

SOME EXISTENCE THEOREMS FOR FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMING

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ABSTRACT. The existence, uniqueness and iterative approximation of solutions for a few classes of functional equations arising in dynamic programming of multistage decision processes are discussed. The results presented in this paper extend, improve and unify the results due to Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

1. Introduction and preliminaries

Bellman [2, 3] first studied the existence of solutions for several classes of functional equations arising in dynamic programming. Bellman and Roosta [5] constructed an approximation solution for a class of the infinite-stage equation arising in dynamic programming. Bellman and Lee [4] pointed out that the basic form of the functional equations of dynamic programming is

$$(1.1) \quad f(x) = \sup_{y \in D} H(x, y, f(T(x, y))), \quad x \in S,$$

where x and y represent the state and decision vectors, respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x . Baskaran and Subrahmanyam [1], Bhakta and Choudhury [6], Bhakta and Mitra [7], Chang [8], Chang and Ma [9], Liu [10]-[12] and others extended the results of [2]-[5] in

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various directions. Bhakta and Mitra [7] established the existence and uniqueness of solutions for the functional equation:

$$(1.2) \quad f(x) = \sup_{y \in D} \{u(x, y) + f(T(x, y))\}, \quad x \in S.$$

Under suitable conditions, Bellman [2], Bhakta and Choudhury [6] and Liu [12] obtained the existence or uniqueness of solutions for the functional equations:

$$(1.3) \quad f(x) = \inf_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S.$$

$$(1.4) \quad f(x) = \inf_{y \geq x} \{u(x, y) + v(x, y) \left[\int_y^{+\infty} p(s-y)q(s)ds \right. \\ \left. + f(0) \int_y^{+\infty} q(s)ds + \int_0^y f(y-s)q(s)ds \right]\}.$$

Inspired and motivated by the work in [1]-[12], in this paper, we prove the existence, uniqueness and iterative approximation of solutions for the functional equations (1.4)-(1.6):

$$(1.5) \quad f(x) = \text{opt}_{y \in D} \{u(x, y) + f(T(x, y))\}, \quad x \in S,$$

$$(1.6) \quad f(x) = \text{opt}_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S,$$

where the opt denotes the sup or inf . The results presented in this paper extend, improve and unify the corresponding results of Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

Throughout this paper, N denotes the set of all positive integers, $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$. Define

$$\Phi_1 = \{\varphi : \varphi : R^+ \rightarrow R^+ \text{ is nondecreasing}\},$$

$$\Phi_2 = \{\varphi : \varphi \in \Phi_1 \text{ and } \lim_{n \rightarrow \infty} \varphi^n(t) = 0 \text{ for } t > 0\},$$

$$\Phi_3 = \{\varphi : \varphi \in \Phi_1, \varphi(0) = 0 \text{ and } \varphi \text{ is right continuous at } 0\}.$$

REMARK 1.1. It is easy to see that $\varphi \in \Phi_2$ implies $\varphi(t) < t$ for any $t > 0$.

Let us recall the following concept. Let X be a nonempty set and let $\{d_n\}_{n \in \mathbb{N}}$ be a countable family of pseudometrics on X such that for any distinct $x, y \in X$, $d_k(x, y) \neq 0$ for some $k \in \mathbb{N}$. Define

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x, y)}{1 + d_k(x, y)} \quad \text{for all } x, y \in X.$$

It is clear that d is a metric on X . A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges to a point $x \in X$ if and only if $d_k(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ and $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$ for each $k \in \mathbb{N}$.

LEMMA 1.1. ([12]) *Let a, b, c be in R . Then*

$$|\text{opt } \{a, c\} - \text{opt } \{b, c\}| \leq |a - b|.$$

2. Existence and uniqueness theorems

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-valued mappings on S that are bounded on bounded subsets of S . It is easy to verify that $BB(S)$ is a linear space over R under usual definitions of addition and multiplication by scalars. For any $k \in \mathbb{N}$ and $a, b \in BB(S)$, let

$$d_k(a, b) = \sup\{|a(x) - b(x)| : x \in \overline{B}(0, k)\},$$

$$d(a, b) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a, b)}{1 + d_k(a, b)},$$

where $\overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k \in \mathbb{N}}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

THEOREM 2.1. *Let $u : S \times D \rightarrow R, T : S \times D \rightarrow S$ be mappings and*

$$(2.1) \quad \begin{aligned} a_0(x) &= \sup_{y \in D} u(x, y), \\ a_n(x) &= \sup_{y \in D} \{u(x, y) + a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in \mathbb{N}. \end{aligned}$$

If there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_1$ such that

$$(2.2) \quad \|T(x, y)\| \leq \varphi(\|x\|) \text{ for all } (x, y) \in S \times D,$$

$$(2.3) \quad |u(x, y)| \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D,$$

and

$$(2.4) \quad \sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty \text{ for all } t > 0,$$

then the functional equation (1.2) possesses a solution $w \in BB(S)$ such that

$$(2.5) \quad \lim_{n \rightarrow \infty} w(x_n) = 0 \text{ for any } x_0 \in S, \\ \{y_n\}_{n \in N} \subseteq D, \quad x_n = T(x_{n-1}, y_n), \quad n \in N.$$

Moreover, the solution w of the functional equation (1.2) is also unique with respect to (2.5).

Proof. Put

$$H(x, y, a) = u(x, y) + a(T(x, y)) \text{ for all } (x, y, a) \in S \times D \times BB(S), \\ fa(x) = \sup_{y \in D} H(x, y, a) \text{ for all } (x, a) \in S \times BB(S).$$

For any $k \in N$, $y \in D$ and $x \in \overline{B}(0, k)$, by (2.2), (2.3), and Remark 1.1, we have

$$(2.6) \quad |u(x, y)| \leq \psi(\|x\|) \leq \psi(k), \|T(x, y)\| \leq \varphi(\|x\|) \leq \varphi(k).$$

Using (2.6) and the definition of f , we infer that $fa \in BB(S)$ for any $a \in BB(S)$. That is, f maps $BB(S)$ into $BB(S)$.

Now we prove that f is a nonexpansive mapping in $BB(S)$. For any $a, b \in BB(S)$, $\varepsilon > 0$, $k \in N$ and $x \in \overline{B}(0, k)$, there exist $y, z \in D$ such that

$$(2.7) \quad fa(x) - \varepsilon < H(x, y, a), \quad fb(x) - \varepsilon < H(x, z, b), \\ fa(x) \geq H(x, z, a), \quad fb(x) \geq H(x, y, b).$$

It follows from (2.7) that

$$\begin{aligned}
 & |fa(x) - fb(x)| \\
 & < \max\{|H(x, z, a) - H(x, z, b)|, |H(x, y, a) - H(x, y, b)|\} + \varepsilon \\
 & = \max\{|a(T(x, z)) - b(T(x, z))|, |a(T(x, y)) - b(T(x, y))|\} + \varepsilon \\
 & \leq d_k(a, b) + \varepsilon,
 \end{aligned}$$

which implies that $d_k(fa, fb) \leq d_k(a, b) + \varepsilon$. Letting $\varepsilon \rightarrow 0$, we have

$$d_k(fa, fb) \leq d_k(a, b) \text{ for all } a, b \in BB(S), k \in N,$$

which yields that

$$d(fa, fb) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(fa, fb)}{1 + d_k(fa, fb)} \leq \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a, b)}{1 + d_k(a, b)} = d(a, b).$$

That is,

$$(2.8) \quad d(fa, fb) \leq d(a, b) \text{ for all } a, b \in BB(S).$$

We claim that for any $n \geq 0$,

$$(2.9) \quad |a_n(x)| \leq \sum_{i=0}^n \psi(\varphi^i(\|x\|)) \text{ for all } x \in S.$$

In fact, by (2.3) we conclude that

$$-\psi(\|x\|) \leq u(x, y) \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D,$$

which means that

$$|a_0(x)| = \left| \sup_{y \in D} u(x, y) \right| \leq \psi(\|x\|) \text{ for all } x \in S.$$

Assume that (2.9) holds for some $n \geq 0$. It follows from (2.2) that

$$(2.10) \quad |a_n(T(x, y))| \leq \sum_{i=0}^n \psi(\varphi^i(\|T(x, y)\|)) \leq \sum_{i=0}^n \psi(\varphi^{i+1}(\|x\|))$$

for all $(x, y) \in S \times D$. From (2.3) and (2.10) we know that

$$\begin{aligned} -\sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) &\leq u(x, y) + a_n(T(x, y)) \\ &\leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \text{ for all } (x, y) \in S \times D. \end{aligned}$$

This yields that

$$\begin{aligned} |a_{n+1}(x)| &= \left| \sup_{y \in D} \{u(x, y) + a_n(T(x, y))\} \right| \\ &\leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \text{ for all } x \in S. \end{aligned}$$

That is, (2.9) holds for all $n \geq 0$.

Next we prove that $\{a_n\}_{n \geq 0}$ is a Cauchy sequence in $BB(S)$. Let $k \in N$, $\varepsilon > 0$ and $x_0 \in \overline{B}(0, k)$ be given. (2.4) ensures that there exists some $m \in N$ such that

$$(2.11) \quad \sum_{i=n}^{n+p} \psi(\varphi^i(k)) < \varepsilon \text{ for all } n \geq m \text{ and } p \in N.$$

For any $n \geq m$ and $p \in N$, by (2.1) we know that there exist $v_1, w_1 \in D$ and $y_1 = T(x_0, v_1)$, $z_1 = T(x_0, w_1)$ satisfying

$$(2.12) \quad \begin{aligned} a_{n+p}(x_0) &< H(x_0, v_1, a_{n+p-1}) + 2^{-1}\varepsilon, \quad a_n(x_0) \geq H(x_0, v_1, a_{n-1}), \\ a_n(x_0) &< H(x_0, w_1, a_{n-1}) + 2^{-1}\varepsilon, \quad a_{n+p}(x_0) \geq H(x_0, w_1, a_{n+p-1}). \end{aligned}$$

From (2.12) we have

$$(2.13) \quad \begin{aligned} &|a_{n+p}(x_0) - a_n(x_0)| \\ &< \max\{|H(x_0, v_1, a_{n+p-1}) - H(x_0, v_1, a_{n-1})|, \\ &\quad |H(x_0, w_1, a_{n+p-1}) - H(x_0, w_1, a_{n-1})|\} + 2^{-1}\varepsilon \\ &= \max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\} + 2^{-1}\varepsilon \\ &= |a_{n-1}(x_1) - a_{n+p-1}(x_1)| + 2^{-1}\varepsilon, \end{aligned}$$

where $x_1 = y_1$ or z_1 and

$$\begin{aligned} & |a_{n-1}(x_1) - a_{n+p-1}(x_1)| \\ &= \max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\}. \end{aligned}$$

Similarly, we conclude that there exist $v_i, w_i \in D$, $y_i = T(x_{i-1}, v_i)$, $z_i = T(x_{i-1}, w_i)$, $x_i = y_i$ or z_i for $2 \leq i \leq n$ satisfying

$$\begin{aligned} & |a_{n+p-1}(x_1) - a_{n-1}(x_1)| < |a_{n+p-2}(x_2) - a_{n-2}(x_2)| + 2^{-2}\varepsilon, \\ & |a_{n+p-2}(x_2) - a_{n-2}(x_2)| < |a_{n+p-3}(x_3) - a_{n-3}(x_3)| + 2^{-3}\varepsilon, \\ (2.14) \quad & \vdots \\ & |a_{p+1}(x_{n-1}) - a_1(x_{n-1})| < |a_p(x_n) - a_0(x_n)| + 2^{-n}\varepsilon. \end{aligned}$$

It follows from $\varphi \in \Phi_2$, (2.2) and Remark 1.1 that

$$(2.15) \quad \|x_n\| \leq \varphi(\|x_{n-1}\|) \leq \varphi^2(\|x_{n-2}\|) \leq \cdots \leq \varphi^n(\|x_0\|) \leq \|x_0\| \leq k$$

for any $n \in N$. In the light of (2.9), (2.11), and (2.13)-(2.15), we obtain that

$$\begin{aligned} & |a_{n+p}(x_0) - a_n(x_0)| < |a_p(x_n) - a_0(x_n)| + \varepsilon, \\ & \leq |a_p(x_n)| + |a_0(x_n)| + \varepsilon \\ & \leq \psi(\|x_n\|) + \sum_{i=0}^p \psi(\varphi^i(\|x_n\|)) + \varepsilon \\ (2.16) \quad & \leq \psi(\varphi^n(\|x_0\|)) + \sum_{i=0}^p \psi(\varphi^{i+n}(\|x_0\|)) + \varepsilon \\ & \leq 2 \sum_{i=n}^{n+p} \psi(\varphi^i(\|k\|)) + \varepsilon \\ & < 3\varepsilon \end{aligned}$$

for any $n \geq m$ and $p \in N$. Thus (2.15) and (2.16) yield that $d_k(a_n, a_{n+p}) \leq 3\varepsilon$. That is, $\{a_n\}_{n \geq 0}$ is a Cauchy sequence in $(BB(S), d)$ and hence it converges to some $w \in BB(S)$. By virtue of (2.8), we get that

$$d(fw, w) \leq d(fw, fa_n) + d(a_{n+1}, w) \leq d(w, a_n) + d(a_{n+1}, w) \rightarrow 0$$

as $n \rightarrow \infty$. That is, $w = fw$ is a fixed point of f and hence the functional equation (1.2) possesses a solution $w \in BB(S)$.

We prove that (2.5) holds. For any $x_0 \in S$, $\{y_n\}_{n \in N} \subseteq D$, $x_n = T(x_{n-1}, y_n)$, $n \in N$, we have by (2.2)

$$(2.17) \quad \|x_n\| = \|T(x_{n-1}, y_n)\| \leq \varphi(\|x_{n-1}\|) \leq \cdots \leq \varphi^n(\|x_0\|) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Put $k = \lceil \|x_0\| \rceil + 1$, where $[t]$ denotes the largest integer not exceeding t . From Remark 1.1 and (2.17) we conclude that $\{x_n\}_{n \geq 0} \subseteq \overline{B}(0, k)$. Let k be in N . Note that $\lim_{m \rightarrow \infty} d_k(w, a_m) = 0$. For given $\varepsilon > 0$, by (2.4) and (2.17) we know that there exists some $m \in N$ such that

$$(2.18) \quad \max \left\{ d_k(w, a_m), \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|)) \right\} < \varepsilon \text{ for any } n \geq m.$$

By virtue of (2.9) and (2.18), we infer that

$$\begin{aligned} |w(x_n)| &\leq |w(x_n) - a_m(x_n)| + |a_m(x_n)| \\ &\leq d_k(w, a_m) + \sum_{i=0}^m \psi(\varphi^i(\|x_n\|)) \\ &\leq d_k(w, a_m) + \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|)) \\ &< 2\varepsilon \end{aligned}$$

for all $n \geq m$. Therefore $\lim_{m \rightarrow \infty} w(x_n) = 0$.

Finally we prove that w is a unique solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Suppose that v is also a solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of w and v , we conclude that there exist $\{y_n\}_{n \geq 1} \subseteq D$, $\{z_n\}_{n \geq 1} \subseteq D$, $\{x_n\}_{n \geq 1} \subseteq S$ and $\{t_n\}_{n \geq 1} \subseteq S$ with $x_n = T(x_{n-1}, y_n)$, $t_n = T(t_{n-1}, z_n)$ for all $n \in N$, such that

$$(2.19) \quad \begin{aligned} w(x_i) &< u(x_i, y_{i+1}) + w(x_{i+1}) + 2^{-i-1}\varepsilon, \\ v(t_i) &< u(t_i, z_{i+1}) + v(t_{i+1}) + 2^{-i-1}\varepsilon, \\ w(t_i) &\geq u(t_i, z_{i+1}) + w(t_{i+1}), \quad v(x_i) \geq u(x_i, y_{i+1}) + v(x_{i+1}) \end{aligned}$$

for all $i \geq 0$. By (2.19) we easily deduce that

$$(2.20) \quad \begin{aligned} w(x_0) &< u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) \\ &\quad + w(x_n) + (1 - 2^{-n})\varepsilon, \\ v(t_0) &< u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) \\ &\quad + v(t_n) + (1 - 2^{-n})\varepsilon, \\ w(t_0) &\geq u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) + w(t_n), \\ v(x_0) &\geq u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) + v(x_n) \end{aligned}$$

for any $n \in N$. Using (2.20) and $x_0 = t_0$, we have

$$|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n})\varepsilon.$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain that $|w(x_0) - v(x_0)| \leq \varepsilon$, which implies that $w(x_0) = v(x_0)$ by letting $\varepsilon \rightarrow 0$. This completes the proof. \square

REMARK 2.1. Theorem 2.4 in [7] is a special case of Theorem 2.1 with $\psi(t) = Mt$ for all $t \in R^+$, where M is a positive constant. The following example reveals that Theorem 2.1 generalizes properly Theorem 2.4 in [7].

EXAMPLE 2.1. Let $X = Y = R$, $S = D = R^+$. Define $u : S \times D \rightarrow R$, $T : S \times D \rightarrow S$ by

$$u(x, y) = \frac{x^2(1 - xy)}{1 + xy}, \quad T(x, y) = \frac{x \sin^2(x + y)}{2 + y^2} \quad \text{for all } (x, y) \in S \times D.$$

Choose $\varphi(t) = 2^{-1}t$ and $\psi(t) = t^2$ for all $t \in R^+$. It is easy to verify that the conditions of Theorem 2.1 are satisfied. Hence the functional equation (1.2) possesses a solution in $BB(S)$. But Theorem 2.4 in [7] is not applicable since

$$|u(x, y)| = |u(M + 1, 0)| = (M + 1)^2 > M|x|$$

for any $M > 0$ with $(x, y) = (M + 1, 0) \in S \times D$.

A proof similar to that of Theorem 2.1 gives the following result and is thus omitted.

THEOREM 2.2. *Let $u : S \times D \rightarrow R$, $T : S \times D \rightarrow S$ be mappings and*

$$(2.21) \quad \begin{aligned} a_0(x) &= \inf_{y \in D} u(x, y), a_n(x) \\ &= \inf_{y \in D} \{u(x, y) + a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in N. \end{aligned}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_1$ satisfying (2.2)-(2.4). Then the functional equation

$$(2.22) \quad f(x) = \inf_{y \in D} \{u(x, y) + f(T(x, y))\}, \quad x \in S,$$

possesses a solution $w \in BB(S)$ such that (2.5) holds. Moreover, the solution w of the functional equation (2.22) is also unique with respect to (2.5).

THEOREM 2.3. *Let $u : S \times D \rightarrow R$, $T : S \times D \rightarrow S$ be mappings and*

$$(2.23) \quad \begin{aligned} a_0(x) &= \sup_{y \in D} u(x, y), a_n(x) \\ &= \sup_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in N. \end{aligned}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) and (2.3). Then the functional equation

$$(2.24) \quad f(x) = \sup_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S,$$

possesses a solution $w \in BB(S)$ such that (2.5) holds and

$$(2.25) \quad w(x) \geq 0 \quad \text{for all } x \in S.$$

Moreover, the solution w of the functional equation (2.24) is also unique with respect to (2.5).

Proof. Set

$$\begin{aligned} H(x, y, a) &= \max\{u(x, y), a(T(x, y))\} \text{ for all } (x, y, a) \in S \times D \times BB(S), \\ fa(x) &= \sup_{y \in D} H(x, y, a) \text{ for all } (x, a) \in S \times BB(S). \end{aligned}$$

As in the proof of Theorem 2.1, we can conclude that f maps $BB(S)$ into $BB(S)$ and (2.8) holds. Now we claim that for all $n \geq 0$,

$$(2.26) \quad |a_n(x)| \leq \psi(\|x\|) \text{ for all } x \in S.$$

It is easy to verify that (2.3) implies that (2.26) holds for $n = 0$. Suppose that (2.26) holds for some $n \geq 0$. From (2.2), $\varphi \in \Phi_2$ and Remark 1.1, we infer that

$$(2.27) \quad \begin{aligned} |a_n(T(x, y))| &\leq \psi(\|T(x, y)\|) \\ &\leq \psi(\varphi(\|x\|)) \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D. \end{aligned}$$

Using (2.3) and (2.27), we have

$$-\psi(\|x\|) \leq \max\{u(x, y), a_n(T(x, y))\} \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D,$$

which implies that

$$|a_{n+1}(x)| = \left| \sup_{y \in D} \max\{u(x, y), a_n(T(x, y))\} \right| \leq \psi(\|x\|) \text{ for all } x \in S.$$

Hence (2.26) holds for all $n \geq 0$. On the other hand, (2.23) ensures that

$$(2.28) \quad a_0(x) \leq a_1(x) \leq \cdots \leq a_n(x) \leq a_{n+1}(x) \leq \cdots \text{ for all } x \in S.$$

Next we show that $\{a_n\}_{n \geq 0}$ is a Cauchy sequence in $BB(S)$. Let $k \in N$, $\varepsilon > 0$ and $x_0 \in \overline{B}(0, k)$ be given. Since $\varphi \in \Phi_2$ and $\psi \in \Phi_3$, it follows that there exists some $m \in N$ such that

$$(2.29) \quad \psi(\varphi^n(k)) < \varepsilon \text{ for all } n \geq m.$$

For any $n \geq m$ and $p \in N$, by (2.23) we easily conclude that there exist $y_1 \in D$ and $x_1 = T(x_0, y_1)$ satisfying

$$(2.30) \quad a_{n+p}(x_0) < H(x_0, y_1, a_{n+p-1}) + 2^{-1}\varepsilon, \quad a_n(x_0) \geq H(x_0, y_1, a_{n-1}).$$

By virtue of (2.28), (2.30), and Lemma 1.1, we have

$$(2.31) \quad \begin{aligned} 0 &\leq a_{n+p}(x_0) - a_n(x_0) \\ &< H(x_0, y_1, a_{n+p-1}) - H(x_0, y_1, a_{n-1}) + 2^{-1}\varepsilon \\ &= \max\{u(x_0, y_1), a_{n+p-1}(x_1)\} - \max\{u(x_0, y_1), a_{n-1}(x_1)\} + 2^{-1}\varepsilon \\ &\leq a_{n+p-1}(x_1) - a_{n-1}(x_1) + 2^{-1}\varepsilon. \end{aligned}$$

Similarly, we conclude that there exist $y_i \in D$, $x_i = T(x_{i-1}, y_i) \in S$, $2 \leq i \leq n$ satisfying

$$(2.32) \quad \begin{aligned} 0 &\leq a_{n+p-1}(x_1) - a_{n-1}(x_1) < a_{n+p-2}(x_2) - a_{n-2}(x_2) + 2^{-2}\varepsilon, \\ 0 &\leq a_{n+p-2}(x_2) - a_{n-2}(x_2) < a_{n+p-3}(x_3) - a_{n-3}(x_3) + 2^{-3}\varepsilon, \\ &\vdots \\ 0 &\leq a_{p+1}(x_{n-1}) - a_1(x_{n-1}) < a_p(x_n) - a_0(x_n) + 2^{-n}\varepsilon. \end{aligned}$$

It follows from (2.2), (2.3), (2.26), (2.29), (2.31), and (2.32) that

$$\begin{aligned} 0 &\leq a_{n+p}(x_0) - a_n(x_0) < a_p(x_n) - a_0(x_n) + \varepsilon \\ &\leq |a_p(x_n)| + |a_0(x_n)| + \varepsilon \leq 2\psi(\|x_n\|) + \varepsilon \\ &= 2\psi(\|T(x_{n-1}, y_n)\|) + \varepsilon \leq 2\psi(\varphi(\|x_{n-1}\|)) + \varepsilon \\ &\leq 2\psi(\varphi^n(\|x_0\|)) + \varepsilon \leq 2\psi(\varphi^n(k)) + \varepsilon < 3\varepsilon \end{aligned}$$

for any $n \geq m$ and $p \in N$. This gives that $d_k(a_n, a_{n+p}) \leq 3\varepsilon$ for any $n \geq m$ and $p \in N$. Consequently, $\{a_n\}_{n \geq 0}$ is a Cauchy sequence in $(BB(S), d)$ and it converges to some $w \in BB(S)$. From (2.8), we deduce that $w = fw$. That is, the functional equation (2.24) possesses a solution $w \in BB(S)$.

We prove that (2.5) holds. For any $x_0 \in S$, $\{y_n\}_{n \in N} \subseteq D$, $x_n = T(x_{n-1}, y_n)$, $n \in N$, (2.2) yields that (2.17) holds. Note that $\psi(0) = 0$ and ψ is right continuous at 0. Thus (2.17) means that

$$(2.33) \quad \lim_{n \rightarrow \infty} \psi(\|x_n\|) = \psi(0) = 0.$$

Put $k = [\|x_0\|] + 1$. It is easy to verify that $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{B}(0, k)$. Let k be in N and $\varepsilon > 0$. Since $\{a_n\}_{n \in N}$ converges to w , by (2.33) we know that there exists some $m \in N$ such that

$$(2.34) \quad \max\{d_k(w, a_m), \psi(\|x_n\|)\} < \varepsilon \text{ for any } n \geq m.$$

By virtue of (2.26) and (2.34), we have

$$|w(x_n)| \leq |w(x_n) - a_m(x_n)| + |a_m(x_n)| \leq d_k(w, a_m) + \psi(\|x_n\|) < 2\varepsilon$$

for all $n \geq m$. That is, $\lim_{n \rightarrow \infty} w(x_n) = 0$.

Given $x_0 \in S$ and $\{y_n\}_{n \in N} \subset D$, take $x_n = T(x_{n-1}, y_n)$ for all $n \in N$. Since w is a solution of the functional equation (2.24), by (2.5) we immediately infer that

$$w(x_0) \geq \max\{u(x_0, y_1), w(T(x_0, y_1))\} \geq w(x_1) \geq \cdots \geq w(x_n) \rightarrow 0$$

as $n \rightarrow \infty$. That is, $w(x_0) \geq 0$ for all $x_0 \in S$.

Finally we prove that w is a unique solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Suppose that v is also a solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of w and v , we conclude that there exist $\{y_n\}_{n \geq 1} \subseteq D$, $\{z_n\}_{n \geq 1} \subseteq D$, $\{x_n\}_{n \geq 1} \subseteq S$ and $\{t_n\}_{n \geq 1} \subseteq S$ with $x_n = T(x_{n-1}, y_n)$, $t_n = T(t_{n-1}, z_n)$ for all $n \in N$, such that

$$(2.35) \quad \begin{aligned} w(x_i) &< \max\{u(x_i, y_{i+1}), w(x_{i+1})\} + 2^{-i-1}\varepsilon, \\ v(t_i) &< \max\{u(t_i, z_{i+1}), v(t_{i+1})\} + 2^{-i-1}\varepsilon, \\ w(t_i) &\geq \max\{u(t_i, z_{i+1}), w(t_{i+1})\}, \quad v(x_i) \geq \max\{u(x_i, y_{i+1}), v(x_{i+1})\} \end{aligned}$$

for all $i \geq 0$. By (2.35) we easily deduce that

$$(2.36) \quad \begin{aligned} w(x_0) &< \max\{u(x_0, y_1), u(x_1, y_2), \dots, u(x_{n-1}, y_n), w(x_n)\} + (1 - 2^{-n})\varepsilon, \\ v(t_0) &< \max\{u(t_0, z_1), u(t_1, z_2), \dots, u(t_{n-1}, z_n), v(t_n)\} + (1 - 2^{-n})\varepsilon, \\ w(t_0) &\geq \max\{u(t_0, z_1), u(t_1, z_2), \dots, u(t_{n-1}, z_n), w(t_n)\}, \\ v(x_0) &\geq \max\{u(x_0, y_1), u(x_1, y_2), \dots, u(x_{n-1}, y_n), v(x_n)\} \end{aligned}$$

for any $n \in N$. Using (2.36), Lemma 1.1 and $x_0 = t_0$, we have

$$|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n})\varepsilon.$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain that $|w(x_0) - v(x_0)| \leq \varepsilon$, which implies that $w(x_0) = v(x_0)$ by letting $\varepsilon \rightarrow 0$. This completes the proof. \square

Following a similar argument as in the proof of Theorem 2.3, we have the following.

THEOREM 2.4. Let $u : S \times D \rightarrow R$, $T : S \times D \rightarrow S$ be mappings and

$$(2.37) \quad \begin{aligned} a_0(x) &= \inf_{y \in D} u(x, y), a_n(x) \\ &= \inf_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in N. \end{aligned}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) and (2.3). Then the functional equation

$$(2.38) \quad f(x) = \inf_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S,$$

possesses a solution $w \in BB(S)$ such that (2.5) and (2.25) hold. Moreover, the solution w of the functional equation (2.38) is also unique with respect to (2.5).

REMARK 2.2. Theorem 2.4 extends, improves and unifies Theorem 3.5 of Bhakta and Choudhury [6], Theorem 3.5 of Liu [12] and a result of Bullman [2, p.135]. The example below shows that Theorem 2.4 is indeed a generalization of the results due to Bhakta and Choudhury [6], Liu [12], and Bullman [2].

EXAMPLE 2.2. Let X, Y, S, D be as in Example 2.1. Define $u : S \times D \rightarrow R$, $T : S \times D \rightarrow S$ by

$$u(x, y) = \frac{x^4(1 + xy)}{1 + x^2 + y^2}, \quad T(x, y) = \frac{x|\sin(x + y)|}{1 + x} \quad \text{for all } (x, y) \in S \times D.$$

Put $\varphi(t) = \frac{t}{1+t}$, $\psi(t) = t^4$ for all $t \in R^+$. Then the assumptions of Theorem 2.4 are fulfilled. However, we cannot invoke the results of Bhakta and Choudhury [6], Liu [12], and Bullman [2] to establish that the functional equation (2.38) possesses a solution in $BB(S)$ because

$$|u(x, y)| = \left| u\left(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\right) \right| \geq \frac{2}{3} \left[\frac{3}{2}(M + 1) \right]^2 > M|x|$$

for any $M > 0$ with $(x, y) = \left(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\right) \in S \times D$.

Let $BC(R^+)$ denote the set of all bounded continuous real-valued functions on R^+ . Put $d(a, b) = \sup\{|a(x) - b(x)| : x \in R^+\}$ for all $a, b \in BC(R^+)$. It is easily seen that $(BC(R^+), d)$ is a complete metric space.

THEOREM 2.5. *Let $X = Y = R$, $S = D = R^+$. Let $u, v : S \times D \rightarrow R^+$ be continuous, $u(x, x)$ be bounded on S , $\lim_{y \rightarrow +\infty} u(x, y) = +\infty$, $u(x, \cdot)$ and $v(x, \cdot)$ be nondecreasing with respect to the second argument on $[x, +\infty)$ for every $x \in S$, and*

$$(2.39) \quad 0 \leq v(x, y) \leq r \text{ for all } (x, y) \in S \times D,$$

where r is a positive constant. Let $p, q : S \rightarrow R^+$ satisfy that p is continuous, nondecreasing, $\int_0^{+\infty} p(s)q(s)ds < +\infty$ and

$$(2.40) \quad \int_0^{+\infty} q(s)ds = t > 0.$$

Assume that

$$(2.41) \quad a_0(x) = \inf_{y \geq x} u(x, y), \quad x \in S,$$

$$a_{n+1}(x) = \inf_{y \geq x} \left\{ u(x, y) + v(x, y) \left[\int_y^{+\infty} p(s-y)q(s)ds \right. \right. \\ \left. \left. + a_n(0) \int_y^{+\infty} q(s)ds + \int_0^y a_n(y-s)q(s)ds \right] \right\}, \quad x \in S, n \geq 0.$$

If $rt < 1$, then the functional equation (1.4) possesses a unique solution $w \in BC(R^+)$ and

$$(2.42) \quad d(a_{n+1}, w) \leq (rt)^{n+1}(1-rt)^{-1}d(a_0, a_1) \text{ for all } n \geq 0.$$

Proof. For all $(x, y, b) \in S \times D \times BC(R^+)$, set

$$H(x, y, b) = u(x, y) + v(x, y) \left[\int_y^{+\infty} p(s-y)q(s)ds \right. \\ \left. + b(0) \int_y^{+\infty} q(s)ds + \int_0^y b(y-s)q(s)ds \right].$$

Let

$$(2.43) \quad fb(x) = \inf_{y \geq x} H(x, y, b) \text{ for all } (x, b) \in S \times BC(R^+).$$

It is easy to see that f maps $BC(R^+)$ into itself. Let $\varepsilon > 0$, $x \in S$ and $b, c \in BC(R^+)$ be given. It follows from (2.43) that there exist $y_1 \geq x$ and $y_2 \geq x$ such that

$$(2.44) \quad \begin{aligned} fb(x) &> H(x, y_1, b) - \varepsilon, \quad fc(x) > H(x, y_2, c) - \varepsilon, \\ fb(x) &\leq H(x, y_2, b), \quad fc(x) \leq H(x, y_1, c). \end{aligned}$$

By virtue of (2.39), (2.40), and (2.44), we deduced that

$$\begin{aligned} &|fb(x) - fc(x)| \\ &< \max \left\{ |H(x, y_i, b) - H(x, y_i, c)| : i = 1, 2 \right\} + \varepsilon \\ &\leq \max \left\{ v(x, y_i) \left[|b(0) - c(0)| \int_{y_i}^{+\infty} q(s) ds \right. \right. \\ &\quad \left. \left. + \int_0^{y_i} |b(y_i - s) - c(y_i - s)| q(s) ds \right] : i = 1, 2 \right\} + \varepsilon \\ &\leq \max \left\{ rd(b, c) \left[\int_{y_i}^{+\infty} q(s) ds + \int_0^{y_i} q(s) ds \right] : i = 1, 2 \right\} + \varepsilon \\ &= rtd(b, c) + \varepsilon, \end{aligned}$$

which implies that

$$d(fb, fc) = \sup\{|fb(x) - fc(x)| : x \in S\} \leq rtd(b, c) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we easily conclude that

$$d(fb, fc) \leq rtd(b, c) \text{ for all } b, c \in BC(R^+).$$

It follows from Banach fixed-point theorem that f has a unique fixed point $w \in BC(R^+)$ and (2.42) holds. Obviously, w is a unique solution of the functional equation (1.4). This completes the proof. \square

REMARK 2.3. Theorem 2.5 generalizes Theorem 3.6 of Bhakta and Choudhury [6] and a result of Bellman [2, p.129].

PROBLEM 2.1. If $rt < 1$ is replaced by $rt = 1$ in Theorem 2.5, does the functional equation (1.4) possess a solution in $BC(R^+)$?

PROBLEM 2.2. If the answer to Problem 2.1 is no, then what additional hypotheses on u, v, p, q are needed to guarantee the existence of a solution of the functional equation (1.4)?

References

- [1] R. Baskaran and P. V. Subrahmanyam, *A note on the solution of a class of functional equations*, Appl. Anal. **22** (1986), no. 3-4, 235–241.
- [2] R. Bellman, *Dynamic programming*, Princeton University Press, Princeton, N. J., 1957.
- [3] ———, *Methods of Non-linear analysis, Vol. II*, Mathematics in Science and Engineering, Vol. 61-II, Academic Press, New York-London, 1973.
- [4] R. Bellman and E. S. Lee, *Functional equations in dynamic programming*, Aequationes Math. **17** (1978), no. 1, 1–18.
- [5] R. Bellman and M. Roosta, *A technique for the reduction of dimensionality in dynamic programming*, J. Math. Anal. Appl. **88** (1982), no. 2, 543–546.
- [6] P. C. Bhakta and S. R. Choudhury, *Some existence theorems for functional equations arising in dynamic programming II*, J. Math. Anal. Appl. **131** (1988), no. 1, 217–231.
- [7] P. C. Bhakta and S. Mitra, *Some existence theorems for functional equations arising in dynamic programming*, J. Math. Anal. Appl. **98** (1984), no. 2, 348–362.
- [8] S. S. Chang, *Some existence theorems of common and coincidence solutions for a class of functional equations arising in dynamic programming*, Appl. Math. Mech. (English Ed.) **12** (1991), no. 1, 33–39.
- [9] S. S. Chang and Y. H. Ma, *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **160** (1991), no. 2, 468–479.
- [10] Z. Liu, *Coincidence theorems for expansive mappings with applications to the solutions of functional equations arising in dynamic programming*, Acta Sci. Math. (Szeged) **65** (1999), no. 1-2, 359–369.
- [11] ———, *Compatible mappings and fixed points*, Acta Sci. Math. (Szeged) **65** (1999), no. 1-2, 371–383.
- [12] ———, *Existence theorems of solutions for certain classes of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **262** (2001), no. 2, 529–553.

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