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A RELATIVE REIDEMEISTER ORBIT NUMBER

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ABSTRACT. The Reidemeister orbit set plays a crucial role in the Nielsen type theory of periodic orbits, much as the Reidemeister set does in Nielsen fixed point theory. In this paper, extending Cardona and Wong's work on relative Reidemeister numbers, we show that the Reidemeister orbit numbers can be used to calculate the relative essential orbit numbers. We also apply the relative Reidemeister orbit number to study periodic orbits of fibre preserving maps.

1. Introduction

Nielsen fixed point theory has been extended to a Nielsen type theory of periodic orbits [9, III.3]. In fixed point theory, the computation of the Nielsen number often relies on our knowledge of the Reidemeister set, that is the set of Reidemeister conjugacy classes in the fundamental group. Our aim in this paper is to introduce relative Reidemeister orbit numbers and show that the relative Reidemeister orbit numbers can be used to calculate the relative essential orbit numbers, and as application, we use the relative Reidemeister orbit numbers to study periodic orbits of fibre preserving maps.

In Nielsen fixed point theory, the relative theory was introduced by H. Schirmer [11]. The relative Nielsen number has many connections with Nielsen type theory. The relative Nielsen number of a map $f : (X,A) \to (X,A)$ is defined by $N(f;X,A) = N(f) - N(f,f_A) + N(f_A)$, where $N(f,f_A)$ denotes the number of essential common fixed point classes of f and f_A. The relative Reidemeister number in [2] is defined by $R(f;X,A) = R(f) - R(f,f_A) + R(f_A)$, where $R(f,f_A)$ is the number of Reidemeister classes of homomorphisms induced by f, which satisfies certain conditions. A Jiang-type theorem for a map of pairs was also proved.

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Let X be a compact connected polyhedron, and $A \subset X$ a finite subpolyhedron. Let $f : (X, A) \to (X, A)$ be a map of pairs. We define the relative Reidemeister n-orbit number of f by $RO^{(n)}(f; X, A) = RO^{(n)}(f_A) + RO^{(n)}(f) - RO^{(n)}(f, f_A)$, and prove a Jiang-type theorem for this number as follows: Under certain conditions, we have

$$EO^{(n)}(f; X, A) = RO^{(n)}(f; X, A),$$

where $EO^{(n)}(f; X, A)$ is the relative essential n-orbit number introduced in Section 2. Note that here although in the relative Reidemeister number we can choose the base path in the component of A identical with the base path in X, in the relative Reidemeister n-orbit number the base path for the component does not inherit the base path in X.

We consider a fibre preserving map $f : E \to E$ of a Hurewicz fibration $p : E \to B$ of compact ANR's. It induces a map $\bar{f} : B \to B$. We generalize the relationship between relative Nielsen theory and Nielsen theory for fibre preserving maps as obtained in [4] and [2] as follows: Under suitable conditions, certain relative essential n-orbit number of f can be calculated by a Reidemeister n-orbit number as follows:

$$EO^{(n)}(g; E, F_{\xi_n}) = RO^{(n)}(f) = \sum_{b_i \in \xi} RO^{(m_i)}(g_{b_i}^{\ell_i}),$$

where g is the fibre preserving map from the Reducing Lemma in [10], ξ is a set of essential n-orbit representatives for $\bar{f}$, ξ_n is a reduced $\bar{g}$-invariant set with respect to ξ, ℓ_i is the length of the essential $\bar{f}$-orbit class containing $b_i \in \xi$, $m_i = n/\ell_i$, and $F_{\xi_n} := \cup_{b \in \xi_n} p^{-1}(b)$.

The paper consists of four sections. In Section 2, we define the relative Reidemeister n-orbit numbers and prove some of their properties. The relative essential n-orbit number of f on the pair is defined and a Jiang-type theorem is proved in Section 3. In the last section we apply them to fibre preserving maps.

For the basics of Nielsen fixed point theory, the reader is referred to [1] and [9].

2. Relative Reidemeister n-orbit numbers

Let X be a compact, connected polyhedron, and $A \subset X$ a finite subpolyhedron. Let $f : (X, A) \to (X, A)$ be a map of pairs. Let $A = \cup A_k$ be the disjoint union of all the connected components of A, then we define the restrictions $f_k = f|_{A_k} : A_k \to A_l$ and $f_A = f|_A : A \to A$.

We denote by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of f. Two fixed points $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if there is a path λ from x to y such that $f(\lambda)$ is homotopic to λ by a homotopy keeping the end points fixed. This relation divides $\mathrm{Fix}(f)$ into a finite number of *fixed point classes* of f. The set of fixed point classes will be denoted by $\mathcal{FP}(f)$.

Let $n > 0$ be a given integer. Then f acts on the set $\mathcal{FP}(f^n)$ of n-periodic point classes of f by $\mathbf{F}_{f^n} \mapsto f(\mathbf{F}_{f^n})$. In [10], the f-orbit of a class $\mathbf{F}_{f^n}$ is called an n-orbit class, denoted by $\mathbf{F}_f^{(n)}$. The set of n-orbit classes is denoted by $\mathcal{O}^{(n)}(f)$. The length of the orbit $\mathbf{F}_f^{(n)}$ is the smallest integer $\ell > 0$ such that $\mathbf{F}_{f^n} = f^\ell(\mathbf{F}_{f^n})$ (see [10] for details).

Let x_0 be the base point in X, and take a path w from x_0 to $f(x_0)$ as the base path for f. The induced endomorphism $f_*^w : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is defined by

$$f_*^w(\langle\gamma\rangle) := \langle w f(\gamma) w^{-1}\rangle \qquad \text{for any loop } \gamma \text{ at } x_0.$$

For $n > 1$, we have $(f^n)_*^{w_n} = (f_*^w)^n$ if the base path for f^n is taken to be $w_n := wf(w)\cdots f^{n-1}(w)$. For the sake of convenience, denote the induced endomorphism $f_*^w : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by φ, and $\pi_1(X, x_0) := \pi_X$. Note that the endomorphism f_*^w depends on the homotopy class of w.

NOTATION. Suppose G is a group. For $\alpha \in G$, let $\tau_\alpha : G \to G$ denote the conjugation defined by $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$.

REMARK 2.1. If x_0 is an n-periodic point of f, then w_n is a loop at x_0. Thus we have

$$(f_*^w)^n = \tau_{\langle w_n\rangle} \circ (f^n)_*^{x_0},$$

where x_0 is the constant path at x_0.

Given $\varphi : \pi_X \to \pi_X$, we have the Reidemeister left action of π_X on π_X, given by

$$\beta \cdot \alpha = \beta\alpha\varphi(\beta^{-1}).$$

The Reidemeister classes are the orbits of this action, and the set of Reidemeister classes is denoted by $\mathcal{R}(\varphi)$. The Reidemeister number of f is given by $R(f) = \sharp\mathcal{R}(\varphi)$, where $\sharp$ denotes the cardinality.

Let $n > 0$ be a given integer. Then φ acts on the Reidemeister set $\mathcal{R}(\varphi^n)$ by $[\alpha]_{\varphi^n} \overset{\varphi}{\mapsto} [\varphi(\alpha)]_{\varphi^n}$. In [10], the φ-orbit of a Reidemeister class $[\alpha]_{\varphi^n}$ is called the Reidemeister n-orbit of φ, and denoted by $[\alpha]_\varphi^{(n)}$. The Reidemeister n-orbit set of φ is the set of all such φ-orbits, denoted by $\mathcal{RO}^{(n)}(\varphi)$. The Reidemeister n-orbit number $RO^{(n)}(f)$ is defined to be the cardinality of the set $\mathcal{RO}^{(n)}(\varphi)$. The length of the orbit $[\alpha]_f^{(n)}$ is the smallest integer $\ell > 0$ such that $[\alpha]_{f^n} = [f^\ell(\alpha)]_{f^n}$.

It is well known that every fixed point class of f is assigned a Reidemeister class in $\mathcal{R}(\varphi)$, called its *coordinate.* We get an injection

$$\rho : \mathcal{FP}(f) \hookrightarrow \mathcal{R}(\varphi),$$

defined by

$$\rho(\mathbf{F}_f) := [\langle cf(c^{-1})w^{-1}\rangle]_\varphi$$

for any path c from x_0 to a point x in $\mathbf{F}_f$. Thus we also get an injection

$$\rho : \mathcal{O}^{(n)}(f) \hookrightarrow \mathcal{RO}^{(n)}(\varphi),$$

defined by

$$\rho(\mathbf{F}_f^{(n)}) := [\langle cf^n(c^{-1})f^{n-1}(w^{-1})\cdots f(w^{-1})w^{-1}\rangle]_\varphi^{(n)}$$

for any path c from x_0 to a point x in $\mathbf{F}_f^{(n)}$.

The following example shows that the coordinate of a fixed point class depends on the choice of the base point of the space (and base path for f).

EXAMPLE 2.2. Define $f : S^1 \to S^1$ by $f(e^{i\theta}) = e^{3i\theta}$. Then $\mathrm{Fix}(f) = \{e^{2\pi ik/(3-1)} : k = 1, 2\} := \{-1, 1\}$. Since the degree of f is 3, $N(f) = 2$ (see [9] and [6]). The fundamental group $\pi_1(S^1, 1) \cong \mathbb{Z}$ is generated by the path η obtained by travelling S^1 once, starting at 1, in the counter-clockwise direction. If we take the constant path at 1 as base path for f and let $\varphi = f_*^1$, then $\mathcal{R}(\varphi) := \{[0]_\varphi, [1]_\varphi\}$ and $[0]_\varphi$ is the coordinate of $\{1\} \in \mathcal{FP}(f)$. Now select the path c to be the arc of S^1 from 1 to -1 passing through the north pole. Then $[\langle cf(c^{-1})\rangle]_\varphi = [\langle \eta^{-1}\rangle]_\varphi = [1]_\varphi$ is the coordinate of $\{-1\} \in \mathcal{FP}(f)$. On the other hand, $\pi_1(S^1, -1)$ is generated by the path ζ obtained by travelling S^1 once, starting at -1, in the counter-clockwise direction. If we take the constant path at -1 as base path for f and let $\varphi' = f_*^{-1}$, then $[0]_{\varphi'}$ is the coordinate of $\{-1\} \in \mathcal{FP}(f)$. Now select the path λ to be the arc of S^1 from -1 to 1 passing through the south pole. Then $[\langle \lambda f(\lambda^{-1})\rangle]_{\varphi'} = [\langle \zeta^{-1}\rangle]_{\varphi'} = [1]_{\varphi'}$ is the coordinate of $\{1\} \in \mathcal{FP}(f)$.

Example 2.1 justifies the following remark.

REMARK 2.3. The φ-coordinate of a fixed point class $\mathbf{F}_f \in \mathcal{FP}(f)$ means its coordinate in $\mathcal{R}(\varphi)$.

For $m \mid n$, we have a commutative diagram of pointed sets

$$\begin{array}{ccc} \mathcal{R}(\varphi^m) & \xrightarrow{\iota_{m,n}} & \mathcal{R}(\varphi^n) \\ \downarrow & & \downarrow \\ \mathcal{RO}^{(m)}(\varphi) & \xrightarrow{\iota_{m,n}} & \mathcal{RO}^{(n)}(\varphi), \end{array}$$

where the vertical maps are projections, and the horizontal maps are induced by the level-change function $\iota_{m,n} : \pi_X \to \pi_X$ defined by

$$\iota_{m,n}(\beta) := \beta\varphi^m(\beta)\varphi^{2m}(\beta)\cdots\varphi^{n-m}(\beta).$$

Recall that an φ-orbit $[\alpha]_\varphi^{(n)} \in \mathcal{RO}^{(n)}(\varphi)$ is reducible to level h, if there exists a $[\beta]_\varphi^{(h)} \in \mathcal{RO}^{(h)}(\varphi)$ such that $\iota_{h,n}([\beta]_\varphi^{(h)}) = [\alpha]_\varphi^{(n)}$. The lowest level $d = d([\alpha]_\varphi^{(n)})$ to which $[\alpha]_\varphi^{(n)}$ reduces is its depth. The depth of an n-orbit class is defined by the depth of its coordinate. A Reidemeister orbit $[\alpha]_\varphi^{(n)} \in \mathcal{RO}^{(n)}(\varphi)$ is said to have the full depth property if its depth equals its length, i.e., $d = \ell$ (see [10]).

PROPOSITION 2.4. *For any $m \mid n$, the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{FP}(f^n) & \xrightarrow{\rho} & \mathcal{R}(\varphi^n) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{FP}(f^n) & \xrightarrow{\rho} & \mathcal{R}(\varphi^n) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{O}^{(m)}(f) & \xrightarrow{\rho} & \mathcal{RO}^{(m)}(\varphi) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \iota_{m,n}} \\ \mathcal{O}^{(n)}(f) & \xrightarrow{\rho} & \mathcal{RO}^{(n)}(\varphi), \end{array}$$

where γ is the function induced by inclusion.

PROOF. See [6, Proposition 1.14]. □

PROPOSITION 2.5. *For the base path w_n at x_0 as above, for any path μ from x_0 to $f(x_0)$, let $\mu_n = \mu f(\mu)\cdots f^{n-1}(\mu)$. Then there is an index preserving bijection*

$$r_{\mu_n,w_n} : \mathcal{RO}^{(n)}(f_*^\mu) \to \mathcal{RO}^{(n)}(f_*^w)$$

given by $r_{\mu_n,w_n}([\langle\gamma\rangle]_{f_*^\mu}^{(n)}) = [\langle\gamma\mu_n w_n^{-1}\rangle]_{f_*^w}^{(n)}$. Furthermore, we have $r_{\mu_n,w_n} \circ \rho = \rho$.

PROOF. See [6] and [14]. □

PROPOSITION 2.6. *For the base path w_n at x_0 as above, for any $x \in X$, let u be a path from x to x_0, and λ a path from x to $f(x)$. Then there is an index preserving bijection*

$$u_* : \mathcal{RO}^{(n)}(f_*^\lambda) \to \mathcal{RO}^{(n)}(f_*^w)$$

defined by $u_*([\langle\gamma\rangle]^{(n)}_{f^\lambda_*}) = [\langle u^{-1}\gamma\lambda_n f^n(u)w_n^{-1}\rangle]^{(n)}_{f^w_*}$, *where* $\lambda_n = \lambda f(\lambda)\cdots f^{n-1}(\lambda)$. *Furthermore, we have* $u_*\circ\rho = \rho$.

PROOF. See [6] and [14]. □

Note that the concept of depth of an f-orbit class does not depend on the choice of the base point and the base path for f (see [6, Theorem A]).

Let A_j, with $j = 1, 2, \ldots, r$, be those components of A which belong to an f_A-cycle, $J_r = \{1, \ldots, r\}$, and $C(f_A)$ the set of equivalence classes of J_r. Now let $[A_j]$ be an f_A-cycle of length $c(j)$ for which $m_j := n/c(j)$ is an integer. For each A_k in this cycle there is a commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{f_k^{c(j)}} & A_k \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i_k} \\ X & \xrightarrow{f^{c(j)}} & X \end{array}$$

of path-connected spaces (for full details see [7, 3.3] and [8, Section 2]).

For a given integer $n > 1$, the following example shows that the base path for $f_k^{c(j)}$ does not inherit the base path for f.

EXAMPLE 2.7. Take the function $f : S^1 \to S^1$ of Example 2.2. In the particular case $n = 2$, then we have $\mathrm{Fix}(f^2) = \{e^{2\pi ik/(3^2-1)} : k = 1, \ldots, 8\}$. Let $A = \{1, e^{\pi/4}, e^{3\pi/4}\}$, $A_1 = \{1\}$. Then there are two f_A-cycles such that $\ell([A_1 = \{1\}]) = 1$ and $\ell([A_2 = \{e^{\pi/4}\}]) = 2$. Take the point $e^{\pi/4}$ as the base point then the constant path at base point is the base path for f_2^2. But the base path for f is a path w in S^1 from $e^{\pi/4}$ to $f(e^{\pi/4}) = e^{3\pi/4}$. Note that $w_2 = wf(w)$ is not the constant path at base point, and by using the constant path at base point as the path c in the definition of ρ, then $[\langle w_2^{-1}\rangle]^{(2)}_{f^w_*}$ is the f^w_*-coordinate of $\{e^{\pi/4}\} \in \mathcal{O}^{(2)}(f)$. And similarly $[1]_{(f_2^2)_*^{e^{\pi/4}}}$ is the $(f_2^2)_*^{e^{\pi/4}}$-coordinate of $\{e^{\pi/4}\} \in \mathcal{FP}(f_2^2)$.

For the inclusion map $i_k : (A_k, x) \to (X, x)$, let w be the base path for f from x to $f(x)$ in X and w_n the base path for f^n. Let $\varphi = f^w_*$. Since A_k is $f^{c(j)}$-invariant, we can take the base path μ for $f_k^{c(j)}$ from x to $f_k^{c(j)}(x) = f^{c(j)}(x)$ in A_k. Then $\mu_{m_j} = \mu f^{c(j)}(\mu)\cdots f^{n-c(j)}(\mu)$ is a path from x to $f^n(x)$. Let ψ be the homomorphism $(f^{c(j)})^\mu_*$ on $\pi_1(X, x)$ and ψ_k the homomorphism $(f_k^{c(j)})^\mu_*$ on $\pi_1(A_k, x)$. Then we have the following:

PROPOSITION 2.8. *Under the above conditions, there is a commutative diagram:*

$$\begin{array}{ccccc}
\mathcal{O}^{(m_j)}(f_k^{c(j)}) & \xrightarrow{i_{k,\mathcal{F}P}} & \mathcal{O}^{(m_j)}(f^{c(j)}) & \xrightarrow{\sigma} & \mathcal{O}^{(n)}(f) \\
\downarrow \rho & & \downarrow \rho & & \downarrow \rho \\
\mathcal{RO}^{(m_j)}(\psi_k) & \xrightarrow{i_{k,c(j)}} & \mathcal{RO}^{(m_j)}(\varphi^{c(j)}) & \xrightarrow{\sigma} & \mathcal{RO}^{(n)}(\varphi),
\end{array}$$

where $i_{k,\mathcal{F}P}$ is a function induced by the inclusion, $i_{k,c(j)}$ is a function induced by the inclusion i_k and $r_{\mu_{m_j},w_n}$, and the map σ is induced by the inclusion.

PROOF. Let $(i_k)_\sharp : \pi_1(A_k, x) \to \pi_1(X, x)$ be the homomorphism induced by i_k. Since a fundamental group can be identified with the group of covering translations, we have a well-defined map $i_{k,*} : \mathcal{RO}^{(m_j)}(\psi_k) \to \mathcal{RO}^{(m_j)}(\psi)$ given by $i_{k,*}([\alpha]_{\psi_k}^{(m_j)}) = [(i_k)_\sharp(\alpha)]_{\psi}^{(m_j)}$ (see [9] and [2]). Since $w_n = (w_{c(j)})_{m_j}$, by Proposition 2.5 there is an index preserving bijection $r_{\mu_{m_j},w_n} : \mathcal{RO}^{(m_j)}(\psi) \to \mathcal{RO}^{(m_j)}(\varphi^{c(j)})$. Let $i_{k,c(j)}$ be the composition of $i_{k,*}$ and $r_{\mu_{m_j},w_n}$. The map

$$i_{k,\mathcal{F}P} : \mathcal{F}P((f_k^{c(j)})^{m_j}) \to \mathcal{F}P((f^{c(j)})^{m_j})$$

induced by the inclusion induces a map $i_{k,\mathcal{F}P} : \mathcal{O}^{(m_j)}(f_k^{c(j)}) \to \mathcal{O}^{(m_j)}(f^{c(j)})$. The map σ induced by the inclusion is defined by $\sigma([\beta]_{\varphi^{c(j)}}^{m_j}) = [\beta]_{\varphi}^{(n)}$ for any $[\beta]_{\varphi^{c(j)}}^{m_j} \in \mathcal{RO}^{(m_j)}(\varphi^{c(j)})$. Note that σ is surjective (see [10, 1.7]). It is easy to see the diagram commutes. □

DEFINITION 2.9. The induced homomorphism $\psi = (f^{c(j)})_*^{\mu}$ in Proposition 2.8 will be called an *associated homomorphism with* $\varphi^{c(j)}$. We define

$$\mathcal{RO}^{(n)}(\varphi_A) = \bigsqcup_{[j]} \mathcal{RO}^{(m_j)}(\psi_k) \quad \text{and} \quad \hat{i}_A := \cup\, \sigma \circ i_{k,c(j)},$$

where the disjoint union is taken over all equivalence classes $[j] \in C(f_A)$, and $m_j := n/c(j)$ is an integer. Similarly, we define

$$\mathcal{O}^{(n)}(f_A) := \bigsqcup_{[j]} \mathcal{O}^{(m_j)}(f_k^{c(j)}).$$

If $\mathbf{F}_{f^n} \in \mathcal{F}P(f^n)$ is a weakly common fixed point class of f^n and f_A^n in the sense of [15, 2.2], then the f-image of $\mathbf{F}_{f^n}$ is weakly common. Then we define the set of *weakly common n-orbit classes* of f and

f_A in $\mathcal{O}^{(n)}(f)$, and denoted it by $\mathcal{O}^{(n)}(f, f_A)$. As in [2], we have the corresponding set from an algebraic point of view as

DEFINITION 2.10. A Reidemeister n-orbit $[\alpha]_\varphi^{(n)} \in \mathcal{RO}^{(n)}(\varphi)$ is an *(algebraically) weakly common n-orbit* of φ and φ_A if there exists a Reidemeister m-orbit $[\beta]_{\psi_k}^{(m)}$ in $\mathcal{RO}^{(n)}(\varphi_A)$ such that $\hat{i}_A([\beta]_{\psi_k}^{(m)}) = [\alpha]_\varphi^{(n)}$, where $m := n/c(j)$ and ψ is an associated homomorphism with $\varphi^{c(j)}$. The set of all such weakly common n-orbits of φ and φ_A will be denoted by $\mathcal{RO}^{(n)}(\varphi, \varphi_A)$.

We write $RO^{(n)}(f, f_A)$ for the cardinality of the set $\mathcal{RO}^{(n)}(\varphi, \varphi_A)$.

DEFINITION 2.11. The relative Reidemeister n-orbit number of f on the pair (X, A) is defined as

$$RO^{(n)}(f; X, A) = RO^{(n)}(f_A) + RO^{(n)}(f) - RO^{(n)}(f, f_A),$$

where $RO^{(n)}(f_A)$ is the cardinality of the set $\mathcal{RO}^{(n)}(\varphi_A)$.

Note that if $n = 1$ then we can regard the fixed point of f as the base point in the corresponding component. Thus the associated homomorphism ψ is identical with the original one φ. So $RO^{(1)}(f; X, A) = R(f; X, A)$ is the relative Reidemeister number of f introduced in [2].

3. Nielsen type relative essential n-orbit numbers

Recently we defined the essential n-orbit number $EO^{(n)}(f)$ to be the cardinality of the set $\mathcal{EO}^{(n)}(f)$ of essential n-orbit classes in [10, Definition 3.1]. Since f-images of essential common fixed point classes of f^n and f_A^n are essential common, we can define

$$EO^{(n)}(f, f_A)$$

as the cardinality of the set $\mathcal{EO}^{(n)}(f, f_A)$ of essential common n-orbits of f and f_A.

DEFINITION 3.1. The *Nielsen type relative essential n-orbit number* of f on the pair (X, A) is defined as

$$EO^{(n)}(f; X, A) = EO^{(n)}(f_A) + EO^{(n)}(f) - EO^{(n)}(f, f_A),$$

where $EO^{(n)}(f_A) = \sum_{[j]} EO^{(m_j)}(f_k^{c(j)})$, the summation runs over all equivalence classes $[j] \in C(f_A)$, and $m_j := n/c(j)$ is an integer.

Note that $EO^{(n)}(f; X, \emptyset) = EO^{(n)}(f; X, X) = EO^{(n)}(f)$ is a Nielsen type number in the general sense of [9, III.4.8].

If $n = 1$, then $EO^{(1)}(f; X, A) = N(f; X, A)$ is the relative Nielsen number introduced in [11, 2.4].

As in [11], the relative essential n-orbit number $EO^{(n)}(f; X, A)$ satisfies the basic properties such as homotopy invariance, commutativity and homotopy type invariance. The proofs of Proposition 3.2, 3.3 and 3.4 are standard.

PROPOSITION 3.2. (Homotopy invariance) *If the maps $f, g : (X, A) \to (X, A)$ are homotopic, then $EO^{(n)}(f; X, A) = EO^{(n)}(g; X, A)$.*

PROPOSITION 3.3. (Commutativity) *If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$, then $EO^{(n)}(g \circ f; X, A) = EO^{(n)}(f \circ g; Y, B)$.*

PROPOSITION 3.4. (Homotopy type invariance) *If $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are maps of the same pairwise homotopy type, then $EO^{(n)}(f; X, A) = EO^{(n)}(g; Y, B)$.*

Recall in [13] that a space X is said to be of Jiang-type if the following conditions are satisfied for all selfmaps $f : X \to X$.

$$(C1) \quad L(f) = 0 \Rightarrow N(f) = 0;$$
$$(C2) \quad L(f) \neq 0 \Rightarrow N(f) = R(f).$$

A Jiang-type result was proven in [2] for selfmaps $f : (X, A) \to (X, A)$. Extending that result, we have

THEOREM 3.5. *Suppose $n > 0$ is given. Suppose that (X, A) is a pair of Jiang-type spaces such that $L(f^n) \cdot (\prod_{[j]} L((f_k^{c(j)})^{m_j})) \neq 0$ for which $m_j := n/c(j)$ is an integer, then we have $EO^{(n)}(f; X, A) = RO^{(n)}(f; X, A)$.*

PROOF. By hypothesis, $L(f^n) \neq 0$ and $L((f_k^{c(j)})^{m_j}) \neq 0$ for every $[j]$. Since X is a Jiang-type space and $L(f^n) \neq 0$, it follows that $N(f^n) = R(f^n)$. By Proposition 2.3, the length of an essential n-orbit class is the same as the length of its coordinate. Thus we have $EO^{(n)}(f) = RO^{(n)}(f)$. Also, since A is a Jiang-type space and $L((f_k^{c(j)})^{m_j}) \neq 0$ for every $[j]$, we have $EO^{(m_j)}(f_k^{c(j)}) = RO^{(m_j)}(f_k^{c(j)})$ for every $[j]$.

The equality

$$N(f^n, f_A^n) = R(f^n, f_A^n)$$

was proved in the proof of [2, 3.2](with f and f_A replaced, whenever those occur, by f^n and f_A^n respectively), thus we have $EO^{(n)}(f, f_A) = RO^{(n)}(f, f_A)$. □

4. Fiber-preserving maps

First of all, we introduce our Reducing Lemma in [10] as the main tool of this section.

REDUCING LEMMA [10]. *Suppose X is a compact connected ANR, and $f : X \to X$ is a map. Suppose $x \in \mathrm{Fix}(f^n)$ lies in an n-orbit class $\mathbf{F}_f^{(n)}$ of depth d. Then there exists a homotopy $H = \{h_t : X \to X\}_{0 \le t \le 1}$ connecting $f = h_0$ and $g = h_1$, such that*
(1) $x \in \mathrm{Fix}(g^d)$.
(2) *The loop $H^n(x) = \{h_t^n(x)\}_{0 \le t \le 1}$ is contractible in X.*
(3) *H equals f outside of an arbitrarily given neighborhood of the point $f^{d-1}(x)$.*

Note that in the Reducing Lemma, d is the length of the g-orbit of x, and the g-orbit of x is a subset of f-orbit of x. In other words $f^i(x)$ is equals to $g^i(x)$ for all $i = 0, 1, 2, \ldots, d-1$, i.e.,

$$\{x, g(x), \ldots, g^{d-1}(x)\} \subset \{x, f(x), f^2(x), \ldots\}.$$

In this paper we will assume that all of our fibrations $F \hookrightarrow E \to B$ (with projection $p : E \to B$) are Hurewicz fibrations with typical fibre, E and B path-connected (see [12]). We say that $f : E \to E$ is a fibre preserving map provided there is a well-defined map $\bar{f} : B \to B$ with $pf = \bar{f}p$. When such a map exists it is unique, and when B is a path connected locally path connected space it is enough that for all $b \in B$ the restriction of f takes the fibre $F_b := p^{-1}(b)$ to another fibre. For any $b \in \mathrm{Fix}(\bar{f}^n)$, we will denote the restricted map on F_b by f_b^n.

For $x \in E$ let $j : F_{p(x)} \to E$ be the inclusion and K denote the kernel of the homomorphism

$$j_* : \pi_1(F_{p(x)}, x) \to \pi_1(E, x).$$

An addition formula of Reidemeister orbit sets for an arbitrary group endomorphism was proved in [10]. Applying that result to fibre preserving maps, we have

PROPOSITION 4.1. *Suppose $p : E \to B$ is a fibration of compact connected ANR's with path-connected fibres. Let $f : E \to E$ be a fibre preserving map. Let $b \in \mathrm{Fix}(\bar{f}^n)$ and let $\mathbf{F}_{\bar{f}}^{(n)}$ be the n-orbit class containing b with depth d, and $m := n/d$. For an m-orbit $[\bar{\alpha}]_{\bar{\psi}}^{(m)} \in \mathcal{RO}^{(m)}(\bar{\psi})$, let d_α be the depth of $[\bar{\alpha}]_{\bar{\psi}}^{(m)}$, $m_\alpha := m/d_\alpha$, and $\bar{\iota}_{d_\alpha, m}(\bar{\beta}_\alpha) = \bar{\alpha}$*

for some $\beta_\alpha \in \pi_1(E,x)$. *If* $\mathrm{Fix}(\tau_{\bar\alpha}\bar\psi^m)=\{1\}$ *for all* $[\bar\alpha]^{(m)}_{\bar\psi} \in \mathcal{RO}^{(m)}(\bar\psi)$, *then*

$$RO^{(m)}(f^d) = \sharp\mathcal{RO}^{(m)}(\psi) = \sum_{[\bar\alpha]^{(m)}_{\bar\psi}\in\mathcal{RO}^{(m)}(\bar\psi)} \sharp\mathcal{RO}^{(m_\alpha)}(\tau_{\beta_\alpha}\psi_b),$$

where $p(x)=b$, *for a path* μ *from* x *to* $g^d(x)$ *in* F_b, *the notation* ψ *stands for the induced endomorphism* $(g^d)^\mu_*$, $\bar\psi$ *stands for* $(\bar g^d)^b_*$, *and* ψ_b *stands for* $(g^d_b)^\mu_{*/K}$. *Here* g *is the fibre preserving map from the Reducing Lemma.*

PROOF. By homotopy invariance we have $RO^{(m)}(f^d) = RO^{(m)}(g^d)$. So without loss of generality (by rewriting g^d as f^d) we may assume that $b\in\mathrm{Fix}(\bar f^d)$ and g^d is the same as f^d.

Consider the exact sequence

$$1 \to \pi_1(F_b,x)/K \xrightarrow{j_*} \pi_1(E,x) \xrightarrow{p_*} \pi_1(B,b) \to 1.$$

Since $b\in\mathrm{Fix}(\bar f^d)$, take a path μ from x to $f^d(x)$ in F_b, so we can assume that $\pi_1(F_b,x)/K$ is an $(f^d)^\mu_*$-invariant normal subgroup of $\pi_1(E,x)$. Then we have a commutative diagram of exact sequences of groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(F_b,x)/K & \xrightarrow{j_*} & \pi_1(E,x) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi_b} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar\psi} & & \\ 1 & \longrightarrow & \pi_1(F_b,x)/K & \xrightarrow{j_*} & \pi_1(E,x) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & 1. \end{array}$$

Hence the proof of the theorem is completed by [10, Theorem 1.14]. □

The following was also proved in [3] for $n=1$.

COROLLARY 4.2. *Suppose the* n*-orbit class* $\mathbf{F}^{(n)}_{\bar f}$ *containing* b *is irreducible. If* $\mathrm{Fix}(\tau_{\bar\alpha}\bar\psi)=\{1\}$ *for all* $[\bar\alpha]_{\bar\psi}\in\mathcal{R}(\bar\psi_b)$,

$$R(f^n) = \sharp\mathcal{R}(\psi_b) = \sum_{[\bar\alpha]_{\bar\psi}\in\mathcal{R}(\bar\psi)} \sharp\mathcal{R}(\tau_\alpha\psi_b),$$

where for a path μ *from* x *to* $f^n(x)$ *in* F_b, *the notation* ψ *stands for the induced endomorphism* $(f^n)^\mu_*$ *and similarly for* $\bar\psi$. *The notation* ψ_b *stands for* $(f^n_b)^\mu_{*/K}$.

Note that we do not need the Reducing Lemma in the proof of it.

PROOF. If the n-orbit class $\mathbf{F}^{(n)}_{\bar f}$ containing b is irreducible, then $d=n$, so we have the corollary. □

COROLLARY 4.3. *Suppose the n-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$ containing b is reducible to 1. For a n-orbit $[\bar{\alpha}]_{\bar{\psi}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{\psi})$, let d_α be the depth of $[\bar{\alpha}]_{\bar{\psi}}^{(n)}$, $n_\alpha := n/d_\alpha$, and $\bar{\iota}_{d_\alpha,n}(\bar{\beta}_\alpha) = \bar{\alpha}$ for some $\beta_\alpha \in \pi_1(E,x)$. If $\mathrm{Fix}(\tau_{\bar{\alpha}}\bar{\psi}^n) = \{1\}$ for all $[\bar{\alpha}]_{\bar{\psi}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{\psi})$, then*

$$RO^{(n)}(f) = \sharp\mathcal{RO}^{(n)}(\psi) = \sum_{[\bar{\alpha}]_{\bar{\psi}}^{(n)} \in \mathcal{RO}^{(n)}(\bar{\psi})} \sharp\mathcal{RO}^{(n_\alpha)}(\tau_{\beta_\alpha}\psi_b),$$

where $p(x) = b$, for a path μ from x to $g(x)$ in F_b, the notation ψ stands for the induced endomorphism g_*^μ, $\bar{\psi}$ stands for $\bar{g}_*^b$, and ψ_b stands for $(g_b)_{*/K}^\mu$. Here g is the fibre preserving map from the Reducing Lemma.

In [10], we call a subset $\xi \subset \mathrm{Fix}(\bar{f}^n)$ a set of essential n-orbit representatives for $\bar{f}$ if $\xi = \{b_1, b_2, \ldots, b_k\}$ contains exactly one point from each essential n-orbit class $\mathbf{F}_{\bar{f}}^{(n)} \in \mathcal{EO}^{(n)}(\bar{f})$. By the Reducing Lemma there exists a fibre preserving map g homotopic to f such that $b_i \in \mathrm{Fix}(\bar{g}^{d_i})$ for the depth d_i of an $\bar{f}$-orbit class containing b_i for each $i = 1, 2, \ldots, k$ (see [10]). Thus we can define a *reduced $\bar{g}$-invariant set with respect to* ξ, denoted by ξ_n, i.e.,

$$\xi_n = \{b_i, \bar{g}(b_i), \bar{g}^2(b_i), \ldots, \bar{g}^{d_i-1}(b_i) \mid b_i \in \xi\}.$$

We denote by g_{ξ_n} the map g restricted to $F_{\xi_n} := \cup_{b \in \xi_n} p^{-1}(b)$.

The following proposition is a generalization of [4, Theorem 4.4.(i)].

PROPOSITION 4.4. *Suppose $p : E \to B$ is a fibration of compact connected ANR's with path-connected fibres, and let $f : E \to E$ be a fibre preserving map. Suppose every essential f-orbit class has the full depth property. Let ξ be a set of essential n-orbit representatives for $\bar{f}$, and ξ_n the reduced $\bar{g}$-invariant set with respect to ξ. Then*

$$EO^{(n)}(g; E, F_{\xi_n}) = EO^{(n)}(g_{\xi_n}) = \sum_{b_i \in \xi} EO^{(m_i)}(g_{b_i}^{d_i}),$$

where d_i is the depth of the $\bar{g}$-orbit class containing b_i, and $m_i = n/d_i$.

PROOF. Since every essential n-orbit class has the full depth property, we can assume that ξ_n is the set of the essential representatives of g^n as follows: Let d be the length of $\bar{g}$-orbit class containing b. By Reducing Lemma, the sets $\{F_{g^j(b)} | 0 \leq j < d\}$ are pairwise disjoint and $g(F_{g^j(b)}) = F_{g^{j+1}(b)}$. Thus g_{ξ_n}-cycle $[F_b]$ has length d, and clearly the set $C(g_{\xi_n})$ has the same cardinality as the set ξ. Thus we have the last equality.

Also for the fibre preserving map $g : (E, F_{\xi_n}) \to (E, F_{\xi_n})$, every essential n-periodic point class of g is essential common (see [4]). By the definition of the relative essential n-orbit number, we have the first equality. □

The following theorem is a generalization of [2, Theorem 5.2].

THEOREM 4.5. *Suppose $p : E \to B$ is a Hurewicz fibration with typical fibre F_b, $b \in B$, E and B compact connected ANR's. Let $f : E \to E$ be a fibre preserving map. Suppose that $\pi_2(B)$ is trivial. Let ξ be a set of essential n-orbit representatives for $\bar{f}$, and ξ_n the reduced $\bar{g}$-invariant set with respect to ξ. If $\mathrm{Fix}((\bar{f}^n)_*^{b_i}) = \{1\}$ for every $b_i \in \xi$, then we have*

$$RO^{(n)}(g; E, F_{\xi_n}) = RO^{(n)}(f),$$

where g is the fibre preserving map from the Reducing Lemma. If, in addition, F_b and E are Jiang-type spaces and $L(g^n)\cdot(\prod_{b_i\in\xi} L((g_{b_i}^{d_i})^{m_i})) \neq 0$ for every $b_i \in \xi$, then

$$EO^{(n)}(g; E, F_{\xi_n}) = RO^{(n)}(f) = \sum_{b_i\in\xi} RO^{(m_i)}(g_{b_i}^{d_i}),$$

where d_i is the depth of the n-orbit class containing b_i and $m_i := n/d_i$.

PROOF. By homotopy invariance we have $RO^{(n)}(f) = RO^{(n)}(g)$. So without loss of generality (by rewriting g as f), we may assume that ξ_n is $\bar{f}$-invariant and g is the same as f.

For each $b = p(x) \in \xi$, since $\pi_2(B)$ is trivial, we have the short exact sequence of groups

$$1 \to \pi_1(F_b, x) \xrightarrow{j_*} \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, b) \to 1.$$

Suppose $b \in \xi$ is in the essential n-periodic point class $\mathbf{F}_{\bar{f}^n}$ which in turn is in the essential n-orbit class $\mathbf{F}_{\bar{f}}^{(n)}$ with depth d and $m := n/d$. Since d is the depth of $\mathbf{F}_{\bar{f}}^{(n)}$, $\mathbf{F}_{\bar{f}^n}$ alone constitutes an essential m-orbit class $\mathbf{F}_{\bar{f}^d}^{(m)} \subset \mathbf{F}_{\bar{f}}^{(n)}$.

Let w be the base path for f from x to $f(x)$ in E and $f_*^w = \varphi$. Then $w_n := wf(w)\cdots f^{n-1}(w)$ is the base path for f^n from x to $f^n(x)$ in E and $p(w) = \bar{w}$ is the base path for $\bar{f}$ from b to $\bar{f}(b)$ in B. Then we have $\bar{\varphi} = \bar{f}_*^{\bar{w}}$. Thus by using the constant path at base point as the path $\bar{c}$

in the definition of $\bar{\rho}$, $[\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d} = [\langle(\bar{w}_d)_m^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}$ is the $\bar{\varphi}^d$-coordinate of $\mathbf{F}^{(m)}_{\bar{f}^d}$ and $\bar{\sigma}([\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}) = [\langle\bar{w}_n^{-1}\rangle]^{(n)}_{\bar{\varphi}}$ is the $\bar{\varphi}$-coordinate of $\mathbf{F}^{(n)}_{\bar{f}}$.

Since the path connected fibre F_b is f^d-invariant, we can choose the base path μ for f^d_b in F_b from x to $f^d_b(x) = f^d(x)$. Then $b = p(\mu)$ is the constant path at b, the associated homomorphism with φ^d is $\psi = (f^d)^{\mu}_*$, and $\mu_m := \mu f^d(\mu)\cdots f^{n-d}(\mu)$ is a path from x to $f^n(x)$ in F_b. Thus we have $\bar{\psi} = (\bar{f}^d)^b_*$. Since $p(\mu_m) = b$ is the constant path at b, by Proposition 2.5, $r_{b,\bar{w}_n}([1]^{(m)}_{\bar{\psi}}) = [\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}$, so we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccccc}
(\mathcal{RO}^{(m)}(\psi_b),[1]^{(m)}_{\psi_b}) & \xrightarrow{j^x_*} & (\mathcal{RO}^{(m)}(\psi),[1]^{(m)}_{\psi}) & \xrightarrow{p_*} & (\mathcal{RO}^{(m)}(\bar{\psi}),[1]^{(m)}_{\bar{\psi}}) & \to & 1 \\
 & & \downarrow{\scriptstyle r_{\mu_m,w_n}} & & \downarrow{\scriptstyle r_{b,\bar{w}_n}} & & \\
 & & (\mathcal{RO}^{(m)}(\varphi^d),[\alpha]^{(m)}_{\varphi^d}) & \xrightarrow{p_*} & (\mathcal{RO}^{(m)}(\bar{\varphi}^d),[\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}) & \to & 1,
\end{array}$$

where α stands for $\langle\mu_m w_n^{-1}\rangle$. Since $\mathrm{Fix}((\bar{\psi})^m) = \mathrm{Fix}((\bar{f}^n)^b_*) = \{1\}$, by [10, 1.6] j^x_* is injective. Let $i_{b,d}$ be the composition of j^x_* and r_{μ_m,w_n}.

When $\mathrm{Fix}(\tau_{\langle\bar{w}_n^{-1}\rangle}\bar{\varphi}^n) = \mathrm{Fix}((\bar{f}^n)^b_*) = \{1\}$, by [10, 1.12] the $\bar{\varphi}$-coordinate of $\mathbf{F}^{(n)}_{\bar{f}}$ has the full depth property. Thus d is the length of $\mathbf{F}^{(n)}_{\bar{f}}$, by [10, 1.7] we have a commutative diagram of exact sequences in the category of pointed sets:

$$\begin{array}{ccccc}
(\mathcal{RO}^{(m)}(\varphi^d),[\alpha]^{(m)}_{\varphi^d}) & \xrightarrow{p_*} & (\mathcal{RO}^{(m)}(\bar{\varphi}^d),[\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}) & \longrightarrow & 1 \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \bar{\sigma}} & & \\
(\mathcal{RO}^{(n)}(\varphi),[\alpha]^{(n)}_{\varphi}) & \xrightarrow{p_*} & (\mathcal{RO}^{(n)}(\bar{\varphi}),[\langle\bar{w}_n^{-1}\rangle]^{(n)}_{\bar{\varphi}}) & \longrightarrow & 1.
\end{array}$$

Furthermore, σ restricts to a bijection

$$\sigma : p_*^{-1}([\langle\bar{w}_n^{-1}\rangle]^{(m)}_{\bar{\varphi}^d}) \to p_*^{-1}([\langle\bar{w}_n^{-1}\rangle]^{(n)}_{\bar{\varphi}}).$$

Combining upper two diagrams, then the composite function as in the definition 2.9

$$\sigma \circ i_{b,d} : \mathcal{RO}^{(m)}(\psi_b) \xrightarrow{i_{b,d}} \mathcal{RO}^{(m)}(\varphi^d) \xrightarrow{\sigma} \mathcal{RO}^{(n)}(\varphi)$$

is injective for every $b \in \xi$, in fact, $RO^{(m)}(f^d_b) = |p_*^{-1}(\bar{\rho}(\mathbf{F}^{(n)}_{\bar{f}}))|$.

As the first part in the proof of Proposition 4.4, the reduced $\bar{f}$-invariant set ξ_n tells us

$$RO^{(n)}(f_{\xi_n}) = \sum_{b_i \in \xi} RO^{(m_i)}(f^{d_i}_{b_i}).$$

For $b' \in \xi - \{b\}$, choose a point $x' \in p^{-1}(b')$ and a path λ from x' to $f(x')$ in E, let u be a path from x' to x in E. By Proposition 2.6 there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}O^{(n)}(\bar{f}) & = & \mathcal{E}O^{(n)}(\bar{f}) \\ {\scriptstyle\bar{\rho}}\downarrow & & \downarrow{\scriptstyle\bar{\rho}} \\ \mathcal{R}O^{(n)}(\bar{f}_*^{\bar{\lambda}}) & \xrightarrow{\bar{u}_*} & \mathcal{R}O^{(n)}(\bar{f}_*^{\bar{w}}), \end{array}$$

where $p(u) = \bar{u}$. Since $\bar{u}_*$ is bijective, the Reidemeister n-orbit classes $\{(\bar{u}_* \circ \bar{\rho})(\mathbf{F}_{\bar{f}}^{(n)}) \mid b \in \xi\}$ in $\mathcal{R}O^{(n)}(\bar{\varphi})$ are all different. Also, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}O^{(n)}(f_*^{\lambda}) & \xrightarrow{u_*} & \mathcal{R}O^{(n)}(f_*^{w}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathcal{R}O^{(n)}(\bar{f}_*^{\bar{\lambda}}) & \xrightarrow{\bar{u}_*} & \mathcal{R}O^{(n)}(\bar{f}_*^{\bar{w}}). \end{array}$$

Then $|p_*^{-1}(\bar{\rho}(\mathbf{F}_{\bar{f}}^{(n)}))| = |p_*^{-1}((\bar{u}_* \circ \bar{\rho})(\mathbf{F}_{\bar{f}}^{(n)}))|$ for every $b \in \xi$. Thus we know that $\{(\sigma \circ i_{b,d})(\mathcal{R}O^{(m)}(\psi_b)) \mid b \in \xi\}$ are all different. This means $RO^{(n)}(f, f_{\xi_n}) = RO^{(n)}(f_{\xi_n})$. The first assertion follows from the definition of $RO^{(n)}(f; E, F_{\xi_n})$.

On the other hand, Theorem 3.5 tells us

$$EO^{(n)}(g; E, F_{\xi_n}) = RO^{(n)}(g; E, F_{\xi_n}),$$

then the first equality of the second assertion holds. Since $\pi_2(B)$ is trivial, from [10, Theorem 2.4] we have

$$EO^{(n)}(f) = \sum_{b_i \in \xi} EO^{(m_i)}(g_{b_i}^{d_i}),$$

and E and F_b are Jiang-type spaces, thus we have the last equality. □

For the torus map $\bar{f}$, the conditions of the first assertion of Theorem 4.5 are always satisfied, then we have

COROLLARY 4.6. *Suppose $p : E \to B$ is fibration over a torus. Then for any fibre preserving map $f : E \to E$, we have*

$$RO^{(n)}(f; E, F_{\xi_n}) = RO^{(n)}(f).$$

Under the same Jiang-type conditions, we have

$$EO^{(n)}(f; E, F_{\xi_n}) = RO^{(n)}(f) = \sum_{b_i \in \xi} RO^{(m_i)}(f_{b_i}^{d_i}).$$

Proof. See [10, 2.6]. □

On the Klein bottle, Corollary 4.6 can be applied to establish the following example (cf. [5, Example 4.1]). Also it improves the example 5.3 in [2]. We omit the details.

Example 4.7. (The Klein bottle). Represent the Klein bottle K^2 as the quotient $\mathbb{R}^2/G$, where G is the group of automorphisms on $\mathbb{R}^2$ generated by $\alpha : (x, y) \mapsto (x, y + 1)$ and $\beta : (x, y) \mapsto (-x, 2y)$.

The map $(x, y) \mapsto (-x, 2y)$ on $\mathbb{R}^2$ induces a well-defined self map f of K^2. This f is fibre preserving with respect to the fibration $S^1 \hookrightarrow K^2 \xrightarrow{p} S^1$, where p is induced by the projection on the first factor. Note that f induces a standard map $\bar{f}$ of degree -1 on the base, and so $\bar{f}^3$ has exactly $N(\bar{f}^3) = 2$ fixed points b_0, b_1. This two 3-orbit classes have the same length 1. Since $(f_{b_0})^3$ is of degree 8 and $(f_{b_1})^3$ is of degree -8, it follows from Corollary 4.6 that $RO^{(3)}(f; K^2, F_{\xi_3}) = RO^{(3)}(f) = \sum RO^{(3)}(f_{b_i}) = 10$.

Although K^2 is *not* a Jiang-type space, in this example, we have

$$RO^{(3)}(f; K^2, F_{\xi_3}) = RO^{(3)}(f) = EO^{(3)}(f).$$

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