J. Korean Math. Soc. **43** (2006), No. 1, pp. 29–43

EXTENDING REPRESENTATIONS OF H TO G WITH DISCRETE G/H

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ABSTRACT. The article deals with the problem of extending representations of a closed normal subgroup H to a topological group G. We show that the standard technique using group cohomology to solve the problem in the case of finite groups can be generalized in the category of topological groups if G/H is discrete.

1. Introduction

Let G be a topological group and H a closed normal subgroup of G. It is natural to ask whether a linear H-action on a vector space V extends to a linear G-action on the same space V, in other words, whether a linear representation of H extends to G. The extendibility question, in particular when G is finite, plays an important role in so-called Clifford theory which is concerned with the relationship between representations and normal subgroups, the basic contribution being due to A. H. Clifford [7].

In the category of topological groups the question is divided into two cases according to the topology of G/H, one is the case when G/H is connected and the other is the case when G/H is discrete. More precisely, let us denote by G_0 the connected component of G containing the identity element, which is closed and normal in G. Then the subgroup, say K, generated by G_0 and H is a union of the connected components in G containing elements of H so that it is also closed and normal in G. Thus we have a series of closed normal subgroups, $H \lhd K \lhd G$, such that $K/H \cong G_0/(G_0 \cap H)$ is connected and G/K is discrete.

Received September 2, 2004.
2000 Mathematics Subject Classification: Primary 20C99; Secondary 22E99.
Key words and phrases: extensions of representations, topological group, group cohomology.
The third-named author is partially supported by Korea Research Foundation Grant KRF-2003-015-C00064.

The first case was studied recently in the category of compact Lie groups in [6], and a necessary and sufficient condition was found using topological techniques for every complex irreducible representation of H to extend to G, in particular, the condition holds if the fundamental group of G/H is torsion free.

On the other hand, the second case needs purely algebraic techniques, and the present article is intended to show the standard technique [9, Chapter 11] using projective representations in the category of finite groups fits well in the category of topological groups if G/H is discrete. This generalization may be already known to some specialists because it is not hard to follow, but there is no exact literature as far as the authors know.

The reader may wonder why the authors were led to study this subject. The motivation comes from the study of fiber module structure of equivariant complex vector bundles. Suppose that a compact topological group G acts continuously on a connected Hausdorff topological space X. The set of elements in G acting trivially on X forms a closed normal subgroup H, and it is possible to decompose isotypically a complex G-vector bundle over X according to irreducible representations of H since all fibers of the bundle are isomorphic as representations of H. Moreover, if an irreducible representation of H extends to G, then its isotypical part can be represented by tensoring a trivial bundle with a G/H-vector bundle over X on which G/H acts effectively. We refer the reader to [4, Section 2] for more details.

This paper is organized as follows. In Section 2 we generalize the standard technique using projective representations in the category of topological groups when G/H is discrete, and show how the extendibility question is related to the cohomology of G/H. As an application Section 3 and 4 deal with the extendibility of real and complex representations when G/H is isomorphic to a finite subgroup of $O(2)$ which occurs in studying real and complex G-vector bundles over a circle [4, 5].

2. Projective representations

In this section $\mathbb{F}$ will denote a topological field [11, Section 25] such that the group $\mathrm{GL}(n,\mathbb{F})$ of nonsingular $n\times n$ matrices over $\mathbb{F}$ is a topological group under the usual matrix multiplication. By a *representation* we mean a continuous homomorphism of a topological group into $\mathrm{GL}(n,\mathbb{F})$.

Let G be a topological group and H a closed normal subgroup of G. We call a representation $\rho\colon H \to \mathrm{GL}(n,\mathbb{F})$ *extends to* G if there exists a representation $\widetilde{\rho}\colon G \to \mathrm{GL}(n,\mathbb{F})$ such that $\rho(h) = \widetilde{\rho}(h)$ for all $h \in H$.

$$\begin{array}{ccc} H & \xrightarrow{\ \rho\ } & \mathrm{GL}(n,\mathbb{F}) \\ \Big| & \overset{\widetilde{\rho}}{\nearrow} & \\ G & & \end{array}$$

It is equivalent to saying that there exists a representation $\widetilde{\rho}\colon G \to \mathrm{GL}(n,\mathbb{F})$ such that its restriction to H is isomorphic to ρ, i.e., there exists a matrix $M \in \mathrm{GL}(n,\mathbb{F})$ such that $M^{-1}\widetilde{\rho}(h)M = \rho(h)$ for all $h \in H$.

Given a representation $\rho\colon H \to \mathrm{GL}(n,\mathbb{F})$ the map ${}^{g}\rho\colon H \to \mathrm{GL}(n,\mathbb{F})$ defined by the conjugation ${}^{g}\rho(h) = \rho(g^{-1}hg)$ becomes a representation of H for each $g \in G$. It follows that the set of representations of H has a natural left G-action. We say that ρ is *G-invariant* if it is isomorphic to the conjugate representation ${}^{g}\rho$ for all $g \in G$, which is a necessary condition for ρ to extend to G.

Let us denote by $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ the multiplicative group of $\mathbb{F}$. A continuous map $\overline{\rho}\colon G \to \mathrm{GL}(n,\mathbb{F})$ is called a *projective representation* of G if $\overline{\rho}(g)\overline{\rho}(g') = \overline{\rho}(gg')\alpha(g,g')$ for $g, g' \in G$ and $\alpha(g,g') \in \mathbb{F}^*$. The continuous function $\alpha\colon G \times G \to \mathbb{F}^*$ is called the associated *factor set* of $\overline{\rho}$.

In the following we assume that G/H is discrete, and show that the technique using projective representations works well in the category of topological groups. The arguments are mainly adapted from the book [9, Chapter 11].

STEP 1. Suppose that a representation $\rho\colon H \to \mathrm{GL}(n,\mathbb{F})$ is *absolutely irreducible*, i.e., every endomorphism of ρ is scalar, and G-invariant. We first show that there exists a projective representation of G which extends ρ if G/H is discrete.

LEMMA 2.1. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Then G is homeomorphic to $G/H \times H$.*

Proof. Denote by G_0 the connected component in G containing the identity element. Then the inclusion $G_0 \hookrightarrow G$ induces a continuous monomorphism from the connected space $G_0/(G_0 \cap H)$ to the discrete space G/H. It follows that $G_0 = G_0 \cap H$ and G_0 is the same as the connected component in H containing the identity element. Therefore H is a union of connected components in G so that G is homeomorphic to $G/H \times H$. □

PROPOSITION 2.2. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a representation $\rho\colon H \to \mathrm{GL}(n,\mathbb{F})$ is absolutely irreducible and G-invariant. Then there exists a projective representation $\overline{\rho}\colon G \to \mathrm{GL}(n,\mathbb{F})$ such that, for all $h \in H$ and $g \in G$,*

(1) $\overline{\rho}(h) = \rho(h)$,
(2) $\overline{\rho}(gh) = \overline{\rho}(g)\overline{\rho}(h)$,
(3) $\overline{\rho}(hg) = \overline{\rho}(h)\overline{\rho}(g)$.

Proof. Choose a transversal T for H in G containing the identity element e of G. Since ρ is G-invariant, there exists a nonsingular matrix M_t for each $t \in T$ such that $M_t^{-1}\rho(h)M_t = {}^t\rho(h) = \rho(t^{-1}ht)$ for all $h \in H$. Take the identity matrix for M_e. Since every element of G is uniquely of the form th for $t \in T$ and $h \in H$ we define $\overline{\rho}(th) = M_t\rho(h)$. It is obvious by Lemma 2.1 that $\overline{\rho}$ is continuous because G/H is homeomorphic to T with discrete topology.

Properties (1) and (2) follow immediately from the definition, and (3) follows from the equation

$$\begin{aligned}\overline{\rho}(h')\overline{\rho}(th) = \rho(h')M_t\rho(h) = M_t\rho(t^{-1}h't)\rho(h) &= M_t\rho(t^{-1}h'th)\\ &= \overline{\rho}(tt^{-1}h'th) = \overline{\rho}(h'\cdot th),\end{aligned}$$

for $t \in T$ and $h, h' \in H$. Applying (1), (2), and (3) it is routine to check that

$$(*) \qquad \overline{\rho}(g)^{-1}\rho(h)\overline{\rho}(g) = \rho(g^{-1}hg)$$

for all $g \in G$ and $h \in H$. Since ρ is absolutely irreducible, the equation

$$\begin{aligned}\overline{\rho}(gg')^{-1}\rho(h)\overline{\rho}(gg') = \rho(g'^{-1}(g^{-1}hg)g') &= \overline{\rho}(g')^{-1}\rho(g^{-1}hg)\overline{\rho}(g')\\ &= [\overline{\rho}(g)\overline{\rho}(g')]^{-1}\rho(h)[\overline{\rho}(g)\overline{\rho}(g')]\end{aligned}$$

implies that $\overline{\rho}$ is a projective representation of G. □

The group cohomology $H^n(G/H,\mathbb{F}^*)$, where G/H acts trivially on $\mathbb{F}^*$ is defined, in terms of the standard (or bar) resolution, by the cochain complex $C^n(G/H,\mathbb{F}^*)$ consisting of functions from $(G/H)^n$ to $\mathbb{F}^*$ and the coboundary formula given by

$$\begin{aligned}&(\delta^n\mu)(g_1,\dots,g_{n+1})\\ &= \mu(g_2,\dots,g_{n+1})\cdot\mu(g_1g_2,\dots,g_{n+1})^{-1}\cdot\mu(g_1,g_2g_3,\dots,g_{n+1})\\ &\quad\cdots\mu(g_1,\dots,g_ng_{n+1})^{(-1)^n}\cdot\mu(g_1,\dots,g_n)^{(-1)^{n+1}}.\end{aligned}$$

STEP 2. We next show that the associated factor set of the projective representation $\overline{\rho}$ constructed in Proposition 2.2 determines an element in the group cohomology $H^2(G/H, \mathbb{F}^*)$, which depends only on the representation ρ of H.

It is routine to check that $\alpha(gh, g'h') = \alpha(g, g')$ for $g, g' \in G$ and $h, h' \in H$. Thus α determines an element $\overline{\alpha}\colon G/H \times G/H \to \mathbb{F}^*$ in $C^2(G/H, \mathbb{F}^*)$ defined by $\overline{\alpha}(gH, g'H) = \alpha(g, g')$. Moreover, it is easy to see that the equation

$$\alpha(g_1g_2, g_3)\alpha(g_1, g_2) = \alpha(g_1, g_2g_3)\alpha(g_2, g_3)$$

holds for $g_1, g_2, g_3 \in G$ so that $\overline{\alpha}$ is a cocycle.

Suppose that $\overline{\rho}_0$ is another projective representation of G satisfying the properties (1), (2), and (3) in Proposition 2.2. Then the equation (*) implies that $\overline{\rho}_0(g) = \overline{\rho}(g)\mu(g)$ for some function $\mu\colon G \to \mathbb{F}^*$ such that $\mu(h) = 1$ for all $h \in H$. Since

$$\overline{\rho}(h)\overline{\rho}(g)\mu(g) = \overline{\rho}_0(h)\overline{\rho}_0(g) = \overline{\rho}_0(hg) = \overline{\rho}(hg)\mu(hg)$$

for $h \in H$ and $g \in G$, the function μ is constant on cosets of H. If α_0 denotes the associated factor set of $\overline{\rho}_0$, then

$$\alpha_0(g, g') = \alpha(g, g')\mu(g)\mu(g')\mu(gg')^{-1} = \alpha(g, g')\delta^1(\mu)(g, g')$$

for $g, g' \in G$. Therefore the corresponding cohomology class of $\overline{\alpha}$ in $H^2(G/H, \mathbb{F}^*)$, denoted by $[\overline{\alpha}]$, depends only on the representation ρ.

STEP 3. Finally we show that the extendibility of ρ is completely determined by the cohomology class $[\overline{\alpha}]$.

THEOREM 2.3. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a representation ρ of H is absolutely irreducible and G-invariant. Then ρ extends to G if and only if $[\overline{\alpha}]$ is trivial in $H^2(G/H, \mathbb{F}^*)$.*

Proof. The necessity is obvious since any G-extension of ρ is a projective representation of G with the trivial factor set.

On the other hand, suppose that $[\overline{\alpha}]$ is trivial, i.e., $\overline{\alpha}$ is a coboundary in $C^2(G/H, \mathbb{F}^*)$. Then there exists a function $\nu\colon G/H \to \mathbb{F}^*$ such that

$$\overline{\alpha}(gH, g'H) = \nu(gH)\nu(g'H)\nu(gg'H)^{-1} = \delta^1(\nu)(gH, g'H).$$

Note that the map $\mu\colon G \to \mathbb{F}^*$ given by $\mu(g) = \nu(gH)$ is continuous since G/H is discrete. Define a continuous map $\widetilde{\rho}\colon G \to \mathrm{GL}(n, \mathbb{F})$ by $\widetilde{\rho}(g) = \overline{\rho}(g)\mu(g)^{-1}$. Then $\widetilde{\rho}$ is a projective representation of G with the trivial factor set so that it is a continuous homomorphism. Since $\overline{\rho}(e) = \rho(e)$ is the identity matrix where e is the identity element of G, we have $\mu(h) = \nu(eH) = \mu(e) = \alpha(e,e) = 1$ for all $h \in H$. Therefore $\widetilde{\rho}$ is a desired representation of G extending ρ. □

It is possible to reduce the extendibility question to Sylow subgroups. We assume that G/H is finite. Then $H^2(G/H, \mathbb{F}^*)$ admits a primary decomposition

$$(**) \qquad H^2(G/H, \mathbb{F}^*) = \bigoplus_p H^2(G/H, \mathbb{F}^*)_{(p)}$$

where p ranges over the primes dividing the order of G/H and $H^2(G/H, \mathbb{F}^*)_{(p)}$ denotes the p-primary component of $H^2(G/H, \mathbb{F}^*)$. If S_p is a Sylow p-subgroup of G/H, then the restriction homomorphism $H^2(G/H, \mathbb{F}^*)_{(p)} \to H^2(S_p, \mathbb{F}^*)$ is injective (see [3, Section 10, Chapter III] for example). Therefore, a cohomology class in $H^2(G/H, \mathbb{F}^*)$ is trivial if so is its reduction to a Sylow p-subgroup of G/H for all primes p dividing the order of G/H.

COROLLARY 2.4. *Let H be a closed normal subgroup of a topological group G such that G/H is finite. Suppose that a representation ρ of H is absolutely irreducible and G-invariant. Then ρ extends to G if and only if it extends to a closed subgroup G_p of G for all primes p dividing the order of G/H, G_p/H being a Sylow p-subgroup of G/H.*

3. Extensions in the complex category

We are mainly interested in representations over connected locally compact topological fields. It is well known that every connected locally compact topological field is isomorphic either to the field $\mathbb{R}$ of real numbers or the filed $\mathbb{C}$ of complex numbers [11, Theorem 21]. From now on we shall assume that $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Let G be a topological group. It is sometimes convenient to use the terminology "G-modules" instead of "representations of G". If G is a finite group, a representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{F})$ makes a vector space $\mathbb{F}^n$ into a left module over the group ring $\mathbb{F}[G]$ of G over $\mathbb{F}$. "This module-theoretic point of view presents technical difficulties in the category of topological groups because continuity must play a role in the definition of group ring and module [2, p.66]." However, a vector space $\mathbb{F}^n$ equipped with a continuous linear action (or a representation) of G will be called a *G-module.*

We first state a well known result providing the structure of the set of G-extensions. Here we do not assume that G/H is discrete.

PROPOSITION 3.1. *Let G be a topological group and H a closed normal subgroup of G. Suppose that an absolutely irreducible H-module V extends to a G-module. Then there is a one-to-one correspondence between the set of G-extensions of V and the set of G/H-modules of dimension one.*

Proof. Choose a G-extension W of V. For a G-extension W' of V, we have a natural map

$$\mathrm{Hom}_H(W, W') \otimes_{\mathbb{F}} W \to W'$$

sending $f \otimes v$ to $f(v)$ for $f \in \mathrm{Hom}_H(W, W')$ and $v \in W$, which is in fact an isomorphism. Here, $\mathrm{Hom}_H(W, W')$ has a canonical G-action induced from those of W and W', and it is a G-module of dimension one with trivial H-action. Since we can view $\mathrm{Hom}_H(W, W')$ as a G/H-module of dimension one, the one-to-one correspondence follows immediately from the map sending W' to $\mathrm{Hom}_H(W, W')$. □

REMARK. (1) Note that, given a G-extension W of V, every G-extension of V has the form of the tensor product of W and a G-module of dimension one with trivial H-action, which is induced from a G/H-module of dimension one by the canonical homomorphism $G \to G/H$.

(2) In particular, if G is a compact Lie group, the commutator subgroup G' is closed and normal in G [8, Theorem 6.11] so that every irreducible G/G'-module is one-dimensional. Therefore, there exists a one-to-one correspondence between the set of G-modules of dimension one and the set of G/G'-modules.

As an application of Theorem 2.3 we now characterize G-extensions of complex H-modules when G/H is a finite subgroup of $O(2)$, namely G/H is a finite cyclic group or a dihedral group.

THEOREM 3.2. *Let G be a topological group and H a closed normal subgroup of G such that G/H is finite cyclic. Then every complex irreducible H-module extends to a G-module if it is G-invariant. Moreover, the number of mutually non-isomorphic G-extensions agrees with the order of G/H.*

Proof. Since the group cohomology $H^2(G/H, \mathbb{C}^*)$ is trivial when G/H is finite cyclic (see [1, Corollary 3.5.2] for instance), the proof is immediate by Theorem 2.3 and Proposition 3.1. On the other hand, it is possible to prove directly without Theorem 2.3 as follows.

Let ρ be an irreducible representation of H corresponding to the given H-module. It suffices to find a representation of G whose restriction to

H is ρ. Choose an element a of G which induces a generator of G/H. Since ρ is G-invariant, there exists a nonsingular matrix M such that

$$M\rho(h)M^{-1} = {}^a\rho(h) = \rho(a^{-1}ha)$$

for all $h \in H$. Set $n = |G/H|$. Using the identity above repeatedly, we have

$$M^n\rho(h)M^{-n} = \rho(a^{-n}ha^n) = \rho(a^n)^{-1}\rho(h)\rho(a^n),$$

where the last equality follows from the fact that $a^n \in H$. The identity above shows that $\rho(a^n)M^n$ commutes with $\rho(h)$ for all $h \in H$. Since ρ is irreducible, it follows from the Schur's lemma that $\rho(a^n)M^n$ is a nonzero scalar matrix. Taking a suitable scalar multiple of M, it can be assumed that the nonzero scalar matrix is the identity matrix. We then define $\rho(a)$ to be M^{-1} so that ρ extends to a representation of G. □

REMARK. It is also known in [6, Corollary 3.4] that every complex irreducible H-module extends to a G-module when G/H is connected abelian, in particular, when G/H is isomorphic to the circle group S^1.

Note that every Sylow p-subgroup of a dihedral group is finite cyclic if $p \neq 2$. The following result follows from Corollary 2.4 and Theorem 3.2.

COROLLARY 3.3. *Let G be a topological group and H a closed normal subgroup of G such that G/H is dihedral. Then a complex irreducible H-module which is G-invariant extends to a G-module if and only if it extends to a P-module for some subgroup of G such that P/H is a Sylow 2-subgroup of G/H.*

EXAMPLE 3.4. Let G be the dihedral group of order $8m$ ($m \geq 1$) generated by

$$\{a, b \mid a^{4m} = b^2 = (ab)^2 = 1\},$$

and let H be the central subgroup of G generated by a^{2m} which is an order two subgroup. Then G/H is isomorphic to the dihedral group of order $4m$. If $\widetilde{\rho}\colon G \to \mathrm{GL}(1,\mathbb{C})$ is a one-dimensional representation of G, then $\widetilde{\rho}(a^{2m}) = 1$ because $b^{2m} = (ab)^{2m} = 1$ and $\mathrm{GL}(1,\mathbb{C})$ is abelian. Therefore, the nontrivial representation $\rho\colon H \to \mathrm{GL}(1,\mathbb{C})$ defined by $\rho(a^{2m}) = -1$ does not extend to G.

COROLLARY 3.5. *Let G be a topological group and H a closed normal subgroup of G such that G/H is dihedral of order $2n$. Let V be a complex irreducible H-module which is G-invariant.*

(1) *If n is odd, then there are two mutually non-isomorphic G-extensions of V.*

(2) *If n is even, then there are either four mutually non-isomorphic G-extensions of V or none.*

Proof. If n is odd, then all Sylow 2-subgroups of G/H are finite cyclic of order two. Thus V has a G-extension by Corollary 2.4 and Theorem 3.2. The number of G-extensions follows from Proposition 3.1, since the number of complex G/H-modules of dimension one is two if n is odd and four if n is even. □

4. Extensions in the real category

Let G be a topological group and let U be a real irreducible G-module. Since U is irreducible, every endomorphism of U is an isomorphism or the zero map so that the endomorphism algebra of U is isomorphic either to the real filed $\mathbb{R}$, the complex field $\mathbb{C}$, or the division ring $\mathbb{H}$ of quaternions. Therefore we call U of *real*, *complex*, or *quaternionic* type, respectively, according to its endomorphism algebra $\mathrm{Hom}_H(U,U)$.

Let H be a closed normal subgroup of G and U a real irreducible H-module. In this section it is assumed that G/H is discrete and U is G-invariant. We study how to attack the extendibility question according to the type of U.

CASE 1: U is of real type. Since U is absolutely irreducible, Theorem 2.3 is rephrased as follows.

THEOREM 4.1. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a real irreducible H-module U is G-invariant and of real type. Then U extends to a real G-module if and only if the corresponding cohomology class in $H^2(G/H, \mathbb{R}^*)$ is trivial. Moreover, every G-extension of U is of real type.*

The last statement in Theorem 4.1 is obvious since $\mathrm{Hom}_G(\widetilde{U}, \widetilde{U}) \subset \mathrm{Hom}_H(U,U) \cong \mathbb{R}$ if U extends to a real G-module $\widetilde{U}$.

CASE 2: U is of complex type. In this case, the endomorphism algebra $\mathrm{Hom}_H(U,U) \cong \mathbb{C}$ acts on U so that we may view U as a complex irreducible H-module, say V. Then $U \otimes \mathbb{C} \cong V \oplus \overline{V}$ and $V \not\cong \overline{V}$. Since U is G-invariant, so is $U \otimes \mathbb{C}$. It follows that the conjugate H-module ${}^gV$ is isomorphic to either V or $\overline{V}$.

LEMMA 4.2. *Suppose that U extends to a real G-module. Then the $\mathbb{R}$-linear action of $g \in G$ on V is either $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear, and it is $\mathbb{C}$-linear (resp. conjugate $\mathbb{C}$-linear) if and only if ${}^gV \cong V$ (resp. ${}^gV \cong \overline{V}$).*

Proof. The action of $g \in G$ on $\mathrm{Hom}_H(U, U)$ given by conjugation $\sigma \to g\sigma g^{-1}$ for $\sigma \in \mathrm{Hom}_H(U, U)$ is an $\mathbb{R}$-algebra automorphism, and since $\mathrm{Hom}_H(U, U)$ is isomorphic to $\mathbb{C}$ as $\mathbb{R}$-algebra, the action of g on $\mathrm{Hom}_H(U, U)$ is either the identity or the complex conjugation when $\mathrm{Hom}_H(U, U)$ is identified with $\mathbb{C}$. Accordingly the action of g on V is either $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear.

If the g-action on V is $\mathbb{C}$-linear, then the map ${}^gV \to V$ sending $v \mapsto gv$ is a $\mathbb{C}$-linear H-isomorphism. Conversely, if there is a $\mathbb{C}$-linear H-isomorphism $\varphi\colon V \to {}^gV$, then the composition $g\varphi\colon V \to V$ sending $v \mapsto g\varphi(v)$ is an $\mathbb{R}$-linear H-isomorphism so that $g\varphi \in \mathrm{Hom}_H(U, U)$. For each $\sigma \in \mathrm{Hom}_H(U, U)$, we have $\sigma(g\varphi) = (g\varphi)\sigma = (g\sigma)\varphi$ since the algebra $\mathrm{Hom}_H(U, U)$ is commutative and φ is $\mathbb{C}$-linear. Therefore $\sigma g = g\sigma$ for all $\sigma \in \mathrm{Hom}_H(U, U)$, which means that the action of g on V is $\mathbb{C}$-linear. □

The set

$$G_U := \{g \in G \mid {}^gV \cong V\}$$

is independent of the choice of V and depends only on U. Clearly G_U contains H and forms a subgroup of G of index at most two.

COROLLARY 4.3. (1) *If $G_U = G$, then every G-extension of U is of complex type.*

(2) *If $G_U \neq G$, then every G-extension of U is of real type.*

When U is of complex type, there is a generalized version of Theorem 2.3 developed by I. M. Issacs [10, Corollary 4.4], and it is not difficult to apply the result for topological groups. In the following we shortly state the result.

We identify V with $\mathbb{C}^n$, and define

$${}^gv := \begin{cases} v & \text{if } {}^gV \cong V \\ \overline{v} & \text{if } {}^gV \cong \overline{V} \end{cases}$$

for $v \in \mathbb{C}^n = V$, where $\overline{v}$ denotes the complex conjugate of v. Similarly we define

$${}^gM := \begin{cases} M & \text{if } {}^gV \cong V \\ \overline{M} & \text{if } {}^gV \cong \overline{V} \end{cases}$$

for $M \in \mathrm{GL}(n,\mathbb{C})$. A continuous map $\widehat{\rho}\colon G \to \mathrm{GL}(n,\mathbb{C})$ is called a *crossed representation* of G if $\widehat{\rho}(gg') = \widehat{\rho}(g)^g\widehat{\rho}(g')$ for $g, g' \in G$.

LEMMA 4.4. *Let ρ be a complex representation of H corresponding to V. Then U extends to a real G-module if and only if ρ extends to a crossed representation $\widehat{\rho}$.*

Proof. For the sufficiency it suffices to show that the $\mathbb{C}$-linear H-action on V extends to an $\mathbb{R}$-linear G-action. Note that ${}^g(Mv) = {}^gM{}^gv$ for $v \in V$ and $M \in \mathrm{GL}(n,\mathbb{C})$. Then V has a G-action defined by $g \cdot v = \widehat{\rho}(g){}^gv$ for $g \in G$ and $v \in V$ since

$$(***) \quad \begin{aligned} g(g'v) = g(\widehat{\rho}(g')^{g'}v) &= \widehat{\rho}(g)^g(\widehat{\rho}(g')^{g'}v) \\ &= \widehat{\rho}(g)^g\widehat{\rho}(g')^{gg'}v = \widehat{\rho}(gg')^{gg'}v = (gg')v \end{aligned}$$

for $g, g' \in G$. It is obviously $\mathbb{R}$-linear, continuous, and extends the H-action on V since $h \cdot v = \widehat{\rho}(h)^hv = \rho(h)v$ for all $h \in H$.

On the other hand, suppose that U extends to a real G-module, i.e., the $\mathbb{C}$-linear H-action on V extends to an $\mathbb{R}$-linear G-action. Since the action of $g \in G$ on V is either $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear according as ${}^gV \cong V$ or $\overline{V}$ by Lemma 4.2, the map sending $v \in V$ to $g \cdot {}^gv \in V$ is $\mathbb{C}$-linear; so we have a complex matrix $\widehat{\rho}(g) \in \mathrm{GL}(n,\mathbb{C})$ such that $g \cdot {}^gv = \widehat{\rho}(g)v$, equivalently $g \cdot v = \widehat{\rho}(g){}^gv$ because ${}^g({}^gv) = v$. The map $\widehat{\rho}\colon G \to \mathrm{GL}(n,\mathbb{C})$ is continuous since the two G-actions, $g \cdot v$ and ${}^gv$, on V are continuous. It is obvious that $\widehat{\rho}(h) = \rho(h)$ for all $h \in H$. The equation (***) above shows that $\widehat{\rho}(gg') = \widehat{\rho}(g)^g\widehat{\rho}(g')$. □

Similarly, a continuous map $\overline{\rho}\colon G \to \mathrm{GL}(n,\mathbb{C})$ is called a *projective crossed representation* of G if $\overline{\rho}(g)^g\overline{\rho}(g') = \overline{\rho}(gg')\alpha(g,g')$ for $g, g' \in G$ and $\alpha(g,g') \in \mathbb{C}^*$. Then Proposition 2.2 is generalized as follows.

PROPOSITION 4.5. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Let ρ be a complex representation of H corresponding to a real irreducible H-module which is of complex type and G-invariant. Then there exists a projective crossed representation $\overline{\rho}$ such that (1) $\overline{\rho}(h) = \rho(h)$, (2) $\overline{\rho}(gh) = \overline{\rho}(g)^g\overline{\rho}(h)$, and (3) $\overline{\rho}(hg) = \overline{\rho}(h)\overline{\rho}(g)$ for all $h \in H$ and $g \in G$.*

Let us denote by $\overline{\mathbb{C}}^*$ the non-zero complex numbers $\mathbb{C}^*$ with the G-action defined by

$${}^gz := \begin{cases} z & \text{if } {}^gV \cong V, \\ \overline{z} & \text{if } {}^gV \cong \overline{V} \end{cases}$$

for $z \in \mathbb{C}$. The factor set $\alpha\colon G \times G \to \mathbb{C}^*$ is constant on cosets of H and satisfies the equation

$$\alpha(g_1 g_2, g_3)\alpha(g_1, g_2) = \alpha(g_1, g_2 g_3)^{g_1}\alpha(g_2, g_3)$$

for $g_1, g_2, g_3 \in G$. So the induced map $\overline{\alpha}\colon G/H \times G/H \to \mathbb{C}^*$ is a cocycle in $C^2(G/H, \overline{\mathbb{C}}^*)$, and the corresponding cohomology class $[\overline{\alpha}] \in H^2(G/H, \overline{\mathbb{C}}^*)$ depends only on ρ. Finally Theorem 2.3 is generalized as follows.

PROPOSITION 4.6. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a real irreducible H-module U is G-invariant and of complex type. Then a complex representation of H corresponding to U extends to a crossed representation of G if and only if $[\overline{\alpha}]$ is trivial in $H^2(G/H, \overline{\mathbb{C}}^*)$.*

Combining Proposition 4.6 with Lemma 4.4, we have the following result.

THEOREM 4.7. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a real irreducible H-module U is G-invariant and of complex type. Then U extends to a real G-module if and only if the corresponding cohomology class in $H^2(G/H, \overline{\mathbb{C}}^*)$ is trivial.*

CASE 3: U is of quaternionic type. Similarly, $\mathbb{C} \subset \mathbb{H} \cong \mathrm{Hom}_H(U, U)$ acts on U so that we may view U as a complex irreducible H-module, say V. Then $U \otimes \mathbb{C} \cong V \oplus V$ and $V \cong \overline{V}$. Since U and thus $U \otimes \mathbb{C}$ are G-invariant, V is also G-invariant. Therefore, V determines an element in the group cohomology $H^2(G/H, \mathbb{C}^*)$, and V (resp. U) extends to a complex (resp. real) G-module if the element is trivial by Theorem 2.3.

THEOREM 4.8. *Let H be a closed normal subgroup of a topological group G such that G/H is discrete. Suppose that a real irreducible H-module U is G-invariant and of quaternionic type. Then U extends to a real G-module if the corresponding cohomology class in $H^2(G/H, \mathbb{C}^*)$ is trivial.*

However, the converse is no longer true as the following example shows.

EXAMPLE 4.9. Denote by Q_8 the quaternion group of order 8 generated by x and y under the conditions $x^4 = 1$, $x^2 = y^2$, and $xyx = y$. The subgroup K generated by x^2 is of order two and $Q_8/K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Let $\rho_1 \colon Q_8 \to \mathrm{GL}(2,\mathbb{C})$ be the standard representation given by

$$\rho_1(x) = \begin{bmatrix} e^{\pi i/2} & 0 \\ 0 & -e^{\pi i/2} \end{bmatrix} \qquad \text{and} \qquad \rho_1(y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and let $\rho_2 \colon K \to \mathrm{GL}(1,\mathbb{C})$ be the nontrivial representation given by $\rho_2(x^2) = -1$. Note that ρ_2 does not extend to Q_8 because $xyx = y$ and $\mathrm{GL}(1,\mathbb{C})$ is abelian. Let us denote by $\rho_1^{\mathbb{R}} \colon Q_8 \to \mathrm{GL}(4,\mathbb{R})$ the realification of ρ_1, and by $\rho_2^{\mathbb{R}} \colon K \to \mathrm{GL}(1,\mathbb{R})$ the nontrivial representation given by $\rho_2^{\mathbb{R}}(x^2) = -1$. Note that $\rho_1^{\mathbb{R}}$ is of quaternionic type and $\rho_2^{\mathbb{R}}$ is of real type.

Let $G = Q_8 \times Q_8$ and $H = Q_8 \times K$. We claim that $\rho_1^{\mathbb{R}} \otimes_{\mathbb{R}} \rho_2^{\mathbb{R}}$ extends to G but the corresponding cohomology class in $H^2(G/H, \mathbb{C}^*)$ is not trivial. Note that $\rho_1^{\mathbb{R}} \otimes_{\mathbb{R}} \rho_2^{\mathbb{R}}$ is the realification of the complex representation $\rho_1 \otimes_{\mathbb{C}} \rho_2$ which does not extend to G since ρ_2 does not extend to Q_8. Thus the cohomology class in $H^2(G/H, \mathbb{C}^*)$ corresponding to $\rho_1 \otimes_{\mathbb{C}} \rho_2$ (and so $\rho_1^{\mathbb{R}} \otimes_{\mathbb{R}} \rho_2^{\mathbb{R}}$) is not trivial. Since there is a conjugate $\mathbb{C}$-linear Q_8-endomorphism $\mathcal{J}$ of ρ_1 such that $\mathcal{J}^2 = -\mathrm{id}$, we have a conjugate $\mathbb{C}$-linear G-endomorphism $\mathcal{J} \otimes_{\mathbb{C}} \mathcal{J}$ of $\rho_1 \otimes_{\mathbb{C}} \rho_1$ such that $(\mathcal{J} \otimes_{\mathbb{C}} \mathcal{J})^2 = \mathrm{id}$, in other words, $\rho_1 \otimes_{\mathbb{C}} \rho_1$ has a G-invariant real structure. Therefore, the realification of $\rho_1 \otimes_{\mathbb{C}} \rho_1$ is reducible and its irreducible component is a G-extension of $\rho_1^{\mathbb{R}} \otimes_{\mathbb{R}} \rho_2^{\mathbb{R}}$, since the restriction of $\rho_1 \otimes_{\mathbb{C}} \rho_1$ to H is isomorphic to $(\rho_1 \otimes_{\mathbb{C}} \rho_2) \oplus (\rho_1 \otimes_{\mathbb{C}} \rho_2)$.

We now apply the results developed in this section to the case when G/H is a finite subgroup of $O(2)$.

LEMMA 4.10. *Let G be a topological group and H a closed normal subgroup of G such that G/H is finite cyclic of odd order. Then a real irreducible H-module extends to a real G-module if it is G-invariant.*

Proof. Note that both $H^2(G/H, \mathbb{R}^*)$ and $H^2(G/H, \mathbb{C}^*)$ are trivial [1, Corollary 3.5.2]. Moreover, $\overline{\mathbb{C}}^* = \mathbb{C}^*$ since G/H has no index two subgroup. Therefore the lemma follows immediately from Theorems 4.1, 4.7, and 4.8. □

LEMMA 4.11. *Let G be a topological group and H a closed normal subgroup of G such that G/H is dihedral of order $2n$ for odd n. A real irreducible H-module U which is G-invariant extends to a real G-module if it extends to a real P-module for some closed subgroup P of G which contains H as an index two subgroup.*

Proof. Note that every Sylow p-subgroup of G/H is finite cyclic of odd order unless $p = 2$, and that P/H is a Sylow 2-subgroup of G/H. If

U is of real or complex type, then the result follows from Corollary 2.4 or its generalized version using $H^2(G/H, \overline{\mathbb{C}}^*)$, respectively. Since every Sylow p-subgroup of G/H is finite cyclic, the p-primary component $H^2(G/H, \mathbb{C}^*)_{(p)}$ is trivial for all prime p so that $H^2(G/H, \mathbb{C}^*)$ is trivial by the primary decomposition (**) in Section 2. Therefore, if U is of quaternionic type, then U always extends to a real G-module by Theorem 4.8. □

THEOREM 4.12. *Let G be a topological group and H a closed normal subgroup G such that G/H is finite cyclic or dihedral. Then a real irreducible H-module U which is G-invariant extends to a real G-module if and only if it extends to a real P-module for some closed subgroup of G such that P/H is a Sylow 2-subgroup of G/H.*

Proof. Suppose that G/H is finite cyclic of order $n = 2^m k$ for odd k. Choose an element $a \in G$ such that $a^n \in H$. Then G is generated by H and a, and P is generated by H and a^k. Note that G/P is finite cyclic of odd order k. Since $a^{-1}a^k a = a^k$, any P-extension of U is G-invariant so that it extends to G by Lemma 4.10.

Suppose that G/H is dihedral of order $2n = 2^m k$ for odd k. Choose elements $a, b \in G$ such that G is generated by a, b, and H under the conditions a^n, b^2, and $(ab)^2 \in H$. We may assume that P is generated by a^k, b, and H. Then the index two subgroup K of P generated by a^k and H is normal in G, and G/K is dihedral of order $2k$. Since k is odd, it suffices to show that U extends to a real K-module which is G-invariant by Lemma 4.11. By assumption U extends to a real P-module, say $\widetilde{U}$. Then the restriction of $\widetilde{U}$ to K is a desired K-extension of U which is G-invariant, since it is P-invariant and $a^{-1}a^k a = a^k$. □

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