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COHOMOLOGY AND TRIVIAL GOTTLIEB GROUPS

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ABSTRACT. This paper observes that the induced homomorphisms on cohomology groups by a cyclic map are trivial. For a CW-complex X, we use the fact to obtain some conditions of X so that the n-th Gottlieb group $G_n(X)$ is trivial for an even positive integer n. As corollaries, for any positive integer m, we obtain $G_{2m}(S^{2m}) = 0$ and $G_2(CP^m) = 0$ which are due to D. H. Gottlieb and G. Lang respectively, where S^{2m} is the $2m$- dimensional sphere and CP^m is the complex projective m-space. Moreover, we show that $G_4(HP^m) = 0$ and $G_8(\Pi) = 0$, where HP^m is the quaternionic projective m-space for any positive integer m and Π is the Cayley projective space.

1. Introduction and preliminary

Let X be a pointed CW-complex. Consider a continuous map $\phi : X \times S^n \to X$ such that $\phi(x, *) = x$, where $*$ is a base point of S^n. Then $g : S^n \to X$ defined by $g(s) = \phi(*, s)$ represents an element $[g] \in \pi_n(X)$. In this case, ϕ is called an *affiliated map* of g and g is a *cyclic map*. The set of all element $[g] \in \pi_n(X)$ obtained in the above manner from ϕ is denoted by $G_n(X)$ and called a *Gottlieb group* or an *evaluation subgroup* of the homotopy group [1]. That is, the n-th Gottlieb group $G_n(X)$ consists of those $\alpha \in \pi_n(X)$ for which there is a map $\phi : X \times S^n \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{\ \phi\ } & X \\ \uparrow J & & \uparrow \nabla \\ X \vee S^n & \xrightarrow{1_X \vee f} & X \vee X \end{array}$$

where $f : S^n \to X$ is a representative of α and ∇ is a folding map.

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The Gottlieb groups of a space have been the objects of extensive study by D. H. Gottlieb [1, 2], J. Siegel [12], G. Lang [4], J. Oprea [11], G. Lupton [5], S. Smith [13], M. H. Woo [8] and the author [7] et al..

D. H. Gottlieb [1] defined the Gottlieb groups of a space and described its outstanding properties. Specially, he showed that if $G_n(X)$ is trivial, then every fibre space over S^{n+1}, with fibre X, has a cross-section. Thus the triviality of Gottlieb groups is closely connected with the cross-section problem of fibrations. Since Gottlieb's original work, several authors have made an attempt to compute Gottlieb groups of various types of spaces. Especially, there are some examples of the trivial Gottlieb group, while its homotopy group is not trivial. For all positive integer m, D. H. Gottlieb computed $G_{2m}(S^{2m}) = 0$ [1] and G. Lang computed $G_2(CP^m) = 0$ [4], where S^{2m} is the $2m$-dimensional sphere and CP^m is the complex projective m-space.

In this paper, we give a generalization of these facts. Actually, we prove the following two theorems and obtain trivial Gottlieb groups due to D. H. Gottlieb and G. Lang as corollaries. Moreover, we show that $G_4(HP^m) = 0$ and $G_8(\Pi) = 0$, where HP^m is the quaternionic projective m-space and Π is the Cayley projective space. Here we denote the j-times cup product $\overbrace{\beta \cup \cdots \cup \beta}$ of $\beta \in H^*(X;\mathbb{Z})$ by β^j.

THEOREM 1. *Let X be a CW-complex with the n-th cohomology group $H^n(X;\mathbb{Z}) = \mathbb{Z}$ for an even integer n. If there exist a generator $u \in H^n(X;\mathbb{Z})$ and an integer $k \geq 1$ such that the kn-th cohomology group $H^{kn}(X;\mathbb{Z})$ is the integer group $\mathbb{Z}$ with a generator u^k and $u^{k+1} = 0$, then each homomorphism on cohomology group induced by a cyclic map is trivial.*

THEOREM 2. *Let X be a simple $(n-1)$-connected CW-complex whose the n-th cohomology group $H^n(X;\mathbb{Z})$ and the n-th homotopy group $\pi_n(X)$ are integer group $\mathbb{Z}$ for an even integer n. If there exist a generator $u \in H^n(X;\mathbb{Z})$ and an integer $k \geq 1$ such that the kn-th cohomology group $H^{kn}(X;\mathbb{Z})$ is the integer group $\mathbb{Z}$ with a generator u^k and $u^{k+1} = 0$, then the n-th Gottlieb group $G_n(X)$ is trivial.*

Since, for all positive integer m, the $2m$-dimensional sphere S^{2m} comes under the case that $n = 2m$ and $k = 1$ and the complex projective space CP^m comes under the case that $n = 2$ and $k = m$ in Theorem 2, we have the following results due to D. H. Gottlieb and G. Lang, respectively.

COROLLARY 3. *$G_{2m}(S^{2m}) = 0$, $G_2(CP^m) = 0$, for each positive integer m.*

Moreover, the quaternionic projective space HP^m comes under the case that $n = 4$ and $k = m$ and the Cayley projective space Π comes under the case that $n = 8$ and $k = 2$ in Theorem 2. So we have the following corollary.

COROLLARY 4. *$G_4(HP^m) = 0$ and $G_8(\Pi) = 0$, for each positive integer m.*

Throughout this paper, all spaces are connected and based CW-complexes, all maps and all homotopies are based.

2. Induced homomorphisms on cohomology group by an affiliated map

In this section we study the consequences of the induced homomorphism by an affiliated map on cohomology groups.

Let $\phi : X \times S^n \to X$ be an affiliated map. Since $H^*(S^n; Z)$ has no torsion, we have

$$H^*(X \times S^n; \mathbb{Z}) \equiv H^*(X; \mathbb{Z}) \otimes H^*(S^n; \mathbb{Z})$$

by Künneth formula. That is,

$$H^n(X \times S^n; \mathbb{Z}) \equiv \bigoplus_{s+t=n} H^s(X; \mathbb{Z}) \otimes H^t(S^n; \mathbb{Z}).$$

Furthermore, since $H^i(S^n; \mathbb{Z}) = 0$ for $0 < i < n$,

$$H^n(X \times S^n; \mathbb{Z}) \equiv H^0(X; \mathbb{Z}) \otimes H^n(S^n; \mathbb{Z}) \oplus H^n(X; \mathbb{Z}) \otimes H^0(S^n; \mathbb{Z}).$$

Thus if $x \in H^n(X \times S^n; \mathbb{Z})$, then x is of the form $1 \otimes y + z \otimes 1$, where $y \in H^n(S^n; \mathbb{Z})$, $z \in H^n(X; Z)$ and 1 is a generator of $H^0(Y)$ for $Y = S^n$ or X.

PROPOSITION 5. *Let $i_1 : X \to X \times S^n$ and $i_2 : S^n \to X \times S^n$ be the maps given by $i_1(x) = (x, *)$ and $i_2(s) = (*, s)$ respectively, for $x \in H^n(X; \mathbb{Z})$, $s \in H^n(S^n; \mathbb{Z})$ and $*$ base point and $p_1 : X \times S^n \to X$ and $p_2 : X \times S^n \to S^n$ natural projections. Then $p_1{}^*(z) = z \otimes 1$, $p_2{}^*(y) = 1 \otimes y$, $i_1{}^*(z \otimes 1) = z$, $i_1{}^*(z \otimes y) = 0$ unless $y \in H^0(S^n; \mathbb{Z})$, $i_2{}^*(1 \otimes y) = y$ and $i_2{}^*(z \otimes y) = 0$ unless $z \in H^0(X; \mathbb{Z})$, where $i_k{}^*$ and $p_k{}^*$ are the homomorphisms induced by i_k and p_k respectively, on cohomology groups for $k = 1, 2$.*

PROOF. See p.249 [14]. □

PROPOSITION 6. *Let $\phi : X \times S^n \to X$ be an affiliated map. Then the induced homomorphism $\phi^* : H^n(X;\mathbb{Z}) \to H^n(X \times S^n;\mathbb{Z})$ is given by*

$$\phi^*(x) = x \otimes 1 + 1 \otimes m\sigma$$

for some integer m, where σ is a generator of $H^n(S^n;\mathbb{Z})$.

PROOF. Since $\phi^*(x) \in H^n(X \times S^n;\mathbb{Z})$, $\phi^*(x) = 1 \otimes y + z \otimes 1$ for some $y \in H^n(S^n;\mathbb{Z})$, $z \in H^n(X;Z)$, where 1 is a generator of $H^0(Y)$ for $Y = S^n$ or X. By Proposition 5,

$$\begin{aligned} x = (\phi i_1)^*(x) &= i_1^*\phi^*(x) = i_1^*(z \otimes 1 + 1 \otimes y) \\ &= i_1^*(z \otimes 1) + i_1^*(1 \otimes y) \\ &= i_1^*(z \otimes 1) = z \end{aligned}$$

and

$$i_2^*\phi^*(x) = i_2^*(z \otimes 1 + 1 \otimes y) = i_2^*(z \otimes 1) + i_2^*(1 \otimes y) = y.$$

Since $i_2^*\phi^*(x)$ belongs to $H^n(S^n;\mathbb{Z})$ and σ is a generator of $H^n(S^n;\mathbb{Z})$, $y = i_2^*\phi^*(x) = m\sigma$ for some integer m. Therefore we have

$$\phi^*(x) = x \otimes 1 + 1 \otimes m\sigma.$$

□

PROOF OF THEOREM 1. Let $\alpha : S^n \to X$ be a cyclic map with an affiliated map $\phi : X \times S^n \to X$. Consider the induced homomorphism $\alpha^* : H^n(X;\mathbb{Z}) \to H^n(S^n;\mathbb{Z})$. Since $\alpha^*(u) \in H^n(S^n;\mathbb{Z})$ for the generator $u \in H^n(X;\mathbb{Z})$, $\alpha^*(u) = m\sigma$ for some integer m, where σ is a generator of $H^n(S^n;\mathbb{Z})$. Thus, in order to show that α^* is trivial, it is sufficient to show that $m = 0$.

By Proposition 6 and the facts that n is even, $u^{k+1} = 0$ and $(m\sigma)^i = 0 \in H^{ni}(S^n)$ for $i \geq 2$, we have

$$\begin{aligned} 0 = \phi^*(u^{k+1}) &= (\phi^*(u))^{k+1} = (u \otimes 1 + 1 \otimes m\sigma)^{k+1} \\ &= (u \otimes 1)^{k+1} + \binom{k+1}{1}((u \otimes 1)^k \cup (1 \otimes m\sigma)) + \\ &\quad \binom{k+1}{2}((u \otimes 1)^{k-1} \cup (1 \otimes m\sigma)^2) + \cdots + (1 \otimes m\sigma)^{k+1} \\ &= u^{k+1} \otimes 1 + (k+1)(u^k \otimes m\sigma) + \\ &\quad \frac{(k+1)k}{2}(u^{k-1} \otimes (m\sigma)^2) + \cdots + 1 \otimes (m\sigma)^{k+1} \\ &= (k+1)(u^k \otimes m\sigma), \end{aligned}$$

where $\binom{j}{l}$ is the number of combinations of j things taken l at a time.

Since u^k and σ are generators of $H^n(X;\mathbb{Z})$ and $H^n(S^n;\mathbb{Z})$ respectively, we have $m = 0$. So α^* is a zero homomorphism. □

3. Proof of the theorem 2 and corollaries

First we introduce a well-known theorem in [14]. We need the theorem to develop our assertions.

THEOREM 7. *Let $\iota \in H^n(X;\mathbb{Z})$ be n-characteristic for a simple $(n-1)$-connected space X, where $n \geq 1$, and let Y be a CW-complex such that $H^q(Y;\pi_q(X)) = 0$ for $q > n$. Then $f_0, f_1 : Y \to X$ are homotopic if and only if ${f_0}^*(\iota) = {f_1}^*(\iota)$.*

PROOF. See p447, [14]. □

PROOF OF THEOREM 2. Let $\alpha \in G_n(X)$. It is sufficient to show that α is null-homotopic. Since X is a simple $(n-1)$-connected CW-complex whose the n-th homotopy group $\pi_n(X)$ is the integer group $\mathbb{Z}$, there is an n-characteristic $\iota \in H^n(X;\mathbb{Z})$ by the absolute Hurewicz isomorphism theorem and the universal-coefficient theorem for cohomology. Since α is a cyclic map, the induced homomorphisms on cohomology groups of X are trivial by Theorem 1. Thus we have

$$\alpha^*(\iota) = 0 = c^*(\iota),$$

where c is the constant map from S^n into the base point of X. Furthermore, $H^q(S^n;\pi_q(X) = 0$ for $q > n$. So α and c are homotopic by Theorem 7. □

For any positive integer m, S^{2m} is $(2m-1)$-connected and $H^{2m}(S^{2m}) = \mathbb{Z} = \pi_{2m}(S^{2m})$. Let u be a generator of $H^{2m}(S^{2m})$, then $u^2 = 0$. Therefore, S^{2m} comes under the case that $n = 2m$ and $k = 1$ in Theorem 2.

For any positive integer m, let CP^m be the complex projective m-space. If $u \in H^2(CP^m) = \mathbb{Z}$ is a generator, then $u^k \in H^{2k}(CP^m)$ is a generator for $1 \leq k \leq m$ and $u^{k+1} = 0$. That is, $H^*(CP^m)$ is a polynomial ring over the integers with one generator u in dimension two, subject to the relation $u^{k+1} = 0$. Moreover, $\pi_2(CP^m) = \mathbb{Z}$ and $\pi_1(CP^m) = 0$. Thus CP^m comes under the case $n = 2$ and $k = m$ in Theorem 2.

So we have the following results due to D. H. Gottlieb and G. Lang, respectively.

COROLLARY 3. *$G_{2m}(S^{2m}) = 0$ and $G_2(CP^m) = 0$ for each positive integer m.*

Similar arguments may be used to compute the cohomology ring of the quaternionic projective space HP^m. $H^*(HP^m)$ is a polynomial ring over the integers with one generator u in dimension four subject to the relation $u^{n+1} = 0$. Moreover, $\pi_4(HP^m) = \mathbb{Z}$ and $\pi_j(HP^m) = 0$ for $i = 1, 2, 3$. Thus HP^m comes under the case $n = 4$ and $k = m$ in Theorem 2.

Let $\alpha : S^{2n-1} \longrightarrow S^n$. Then $\widetilde{H}^*(S^n \cup_\alpha e^{2n})$ has generators ι_1 and ι_2 in dimension n and $2n$. Hence ${\iota_1}^2 = k\iota_2$ for some integer k. k depends only on the choice of generators up to sign, but otherwise depends only on the type of $S^n \cup_\alpha e^{2n}$ and hence only on the homotopy class of α. Thus there is a transformation $H : \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}$ called the *Hopf invariant.* A map $\sigma : S^{15} \longrightarrow S^8$ with $H(\sigma) = 1$ can be constructed using the Cayley number multiplication. The space $\Pi = S^8 \cup_\sigma e^{16}$ is called the *Cayley projective space.* Thus the Cayley projective space Π comes under the case the $n = 8$ and $k = 2$ in Theorem 2.

Therefore we have the following result.

COROLLARY 4. *For each positive integer m, $G_4(HP^m) = 0$ and $G_8(\Pi) = 0$.*

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