2.1) and after straightforward calculations, we arrive at the desired result. For detail proof see [20]. Thus we choose to omit the details involved. □

THEOREM 5. *Let h and k are integers with $(h,k)=1$ and let p be an odd prime with $p \geq 1$. If k is even,*

$$\begin{aligned} s_2(h,k:p) &= \frac{-4ip!}{k(2\pi i)^p}\sum_{\mu=1}^{k-1}\mu\sum_{m=1}^{\infty}m^{-p} \\ &\quad\times \sin\left(\frac{\pi m h\mu}{k}\right)\left(2\cos(\frac{3\pi m h\mu}{k}\right)+\cos(\frac{\pi m h\mu}{k})\Big), \end{aligned} \tag{2.5}$$

If k is odd,

(2.6)

$$\begin{aligned} s_3(h,k:p) &= \frac{8ip!}{k(2\pi i)^p}\sum_{\mu=1}^{k-1}\mu\sum_{m=1}^{\infty}m^{-p} \\ &\quad\times \sin\left(\frac{\pi m h\mu}{k}\right)\left(2\cos\left(\frac{3\pi m h\mu}{k}\right)+\cos(\frac{\pi m h\mu}{k})\right). \end{aligned}$$

Proof. By using (1.1) in (2.7), we have

$$s_2(h,k:p) = -\frac{1}{k}\sum_{\mu=1}^{k-1}\mu\left(\overline{B}_p\left(\frac{h\mu}{k}\right)-2\overline{B}_p\left(\frac{2h\mu}{k}\right)\right).$$

Substituting (1.2) into the above and using well-known identity $2i\sin a = e^{ia}-e^{-ia}$ and after some elementary calculations, we obtain the desired result (for detail proof, see [20]). Thus we choose to omit the details involved. □

REMARK 1. By (1.11), we set

$$s_2(h,k:p) = -s(h,k:p) + 2s(2h,k:p), \tag{2.7}$$

if k is even. Observe that when $p=1$,

$$s_2(h,k:1) = -s(h,k:1) + 2s(2h,k:1),$$

where the sum $s_2(h,k:1)$ is known as the sum $s_2(h,k)$ and the sum $s(h,k:1)$ is known Dedekind sum which is given by (1.1).

LEMMA 1. *Let $p \equiv 1 \pmod{k}$ (p odd, as always) and h and k be relatively prime positive integers such that $p \nmid k$. If $h+k$ is odd, then*

$$\begin{aligned} S(m,h,k:p) = &\left(1-p^{m-1}\right)k^m 2^{m+3} s(h,2k:m) \\ &+ 4\left(1-p^{m-1}\right)k^m (2s(2h,k:m) - 5s(h,k:m)); \end{aligned}$$

if h is even, then

$$s_1(m,h,k:p) = 2\left(1-p^{m-1}\right)k^m\left(s(h,k:m) - 2^{m+1}s(h,2k:m)\right);$$

if k is even, then

$$s_2(m,h,k:p) = -\left(1-p^{m-1}\right)k^m\left(s(h,k:m) - 2s(2h,k:m)\right);$$

if k is odd, then

$$s_3(m,h,k:p) = 2\left(1-p^{m-1}\right)k^m\left(s(h,k:m) - 2s(2h,k,m)\right);$$

if h is odd, then

$$s_4(m,h,k:p) = -4\left(1-p^{m-1}\right)k^m\left(s(h,k:m) - 2^{m+1}s(h,2k:m)\right);$$

if $h+k$ is even, then

$$\begin{aligned} s_5(m,h,k:p) = &-10\left(1-p^{m-1}\right)k^m s(h,k:m) \\ &+ 4\left(1-p^{m-1}\right)k^m (s(2h,k:m) + 2^m s(h,2k:m)), \end{aligned}$$

for all m such that $m+1 \equiv 0 \pmod{p-1}$. Each one of $S(m,h,k:p)$ ($h+k$ even), $s_1(m,h,k:p)$, (h odd), $s_2(m,h,k:p)$ (k odd), $s_3(m,h,k:p)$ (k even), $s_4(m,h,k:p)$ (h even) and $s_5(m,h,k:p)$ ($h+k$ odd) is zero.

Proof. By using (1.8), after straightforward calculations, we arrive at the desired result. For detail proof see [20]. Thus we choose to omit the details involved. □

3. Relations between Hardy sums and Lambert series

The sums $s(h,k:p)$ are related to the Lambert series, $G_p(x)$. The special case $p=1$ gives $G_1(x) = -\log\prod_{m=1}^{\infty}(1-x^m)$. Thus, $\log\eta(z)$ is the same as $\frac{\pi iz}{12} - G_1(e^{2\pi iz})$ ([1]-[2], [18]). Generalized Dedekind sums are expressible as infinite series related to certain Lambert series ([18]). In this section we shall establish new relations connection between the sums $s(h,k:p)$ and the Lambert series $G_p(e^{2\pi ih/k})$.

Relation between $s(h,k:p)$ and $G_p(e^{2\pi iz})$ are given as follows.

LEMMA 2. *Let $p \equiv 1 \,(\mathrm{mod}\, k)$ (p odd, as always) and h and k be relatively prime positive integers. Then we have*

$$s(h,k:p) = \frac{p!}{(2\pi i)^p} \sum_{\substack{n=1 \\ n\not\equiv 0(k)}}^{\infty} \sum_{m=1}^{\infty} n^{-p}\left(e^{\frac{2\pi imnh}{k}} - e^{-\frac{2\pi imnh}{k}}\right)$$

$$= \frac{p!}{(2\pi i)^p}\left(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})\right).$$

Proof. By using (1.3) and definition of Lambert series and after straightforward calculations, we arrive at the desired result. □

THEOREM 6. *Let $p \equiv 1 \,(\mathrm{mod}\, k)$ (p odd, as always) and h and k be relatively prime positive integers such that $p \nmid k$. If $h+k$ is odd, then*

$$S(m,h,k:p)$$
$$= \frac{4m!(1-p^{m-1})k^m}{(2\pi i)^m}\Big(2^{m+1}\big(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})\big)$$
$$+ 2\big(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}})\big) - 5\big(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})\big)\Big);$$

if h is even,

$$s_1(m,h,k:p) = \frac{2m!(1-p^{m-1})k^m}{(2\pi i)^m}\Big(-2^{m+1}\big(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})\big)$$
$$+ \big(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})\big)\Big);$$

if k is even, then

$$s_2(m,h,k:p) = -\frac{m!(1-p^{m-1})k^m}{(2\pi i)^m}\Big(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}\big)$$
$$- 2\big(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}})\big)\Big);$$

if k is odd, then

$$s_3\,(m,h,k:p) = \frac{2m!(1-p^{m-1})k^m}{(2\pi i)^m}(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}) \\ - 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}})));$$

if h is odd, then

$$s_4\,(m,h,k:p) = -\frac{4m!(1-p^{m-1})k^m}{(2\pi i)^m}(-2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) \\ + G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}}));$$

if $h+k$ is even, then

$$s_5\,(m,h,k:p) = -\frac{2m!(1-p^{m-1})k^m}{(2\pi i)^m}(-2^{m+1}(G_p(e^{\frac{\pi ih}{k}}) - G_p(e^{-\frac{\pi ih}{k}})) \\ + 5(G_p(e^{\frac{2\pi ih}{k}}) - G_p(e^{-\frac{2\pi ih}{k}})) \\ - 2(G_p(e^{\frac{4\pi ih}{k}}) - G_p(e^{-\frac{4\pi ih}{k}}))),$$

for all m such that $m+1 \equiv 0 \,(\mathrm{mod}\, p-1)$. Each one of $S\,(m,h,k:p)$ ($h+k$ even), $s_1\,(m,h,k:p)$, (h odd), $s_2\,(m,h,k:p)$ (k odd), $s_3\,(m,h,k:p)$ (k even), $s_4\,(m,h,k:p)$ (h even) and $s_5\,(m,h,k:p)$ ($h+k$ odd) is zero.

Proof. By using Lemma 1 and Lemma 2, after straightforward calculations, we arrive at the desired result. □

4. Generalized p-adic q-Hardy sums

By using some simple properties of $((x))$, $s(h,k)$, some well-known results due to [4], [15], and [21], and the following basic observation, we define generalized Hardy sums.

$$(-1)^{[x]_G} = 2((x)) - 4((\frac{x}{2}))$$

if x is not an integer. To give this, observe that if x is not an integer, then

$$2((x)) - 4((\frac{x}{2})) = 2(x - [x]_G - \frac{1}{2}) - 4(\frac{x}{2} - [\frac{x}{2}]_G - \frac{1}{2}) \\ \equiv 1 - 2[x]_G + 4[\frac{x}{2}]_G = 1 - 2[x]_G + 4[\frac{[x]_G}{2}]_G$$

$$= \begin{cases} 1-2[x]_G+4\frac{[x]_G}{2}, & \text{if } [x]_G \text{ is even} \\ 1-2[x]_G+4(\frac{[x]_G-1}{2}), & \text{if } [x]_G \text{ is odd} \end{cases}$$
$$= (-1)^{[x]_G}.$$

By using the above equation, Cenkci, Can and Kurt [4] gave the following definition:

Let h and k be integers with $k > 0$, the Hardy sums are defined as follows

$$S(h,k) = 4\sum_{j=1}^{k-1}\left(\left(\frac{(h+k)j}{2k}\right)\right),$$
$$s_1(h,k) = \sum_{j=1}^{k}\left(\left(\frac{j}{k}\right)\right)\left\{2\left(\left(\frac{hj}{k}\right)\right) - 4\left(\left(\frac{hj}{2k}\right)\right)\right\},$$
$$s_2(h,k) = -4\sum_{j=1}^{k-1}\left(\left(\frac{hj}{2k}\right)\right),$$
$$s_5(h,k) = \sum_{j=1}^{k}\left(\left(\frac{j}{k}\right)\right)\left\{2\left(\left(\frac{hj}{k}\right)\right) - 4\left(\left(\frac{(h+k)j}{2k}\right)\right)\right\}.$$

By using (1.1), due to Cenkci, Can and Kurt [4], we arrive at the following definition:

DEFINITION 3. Let h and k be integers with $k > 0$, the Hardy sums are defined as follows:

$$S(h,k:m) = 4\sum_{j=1}^{k-1}\overline{B}_m\left(\frac{(h+k)j}{2k}\right),$$
$$s_1(h,k:m) = \sum_{j=1}^{k-1}\overline{B}_1\left(\frac{j}{k}\right)\left(2\overline{B}_m\left(\frac{hj}{k}\right) - 4\overline{B}_m\left(\frac{hj}{2k}\right)\right)$$
$$= 2s(h,k:m) - 4\sum_{j=1}^{k-1}\overline{B}_1\left(\frac{j}{k}\right)\overline{B}_m\left(\frac{hj}{2k}\right),$$
$$s_2(h,k:m) = \sum_{j=1}^{k-1}(-1)^j\overline{B}_1\left(\frac{j}{k}\right)\overline{B}_m\left(\frac{hj}{k}\right),$$

$$s_3(h,k:m) = -4\sum_{j=1}^{k-1}(-1)^j \overline{B}_m\left(\frac{hj}{k}\right),$$

$$s_4(h,k:m) = -4\sum_{j=1}^{k-1}\overline{B}_m\left(\frac{hj}{k}\right),$$

$$s_5(h,k:m) = \sum_{j=1}^{k-1}\overline{B}_1\left(\frac{j}{k}\right)\left(2\overline{B}_m\left(\frac{hj}{k}\right) - 4\overline{B}_m\left(\frac{(h+k)j}{2k}\right)\right)$$

$$= 2s(h,k:m) - 4\sum_{j=1}^{k-1}\overline{B}_1\left(\frac{j}{k}\right)\overline{B}_m\left(\frac{(h+k)j}{2k}\right),$$

where $s(h,k:m)$ is generalized Dedekind sums.

REMARK 2. Observe that when $0 \le x < 1$, the Bernoulli function $\overline{B}_m(x)$ reduces to the m-th Bernoulli polynomial(our notation is the same as that in Apostol [1], [2]).

By using Theorem 2, Definition 2, and Definition 3, we construct a p-adic continuous function for an odd prime to contain a p-adic q-analogue of higher-order Hardy sums in this chapter. It is the aim of this chapter to give a q-analogue of p-adic Hardy type sums by using an invariant p-adic q-integral on $\mathbb{Z}_p$ approach the p-adic analogue of higher-order Dedekind sums at $q = 1$ as follows:

THEOREM 7. *Assume that h, k are fixed integers with $(h,k) = (p,k) = 1$. Let*

$$S_q(h,k:m;q^l) = 4\sum_{j=1}^{k-1}\int_{\mathbb{Z}_p} q^{-lx}\left[x + \left\{\frac{j(h+k)}{2k}\right\} : q^l\right]^m d\mu_{q^l}(x),$$

$$s_{1,q}(h,k:m;q^l) = 2s_q(h,k:m;q^l)$$
$$- 4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]} - \frac{1}{2}\right)\int_{\mathbb{Z}_p} q^{-lx}\left[x + \left\{\frac{hj}{2k}\right\} : q^l\right]^m d\mu_{q^l}(x),$$

$$s_{2,q}(h,k:m;q^l) = \sum_{j=1}^{k-1}(-1)^j\left(\frac{[j]}{[k]} - \frac{1}{2}\right)$$
$$\times \int_{\mathbb{Z}_p} q^{-lx}\left[x + \left\{\frac{hj}{k}\right\} : q^l\right]^m d\mu_{q^l}(x),$$

$$s_{3,q}(h,k:m;q^l) = \sum_{j=1}^{k-1}(-1)^j \int_{\mathbb{Z}_p} q^{-lx}\left[x+\left\{\frac{hj}{k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{4,q}(h,k:m;q^l) = -4\sum_{j=1}^{k-1} \int_{\mathbb{Z}_p} q^{-lx}\left[x+\left\{\frac{hj}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{5,q}(h,k:m;q^l) = -4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right) \int_{\mathbb{Z}_p} q^{-lx}\left[x+\left\{\frac{j(h+k)}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x) + 2s_q(h,k:m;q^l),$$

where $\{x\}$ denotes the fractional part of x. Then, there exist continuous functions $S_q(s,h,k:p;q^k)$, and $s_{y,q}(s,h,k:p;q^k)$, $1 \le y \le 4$, on $\mathbb{Z}_p$ which satisfy :
if $h+k$ be odd, then

$$S_q\left(m+1,h,k:p;q^k\right) = 8s_q(m+1;h,2k:p;q^k)+8s_q(m+1;2h,k:p;q^k) - 20s_q(m+1;h,k:p;q^k); \tag{4.1}$$

if h be even, then

$$s_{1,q}\left(m+1,h,k:p;q^k\right) = 2s_q(m+1;h,k:p;q^k)-4s_q(m+1;h,2k:p;q^k); \tag{4.2}$$

if k be even, then

$$s_{2,q}\left(m+1,h,k:p;q^k\right) = -s_q(m+1;h,k:p;q^k)+2s_q(m+1;2h,k:p;q^k); \tag{4.3}$$

if k be odd, then

$$s_{3,q}\left(m+1,h,k:p;q^k\right) = 2s_q(m+1;h,k:p;q^k)-4s_q(m+1;2h,k:p;q^k); \tag{4.4}$$

if h be odd, then

$$s_{4,q}\left(m+1,h,k:p;q^k\right) = -4s_q(m+1;h,k:p;q^k)+8s_q(m+1;h,2k:p;q^k); \tag{4.5}$$

if $h+k$ be even, then

(4.6)
$$s_{5,q}\left(m,h,k:p;q^k\right) = 4s_q(m+1;2h,k:p;q^k) + 4s_q(m+1;h,2k:p;q^k) - 10s_q(m+1;h,k:p;q^k),$$

for all $m+1 \equiv 0 \pmod{p-1}$, where $s_q(m+1;h,k:p;q^k)$ is q-analogue of p-adic Dedekind-type sums.

Proof. Proof of this Theorem is similar to that of Theorem 5, which is given by Kim [8]. q-analogue of (1.4) is given by [8]: Let w denote Teichmuller character $(\bmod p)$. For $x \in \mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p$, we set $\langle x \rangle = \langle x : q\rangle = w^{-1}(x)[x]$. Let p be an odd prime, a and N positive integers such that $(p,a)=1$ and $p|N$. Set

$$F_q(s,a,N;q^N) = \frac{w^{-1}(a)\langle a\rangle^s}{[N]}\sum_{j=0}^{\infty}\binom{s}{j}q^{aj}\frac{[N]^j}{[a]^j}\beta_j(q^N)$$

for $s \in \mathbb{Z}_p$, and let

$$Y_m(a,N,w^{-1};q^l) = \frac{w^{-1}(a)}{[N]^{m-1}}\int_{\mathbb{Z}_p} q^{-lx}\left[x+\frac{a}{l}:q^l\right]^m d\mu_{q^l}(x),$$

where m, l, $N \in \mathbb{N}$. It is easy to see in [6] that

$$\sum_{j=0}^{\infty}\binom{s}{j}q^{aj}\frac{[N]^j}{[a]^j}\beta_j(q^N)$$

converges to continuous p-adic function of s on $\mathbb{Z}_p$. Set $s=m$ in the above, we get

$$F_q(s,a,N;q^N) = Y_m(a,N,w^{-1};q^{-N}).$$

If $m+1 \equiv 0 \pmod{p-1}$, then by using (1.9), we obtain

$$F_q(m,a,N;q^N) = \frac{1}{[N]^{m-1}}\int_{\mathbb{Z}_p} q^{-Nx}\left[x+\frac{a}{N}:q^N\right]^m d\mu_{q^N}(x) = \frac{\beta_m(\frac{a}{N},q^N)}{[N]^{m-1}}.$$

Consequently, $F_q(m,a,N;q^N)$ is a continuous p-adic extension of $[N]^{1-m}$ $\beta_m(\frac{a}{N},q^N)$.

By (1.6) and (1.7), higher-order Hardy-type sums $S_q\left(h,k:m;q^l\right)$ and $s_{y,q}\left(h,k:m;q^l\right)$, $1\leq y\leq 4$, are given by

$$S_q(h,k:m;q^l)=4\sum_{j=1}^{k-1}\int_{\mathbb{Z}_p}q^{-lx}\left[x+\left\{\frac{j(h+k)}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{1,q}(h,k:m;q^l)=2s_q(h,k:m;q^l)$$
$$-4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\int_{\mathbb{Z}_p}q^{-lx}\left[x+\left\{\frac{hj}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x)$$

$$s_{2,q}(h,k:m;q^l)=\sum_{j=1}^{k-1}(-1)^j\left(\frac{[j]}{[k]}-\frac{1}{2}\right)$$
$$\cdot\int_{\mathbb{Z}_p}q^{-lx}\left[x+\left\{\frac{hj}{k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{3,q}(h,k:m;q^l)=\sum_{j=1}^{k-1}(-1)^j\int_{\mathbb{Z}_p}q^{-lx}\left[x+\left\{\frac{hj}{k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{4,q}(h,k:m;q^l)=-4\sum_{j=1}^{k-1}\int_{\mathbb{Z}_p}q^{-lx}\left[x+\left\{\frac{hj}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x),$$

$$s_{5,q}(h,k:m;q^l)=-4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\int_{\mathbb{Z}_p}q^{-lx}$$
$$\cdot\left[x+\left\{\frac{j(h+k)}{2k}\right\}:q^l\right]^m d\mu_{q^l}(x)+2s_q(h,k:m;q^l),$$

where $s_q\left(h,k:m;q^l\right)$ is higher-order Dedekind sums.

If $m+1\equiv 0\,(\mathrm{mod}\,p-1)$, we have

$$[k]^m s_q\left(h,k:m;q^k\right)=\sum_{\mu=1}^{k-1}[\mu]F_q\left(m;(h\mu)_k,k;q^k\right).$$

If $h+k$ is odd, then

$$[k]^m S_q\left(h,k:m;q^k\right)$$

$$= 8\sum_{\mu=1}^{k-1}[\mu]\left(F_q\left(m;(h\mu)_k,2k;q^k\right)+F_q\left(m;(2h\mu)_k,k;q^k\right)\right)$$
$$-20\sum_{\mu=1}^{k-1}[\mu]F_q\left(m;(h\mu)_k,k;q^k\right);$$

if h is even, then

$$[k]^m s_{1,q}\left(h,k:m;q^k\right)$$
$$=\sum_{\mu=1}^{k-1}[\mu]\left(2F_q\left(m;(h\mu)_k,k;q^k\right)-4F_q\left(m;(h\mu)_k,2k;q^k\right)\right);$$

if k is even, then

$$[k]^m s_{2,q}\left(h,k:m;q^k\right)$$
$$=\sum_{\mu=1}^{k-1}[\mu]\left(-F_q\left(m;(h\mu)_k,k;q^k\right)+2F_q\left(m;(2h\mu)_k,k;q^k\right)\right);$$

if k is odd, then

$$[k]^m s_{3,q}\left(h,k:m;q^k\right)$$
$$=\sum_{\mu=1}^{k-1}[\mu]\left(2F_q\left(m;(h\mu)_k,k;q^k\right)-4F_q\left(m;(2h\mu)_k,k;q^k\right)\right);$$

if h is odd, then

$$[k]^m s_{4,q}\left(s,h,k:p;q^k\right)$$
$$=\sum_{\mu=1}^{k-1}[\mu]\left(-4F_q\left(m;(h\mu)_k,k;q^k\right)+8F_q\left(m;(h\mu)_k,2k;q^k\right)\right);$$

if $h+k$ is even, then

$$[k]^m s_{5,q}\left(s,h,k:p;q^k\right)$$
$$=\sum_{\mu=1}^{k-1}[\mu]\left(4F_q\left(m;(2h\mu)_k,k;q^k\right)+4F_q\left(m;(h\mu)_k,2k;q^k\right)\right)$$
$$-10\sum_{\mu=1}^{k-1}[\mu]F_q\left(m;(h\mu)_k,k;q^k\right),$$

where $(y)_k$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha(\operatorname{mod} k)$. It is easy to check in [8] that for $s \in \mathbb{Z}_p$, p-adic q-Hardy-type sums defined as follows: if $h+k$ is odd,

$$\begin{aligned}
& S_q\left(s, h, k: p; q^k\right) \\
& =\sum_{\mu=1}^{k-1}[\mu]\left(8 F_q\left(s; h\mu, 2k; q^k\right)+8 F_q\left(s; 2h\mu, k; q^k\right)\right. \\
& \left.-20 F_q\left(s; h\mu, k; q^k\right)\right), \\
& s_{1,q}\left(s, h, k: p; q^k\right) \\
& =\sum_{\mu=1}^{k-1}[\mu]\left(2 F_q\left(s; h\mu, k; q^k\right)-4 F_q\left(s; h\mu, 2k; q^k\right)\right), \text{ if } h \text{ is even}, \\
& s_{2,q}\left(s, h, k: p; q^k\right) \\
& =\sum_{\mu=1}^{k-1}[\mu]\left(-F_q\left(s; h\mu, k; q^k\right)+2 F_q\left(s; 2h\mu, k; q^k\right)\right), \text{ if } k \text{ is even}, \\
& s_{3,q}\left(s, h, k: p; q^k\right) \\
& =\sum_{\mu=1}^{k-1}[\mu]\left(2 F_q\left(s; h\mu, k; q^k\right)-4 F_q\left(s; 2h\mu, k; q^k\right)\right), \text{ if } k \text{ is odd}, \\
& s_{4,q}\left(s, h, k: p; q^k\right) \\
& =\sum_{\mu=1}^{k-1}[\mu]\left(-4 F_q\left(s; h\mu, k; q^k\right)+8 F_q\left(s; h\mu, 2k; q^k\right)\right), \text{ if } h \text{ is odd},
\end{aligned}$$

and if $h+k$ is even

$$\begin{aligned}
s_{5,q}\left(s, h, k: p; q^k\right)= & \sum_{\mu=1}^{k-1}[\mu]\left(4 F_q\left(s; h\mu, 2k; q^k\right)\right. \\
& \left.+4 F_q\left(s; 2h\mu, k; q^k\right)-10 F_q\left(s; h\mu, k; q^k\right)\right).
\end{aligned}$$

Thus, there exist continuous functions $S_q\left(s, h, k: p; q^k\right)$, and $s_{y,q}(s, h, k: p; q^k)$, $1 \leq y \leq 4$, on $\mathbb{Z}_p$ which satisfy (1.10) and (4.1)-(4.6) for $m+1 \equiv 0\,(\operatorname{mod} p-1)$. So we arrive at the desired result. □

REMARK 3. In [8], Kim defined the sum $s_q(h,k:m;q^l)$ as follows

$$s_q(h,k:m;q^l)=\sum_{j=0}^{k-1}\frac{[j]}{[k]}\beta_m(\frac{hj}{k},q^l).$$

Observe that when $q\to 1$, the sum $s_1(s,h,k:p;1)$ reduces to p-adic analogue of higher-order Dedekind sums $k^m s_{m+1}(h,k)$. Here we note that

$$\begin{aligned}
S_q(h,k:m;q^l) &= 4\sum_{j=1}^{k-1}\beta_m(\frac{j(h+k)}{2k},q^l),\\
s_{1,q}(h,k:m;q^l) &= 2\sum_{j=0}^{k-1}\frac{[j]}{[k]}\beta_m(\frac{hj}{k},q^l)-4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\beta_m(\frac{jh}{2k},q^l),\\
&= 2s_q(h,k:m;q^l)-4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\beta_m(\frac{jh}{2k},q^l),\\
s_{2,q}(h,k:m;q^l) &= \sum_{j=0}^{k-1}(-1)^j\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\beta_m(\frac{hj}{k},q^l),\\
s_{3,q}(h,k:m;q^l) &= \sum_{j=0}^{k-1}(-1)^j\beta_m(\frac{hj}{k},q^l),\\
s_{4,q}(h,k:m;q^l) &= -4\sum_{j=1}^{k-1}\beta_m(\frac{jh}{2k},q^l),\\
s_{5,q}(h,k:m;q^l) &= 2\sum_{j=0}^{k-1}\frac{[j]}{[k]}\beta_m\left(\frac{hj}{k},q^l\right)\\
&\quad -4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\beta_m\left(\frac{j(h+k)}{2k},q^l\right)\\
&= 2s_q(h,k:m;q^l)\\
&\quad -4\sum_{j=1}^{k-1}\left(\frac{[j]}{[k]}-\frac{1}{2}\right)\beta_m\left(\frac{j(h+k)}{2k},q^l\right).
\end{aligned}$$

It is easy to see that if $q\to 1$, then $S_1(s,h,k:p;1)$ and $s_{y,1}(s,h,k:p;1)$, $1\le y\le 4$ are the p-adic analogue of higher-order Hardy-type sums $k^m S_{m+1}(h,k)$ and $k^m s_{y,m+1}(h,k)$, $1\le y\le 4$, respectively.

Finally we conclude this paper by raising the following question:

Find the reciprocity law for p-adic q-higher-order Hardy-type sums?

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