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LOCALIZATION PROPERTY AND FRAMES II

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ABSTRACT. Localization of sequences with respect to Riesz bases for Hilbert spaces are comparable with perturbation of Riesz bases or frames. Gröchenig first introduced the notion of localization. We introduce more general definition of localization and show that exponentially localized sequences and polynomially localized sequences with respect to Riesz bases are Bessel sequences. Furthermore, they are frames provided some additional conditions are satisfied.

1. Introduction

The main feature of a basis $\{f_i\}_{i=1}^{\infty}$ for a Hilbert space H is that every $f \in H$ can be represented as a convergent series in terms of the elements f_i, that is, $f = \sum_{i=1}^{\infty} c_i f_i$ with unique coefficients c_i. A frame is also a sequence $\{f_i\}_{i=1}^{\infty}$ in H which allows every $f \in H$ to be written as a series. But, it is overcomplete and so the series representations need not be unique. This redundancy may be useful, for example, in applications to noise reduction or for reconstruction from lossy data. In addition, construction of a frame is easier and more flexible than construction of a basis.

A sequence $\{f_i\}_{i=1}^{\infty}$ of elements in a Hilbert space H is called a *Bessel sequence* if there exists a constant $B < \infty$ such that for every $f \in H$,

$$\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

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A Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is called a *frame* if it satisfies an additional condition : there exists a constant $A > 0$ such that for every $f \in H$,

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2.$$

The numbers A and B are called a *lower* and an *upper frame bound* of the frame, respectively. If $\{f_i\}_{i=1}^{\infty}$ is a frame, then it is well-known that for every $f \in H$, $f = \sum_{i=1}^{\infty} c_i f_i$ for some coefficients $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$.

A sequence $\{f_i\}_{i=1}^{\infty}$ of elements in a Hilbert space H is called a *Riesz basis* if it is complete and there exist positive constants A and B such that for every $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$,

$$A \sum_{i=1}^{\infty} |c_i|^2 \leq \|\sum_{i=1}^{\infty} c_i f_i\|^2 \leq B \sum_{i=1}^{\infty} |c_i|^2.$$

The numbers A and B are called a *lower* and an *upper Riesz bound* of the Riesz basis, respectively. If $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for a Hilbert space H, then there exists a unique sequence $\{g_i\}_{i=1}^{\infty}$ in H such that for every $f \in H$, $f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i$. Such a sequence $\{g_i\}_{i=1}^{\infty}$ is called the dual Riesz basis of $\{f_i\}_{i=1}^{\infty}$. There are well-known characterizations for Riesz bases (see [11]). For instance, a Riesz basis is an ω-independent frame.

Gröchenig introduced a concept of localization of sequences in [5] and [7]. In the next section, we give more general definition of localization. To compare with our definition, let us give Gröchenig's definition of localization in [7]. A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called *exponentially localized* with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ if there exist positive constants α and C such that $|\langle f_i, g_j \rangle| \leq Ce^{-\alpha|i-j|}$ and $|\langle f_i, \tilde{g}_j \rangle| \leq Ce^{-\alpha|i-j|}$ for all $i, j \in \mathbb{N}$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$. Similarly, a sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called *polynomially localized* with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ if there exist positive constants s and C such that $|\langle f_i, g_j \rangle| \leq C(1 + |i-j|)^{-s}$ and $|\langle f_i, \tilde{g}_j \rangle| \leq C(1 + |i-j|)^{-s}$ for all $i, j \in \mathbb{N}$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$. It can be shown that if a sequence $\{f_i\}_{i=1}^{\infty}$ is exponentially or polynomially localized with respect to a Riesz basis then it is a Bessel sequence (see [8]).

The results in this paper are extensions of the results in our previous paper [10]. In [10], we found some sufficient conditions which guarantee that localized sequences are frames. In this paper, we consider the same problem with generalized definitions of localization.

2. Definitions

DEFINITION 2.1. A sequence $\{I_i\}_{i=1}^{\infty}$ is called a *sequence of indexing sets* if I_i are finite and disjoint and $\mathbb{N} = \bigcup_{i=1}^{\infty} I_i$.

DEFINITION 2.2. A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called *exponentially localized* with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ if there exist a sequence of indexing sets $\{I_i\}_{i=1}^{\infty}$, positive constants $r < 1$, and C such that

$$|\langle f_k, g_j\rangle| \le \frac{C}{\sqrt{n_i}} r^{|i-j|} \tag{1}$$

and

$$|\langle f_k, \tilde{g}_j\rangle| \le \frac{C}{\sqrt{n_i}} r^{|i-j|} \tag{2}$$

for all $i, j \in \mathbb{N}$ and $k \in I_i$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_i is the cardinality of I_i.

DEFINITION 2.3. A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called *polynomially localized* with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ if there exist a sequence of indexing sets $\{I_i\}_{i=1}^{\infty}$, positive constants $s > 1$, and C such that

$$|\langle f_k, g_j\rangle| \le \frac{C}{\sqrt{n_i}} (1 + |i-j|)^{-s} \tag{3}$$

and

$$|\langle f_k, \tilde{g}_j\rangle| \le \frac{C}{\sqrt{n_i}} (1 + |i-j|)^{-s} \tag{4}$$

for all $i, j \in \mathbb{N}$ and $k \in I_i$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_i is the cardinality of I_i.

Note that if $\{g_i\}_{i=1}^{\infty}$ is an orthonomal basis for H then the conditions (1) and (2) are identical and the conditions (3) and (4) are identical because $\{g_i\}_{i=1}^{\infty} = \{\tilde{g}_i\}_{i=1}^{\infty}$. Although we use the same terminology, our definitions are more general than Gröchenig's definition.

The following lemma is equivalent to the corresponding lemma in [7]. The only difference is the usage of exponential function. In [7], they use $e^{-\alpha t}, \alpha > 0$, and in this paper we use $r^t, 0 < r < 1$, as exponential functions.

LEMMA 2.4. (1) *For any positive constant $r < 1$, there exists a constant $C > 0$ such that*

$$\sum_{l=1}^{\infty} r^{|l-k|} r^{|l-j|} \leq C r^{\frac{1}{2}|k-j|}$$

for all $k, j \in \mathbb{N}$.

(2) *For any constant $s > 1$, there exists a constant $C > 0$ such that*

$$\sum_{l=1}^{\infty} (1 + |l-k|)^{-s} (1 + |l-j|)^{-s} \leq C(1 + |k-j|)^{-s}$$

for all $k, j \in \mathbb{N}$.

PROOF. (1) Fix $k, j \in \mathbb{N}$ and let $I_1 = \{l \in \mathbb{N} : |l-j| \leq \frac{1}{2}|k-j|\}$ and $I_2 = \mathbb{N} - I_1$. If $l \in I_1$, then $|l-k| \geq \frac{1}{2}|k-j|$. Hence, we have

$$\begin{aligned}
\sum_{l=1}^{\infty} r^{|l-k|} r^{|l-j|} &= \sum_{l \in I_1} r^{|l-k|} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} r^{|l-j|} \\
&\leq \sum_{l \in I_1} r^{\frac{1}{2}|k-j|} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} r^{\frac{1}{2}|k-j|} \\
&= r^{\frac{1}{2}|k-j|} \Big(\sum_{l \in I_1} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} \Big) \\
&\leq r^{\frac{1}{2}|k-j|} \Big(\sum_{l=1}^{\infty} r^{|l-j|} + \sum_{l=1}^{\infty} r^{|l-k|} \Big) \\
&= r^{\frac{1}{2}|k-j|} \Big(\Big(\sum_{l=1}^{j-1} r^l + \sum_{l=0}^{\infty} r^l \Big) + \Big(\sum_{l=1}^{k-1} r^l + \sum_{l=0}^{\infty} r^l \Big) \Big) \\
&\leq r^{\frac{1}{2}|k-j|} 2 \Big(\frac{r}{1-r} + \frac{1}{1-r} \Big) \\
&= r^{\frac{1}{2}|k-j|} 2 \Big(\frac{1+r}{1-r} \Big) \\
&= C r^{\frac{1}{2}|k-j|},
\end{aligned}$$

where $C = 2(1+r)/(1-r)$ and it is independent of k and j.

(2) Fix $k, j \in \mathbb{N}$ and let $I_1 = \{l \in \mathbb{N} : |l-j| \leq \frac{1}{2}|k-j|\}$ and $I_2 = \mathbb{N} - I_1$. If $l \in I_1$, then $|l-k| \geq \frac{1}{2}|k-j|$. Hence, we have

$$\begin{aligned}
&\sum_{l=1}^{\infty}(1+|l-k|)^{-s}(1+|l-j|)^{-s} \\
&= \sum_{l\in I_1}(1+|l-k|)^{-s}(1+|l-j|)^{-s} + \sum_{l\in I_2}(1+|l-k|)^{-s}(1+|l-j|)^{-s} \\
&\leq \sum_{l\in I_1}\Big(1+\frac{1}{2}|k-j|\Big)^{-s}(1+|l-j|)^{-s} \\
&\quad + \sum_{l\in I_2}(1+|l-k|)^{-s}\Big(1+\frac{1}{2}|k-j|\Big)^{-s} \\
&= \Big(1+\frac{1}{2}|k-j|\Big)^{-s}\Big(\sum_{l\in I_1}(1+|l-j|)^{-s} + \sum_{l\in I_2}(1+|l-k|)^{-s}\Big) \\
&\leq 2^s(1+|k-j|)^{-s}\Big(\sum_{l\in I_1}(1+|l-j|)^{-s} + \sum_{l\in I_2}(1+|l-k|)^{-s}\Big) \\
&\leq 2^s(1+|k-j|)^{-s}\Big(\sum_{l=1}^{\infty}(1+|l-j|)^{-s} + \sum_{l=1}^{\infty}(1+|l-k|)^{-s}\Big) \\
&= 2^s(1+|k-j|)^{-s}\Big(\Big(\sum_{l=1}^{j-1}(1+l)^{-s} + \sum_{l=0}^{\infty}(1+l)^{-s}\Big) \\
&\quad + \Big(\sum_{l=1}^{k-1}(1+l)^{-s} + \sum_{l=0}^{\infty}(1+l)^{-s}\Big)\Big) \\
&\leq \Big(2^s(1+|k-j|)^{-s}\Big)2\Big(\int_1^{\infty}x^{-s}dx + \int_1^{\infty}x^{-s}dx + 1\Big) \\
&= \Big(2^s(1+|k-j|)^{-s}\Big)2\Big(\frac{s+1}{s-1}\Big) \\
&= C(1+|k-j|)^{-s},
\end{aligned}$$

where $C = 2^{s+1}(s+1)/(s-1)$ and it is independent of k and j. □

3. Main results

PROPOSITION 3.1. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a sequence in a Hilbert space* H *and* $\{g_i\}_{i=1}^{\infty}$ *be a Riesz basis for* H *with bounds* A, B. *If* $\{f_i\}_{i=1}^{\infty}$ *is* *exponentially or polynomially localized with respect to* $\{g_i\}_{i=1}^{\infty}$, *then* $\{f_i\}_{i=1}^{\infty}$ *is a Bessel sequence.*

PROOF. Since $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis, we can write $f_i = \sum_{l=1}^{\infty}\langle f_i, \tilde{g}_l\rangle g_l$ for every $i \in \mathbb{N}$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$.

Let $f \in H$.

(1) Suppose that $\{f_i\}_{i=1}^{\infty}$ is exponentially localized. Then there exist a sequence of indexing sets $\{I_k\}_{k=1}^{\infty}$, positive constants $r < 1$ and C such that $|\langle f_i, g_j\rangle| \le \frac{C}{\sqrt{n_k}} r^{|k-j|}$ and $|\langle f_i, \tilde{g}_j\rangle| \le \frac{C}{\sqrt{n_k}} r^{|k-j|}$ for all $k, j \in \mathbb{N}$ and $i \in I_k$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_k is the cardinality of I_k. So, by Lemma 2.4 (1), we obtain

$$
\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 &= \sum_{k=1}^{\infty}\sum_{i\in I_k} |\langle f, f_i\rangle|^2 \\
&= \sum_{k=1}^{\infty}\sum_{i\in I_k} \Big|\sum_{l=1}^{\infty}\langle f, g_l\rangle \overline{\langle f_i, \tilde{g}_l\rangle}\Big|^2 \\
&\le \sum_{k=1}^{\infty}\sum_{i\in I_k} \Big(\sum_{l=1}^{\infty} |\langle f, g_l\rangle||\langle f_i, \tilde{g}_l\rangle|\Big)^2 \\
&\le \sum_{k=1}^{\infty}\sum_{i\in I_k} \Big(\sum_{l=1}^{\infty} |\langle f, g_l\rangle| \frac{C}{\sqrt{n_k}} r^{|k-l|}\Big)^2 \\
&= C^2 \sum_{k=1}^{\infty} \Big(\sum_{l=1}^{\infty} |\langle f, g_l\rangle| r^{|k-l|}\Big)^2 \\
&= C^2 \sum_{k=1}^{\infty} \Big(\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} |\langle f, g_l\rangle||\langle f, g_m\rangle| r^{|k-l|} r^{|k-m|}\Big) \\
&= C^2 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} |\langle f, g_l\rangle||\langle f, g_m\rangle| \sum_{k=1}^{\infty} r^{|k-l|} r^{|k-m|} \\
&\le C^2 C_1 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} |\langle f, g_l\rangle||\langle f, g_m\rangle| r^{\frac{|l-m|}{2}} \\
&\le \frac{C^2 C_1}{2} \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \Big(|\langle f, g_l\rangle|^2 + |\langle f, g_m\rangle|^2\Big) r^{\frac{|l-m|}{2}} \\
&= \frac{C^2 C_1}{2} \Big(\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} r^{\frac{|l-m|}{2}} |\langle f, g_l\rangle|^2 + \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} r^{\frac{|l-m|}{2}} |\langle f, g_m\rangle|^2\Big)
\end{aligned}
$$

$$\leq \frac{C^2C_1}{2}\Big(\sum_{l=1}^{\infty}\Big(\frac{1+\sqrt{r}}{1-\sqrt{r}}\Big)|\langle f,g_l\rangle|^2+\sum_{m=1}^{\infty}\Big(\frac{1+\sqrt{r}}{1-\sqrt{r}}\Big)|\langle f,g_m\rangle|^2\Big)$$
$$= C^2C_1C_2\sum_{l=1}^{\infty}|\langle f,g_l\rangle|^2$$
$$\leq C^2C_1C_2B\|f\|^2,$$

where $C_1 = 2(1+r)/(1-r)$ and $C_2 = (1+\sqrt{r})/(1-\sqrt{r})$. So,

$$\sum_{i=1}^{\infty}|\langle f,f_i\rangle|^2 \leq M\|f\|^2,$$

where $M = C^2C_1C_2B$. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence.

(2) Suppose that $\{f_i\}_{i=1}^{\infty}$ is polynomially localized. Then there exist a sequence of indexing sets $\{I_k\}_{k=1}^{\infty}$, positive constants $s>1$ and C such that $|\langle f_i,g_j\rangle| \leq \frac{C}{\sqrt{n_k}}(1+|k-j|)^{-s}$ and $|\langle f_i,\tilde{g}_j\rangle| \leq \frac{C}{\sqrt{n_k}}(1+|k-j|)^{-s}$ for all $k,j\in\mathbb{N}$ and $i\in I_k$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_k is the cardinality of I_k. So, by Lemma 2.4 (2), we obtain

$$\sum_{i=1}^{\infty}|\langle f,f_i\rangle|^2$$
$$= \sum_{k=1}^{\infty}\sum_{i\in I_k}|\langle f,f_i\rangle|^2$$
$$= \sum_{k=1}^{\infty}\sum_{i\in I_k}\Big|\sum_{l=1}^{\infty}\langle f,g_l\rangle\overline{\langle f_i,\tilde{g}_l\rangle}\Big|^2$$
$$\leq \sum_{k=1}^{\infty}\sum_{i\in I_k}\Big(\sum_{l=1}^{\infty}|\langle f,g_l\rangle||\langle f_i,\tilde{g}_l\rangle|\Big)^2$$
$$\leq \sum_{k=1}^{\infty}\sum_{i\in I_k}\Big(\sum_{l=1}^{\infty}|\langle f,g_l\rangle|\frac{C}{\sqrt{n_k}}(1+|k-l|)^{-s}\Big)^2$$
$$= C^2\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{\infty}|\langle f,g_l\rangle|(1+|k-l|)^{-s}\Big)^2$$
$$= C^2\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}|\langle f,g_l\rangle||\langle f,g_m\rangle|(1+|k-l|)^{-s}(1+|k-m|)^{-s}\Big)$$

$$
\begin{aligned}
&= C^2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} |\langle f, g_l\rangle||\langle f, g_m\rangle| \sum_{k=1}^{\infty} (1+|k-l|)^{-s}(1+|k-m|)^{-s} \\
&\le C^2 C_1 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} |\langle f, g_l\rangle||\langle f, g_m\rangle|(1+|l-m|)^{-s} \\
&\le \frac{C^2C_1}{2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \Big(|\langle f, g_l\rangle|^2 + |\langle f, g_m\rangle|^2\Big)(1+|l-m|)^{-s} \\
&= \frac{C^2C_1}{2} \Big(\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (1+|l-m|)^{-s} |\langle f, g_l\rangle|^2 \\
&\quad + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} (1+|l-m|)^{-s} |\langle f, g_m\rangle|^2 \Big) \\
&\le \frac{C^2C_1}{2} \Big(\sum_{l=1}^{\infty} \Big(\frac{s+1}{s-1}\Big) |\langle f, g_l\rangle|^2 + \sum_{m=1}^{\infty} \Big(\frac{s+1}{s-1}\Big) |\langle f, g_m\rangle|^2 \Big) \\
&= C^2 C_1 C_2 \sum_{l=1}^{\infty} |\langle f, g_l\rangle|^2 \\
&\le C^2 C_1 C_2 B ||f||^2,
\end{aligned}
$$

where $C_1 = 2^{s+1}(s+1)/(s-1)$ and $C_2 = (s+1)/(s-1)$. So,

$$
\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le M||f||^2,
$$

where $M = C^2C_1C_2B$. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence. □

COROLLARY 3.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{e_i\}_{i=1}^{\infty}$ be an orthonomal basis for H. If $\{f_i\}_{i=1}^{\infty}$ is exponentially or polynomially localized with respect to $\{e_i\}_{i=1}^{\infty}$, then $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence.*

To prove Theorem 3.5 and Theorem 3.6, we need the well-known fact that a diagonally dominant matrix is invertible (see [9] for proof).

LEMMA 3.3. *Suppose that a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ defines a bounded self-adjoint operator on $l^2(\mathbb{N})$ and that A satisfies the condition of diagonal dominance for every $i \in \mathbb{N}$; that is, there exists a positive constant δ such that*

$$
|a_{ii}| - \sum_{j:j\ne i} |a_{ij}| \ge \delta
$$

for every $i \in \mathbb{N}$. Then A is invertible on $l^2(\mathbb{N})$.

If $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, we can define a bounded operator T, usually called the *pre-frame operator* associated to $\{f_i\}_{i=1}^{\infty}$:

$$T : l^2(\mathbb{N}) \to H, \qquad T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i.$$

Then its adjoint $T^* : H \to l^2(\mathbb{N})$ is defined by $T^*f = \{\langle f, f_i\rangle\}_{i=1}^{\infty}$ for every $f \in H$. By composing T and T^*, we obtain the *frame operator* S:

$$S : H \to H, \qquad Sf = TT^*f = \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i.$$

If $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, the series defining S converges unconditionally for all $f \in H$, and S is a bounded self-adjoint operator on H.

THEOREM 3.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H. If the matrix $\{\langle Sg_i, g_j\rangle\}_{i,j=1}^{\infty}$ is invertible on $l^2(\mathbb{N})$, then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.*

The proof of Theorem 3.4 can be found in [10].

THEOREM 3.5. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H, and suppose that $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{g_i\}_{i=1}^{\infty}$; that is, there exist a sequence of indexing sets $\{I_i\}_{i=1}^{\infty}$, positive constants $r < 1$, and C_1 such that*

$$|\langle f_k, g_j\rangle| \le \frac{C_1}{\sqrt{n_i}} r^{|i-j|}$$

and

$$|\langle f_k, \tilde{g}_j\rangle| \le \frac{C_1}{\sqrt{n_i}} r^{|i-j|}$$

for all $i, j \in \mathbb{N}$ and $k \in I_i$, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_i is the cardinality of I_i.

If there exists a positive constant C_2 such that

$$\sum_{i=1}^{\infty} |\langle f_i, g_j\rangle|^2 \ge C_2^2$$

and

$$\sqrt{2}C_2 > \left(\frac{1+r}{1-r}\right)C_1,$$

then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

PROOF. By Proposition 3.1, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence. So it suffices to show that $\{f_i\}_{i=1}^{\infty}$ has a lower frame bound. Fix $i \in \mathbb{N}$ and consider

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i} |\langle Sg_i, g_j\rangle|.$$

Since $\langle Sg_i, g_j\rangle = \sum_{k=1}^{\infty}\langle g_i, f_k\rangle\langle f_k, g_j\rangle$ and $\langle Sg_i, g_i\rangle = \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2$, we have

$$\begin{aligned}\sum_{j:j\neq i}|\langle Sg_i, g_j\rangle| &= \sum_{j=1}^{\infty}|\langle Sg_i, g_j\rangle| - |\langle Sg_i, g_i\rangle| \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2 \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - C_2^2.\end{aligned}$$

Since $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{g_i\}_{i=1}^{\infty}$,

$$\begin{aligned}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| &= \sum_{m=1}^{\infty}\sum_{k\in I_m}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| \\ &\leq C_1^2\sum_{m=1}^{\infty}\sum_{k\in I_m}\Big(\frac{r^{|i-m|}}{\sqrt{n_m}}\Big)\Big(\frac{r^{|j-m|}}{\sqrt{n_m}}\Big) \\ &= C_1^2\sum_{m=1}^{\infty}r^{|i-m|}r^{|j-m|}.\end{aligned}$$

From this inequality, we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| &\leq C_1^2\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}r^{|i-m|}r^{|j-m|} \\ &= C_1^2\sum_{m=1}^{\infty}r^{|i-m|}\sum_{j=1}^{\infty}r^{|j-m|} \\ &\leq C_1^2\Big(\frac{r}{1-r}+\frac{1}{1-r}\Big)^2 \\ &= C_1^2\Big(\frac{1+r}{1-r}\Big)^2.\end{aligned}$$

Hence,

$$\sum_{j:j\neq i} |\langle Sg_i, g_j\rangle| \le C_1^2\Big(\frac{1+r}{1-r}\Big)^2 - C_2^2.$$

Finally,

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i} |\langle Sg_i, g_j\rangle| \ge 2C_2^2 - C_1^2\Big(\frac{1+r}{1-r}\Big)^2 > 0,$$

where the constants r, C_1, C_2 are independent of i. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a frame for H. □

THEOREM 3.6. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H, and suppose that $\{f_i\}_{i=1}^{\infty}$ is polynomially localized with respect to $\{g_i\}_{i=1}^{\infty}$; that is, there exists a sequence of indexing sets $\{I_i\}_{i=1}^{\infty}$, positive constants $s > 1$, and C_1 such that*

$$|\langle f_k, g_j\rangle| \le \frac{C_1}{\sqrt{n_i}}(1+|i-j|)^{-s}$$

and

$$|\langle f_k, \tilde{g}_j\rangle| \le \frac{C_1}{\sqrt{n_i}}(1+|i-j|)^{-s}$$

for all $i, j \in \mathbb{N}$ and $k \in I_i$ where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$ and n_i is the cardinality of I_i.

If there exists a positive constant C_2 such that

$$\sum_{i=1}^{\infty} |\langle f_i, g_j\rangle|^2 \ge C_2^2$$

and

$$\sqrt{2}C_2 > \Big(\frac{s+1}{s-1}\Big)C_1,$$

then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

PROOF. By Proposition 3.1, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence. So it suffices to show that $\{f_i\}_{i=1}^{\infty}$ has a lower frame bound. Fix $i \in \mathbb{N}$ and consider

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i} |\langle Sg_i, g_j\rangle|.$$

Since $\langle Sg_i, g_j\rangle = \sum_{k=1}^{\infty}\langle g_i, f_k\rangle\langle f_k, g_j\rangle$ and $\langle Sg_i, g_i\rangle = \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2$, we have

$$\begin{aligned}\sum_{j:j\neq i}|\langle Sg_i, g_j\rangle| &= \sum_{j=1}^{\infty}|\langle Sg_i, g_j\rangle| - |\langle Sg_i, g_i\rangle| \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2 \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - C_2^2.\end{aligned}$$

Since $\{f_i\}_{i=1}^{\infty}$ is polynomially localized with respect to $\{g_i\}_{i=1}^{\infty}$,

$$\begin{aligned}&\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| \\ &= \sum_{m=1}^{\infty}\sum_{k\in I_m}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| \\ &\leq C_1^2\sum_{m=1}^{\infty}\sum_{k\in I_m}\Big(\frac{(1+|i-m|)^{-s}}{\sqrt{n_m}}\Big)\Big(\frac{(1+|j-m|)^{-s}}{\sqrt{n_m}}\Big) \\ &= C_1^2\sum_{m=1}^{\infty}(1+|i-m|)^{-s}(1+|j-m|)^{-s}.\end{aligned}$$

From these inequalities, we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| &\leq \sum_{m=1}^{\infty}(1+|i-m|)^{-s}\sum_{j=1}^{\infty}(1+|j-m|)^{-s} \\ &\leq C_1^2\Big(\frac{s+1}{s-1}\Big)^2.\end{aligned}$$

Hence,

$$\sum_{j:j\neq i}|\langle Sg_i, g_j\rangle| \leq C_1^2\Big(\frac{s+1}{s-1}\Big)^2 - C_2^2.$$

Finally,

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i}|\langle Sg_i, g_j\rangle| \geq 2C_2^2 - C_1^2\Big(\frac{s+1}{s-1}\Big)^2 > 0,$$

where the constants r, C_1, C_2 are independent of i. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a frame for H. □

4. Examples

As an application of our main theorem, we show examples. The first example is a sequence which is exponentially localized with respect to an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H.

EXAMPLE 4.1. Let $0 < r \le \frac{1}{6}$,

$$f_1 = e_1 + re_2 + r^2e_3 + \cdots ,$$
$$f_2 = \frac{1}{\sqrt{2}}\Big(re_1 + e_2 + re_3 + r^2e_4 + \cdots\Big),$$
$$f_3 = \frac{1}{\sqrt{2}}\Big(re_1 + e_2 + re_3 + r^2e_4 + \cdots\Big),$$
$$f_4 = \frac{1}{\sqrt{3}}\Big(r^2e_1 + re_2 + e_3 + re_4 + r^2e_5 + \cdots\Big),$$
$$f_5 = \frac{1}{\sqrt{3}}\Big(r^2e_1 + re_2 + e_3 + re_4 + r^2e_5 + \cdots\Big),$$
$$f_6 = \frac{1}{\sqrt{3}}\Big(r^2e_1 + re_2 + e_3 + re_4 + r^2e_5 + \cdots\Big),$$
$$\ldots,$$

and $I_i = \{\frac{i(i-1)}{2}+1, \frac{i(i-1)}{2}+2, \ldots, \frac{i(i-1)}{2}+i\}$ for each $i \in \mathbb{N}$. Then $\{I_i\}_{i=1}^{\infty}$ is a sequence of indexing sets and $n_i = |I_i| = i$.

For every $i, j \in \mathbb{N}$ and $k \in I_i$,

$$|\langle f_k, e_j\rangle| = \frac{1}{\sqrt{i}} r^{|i-j|},$$

and

$$\sum_{i=1}^{\infty} |\langle f_i, e_j\rangle|^2 = (r^{j-1})^2 + (r^{j-2})^2 + \cdots + r^2 + 1 + r^2 + r^4 + \cdots \ge 1.$$

Let $C_1 = 1$ and $C_2 = 1$. Then,

$$\sum_{i=1}^{\infty} |\langle f_i, e_j\rangle|^2 \ge C_2^2$$

and

$$\sqrt{2}C_2 = \sqrt{2} > \frac{7}{5} = \frac{1+\frac{1}{6}}{1-\frac{1}{6}} \ge \Big(\frac{1+r}{1-r}\Big)C_1.$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

The second example is a sequence which is polynomially localized with respect to an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H.

EXAMPLE 4.2. Let $s \geq 6$,

$$\begin{aligned} f_1 &= e_1 + 2^{-s}e_2 + 3^{-s}e_3 + \cdots, \\ f_2 &= \frac{1}{\sqrt{2}}\Big(2^{-s}e_1 + e_2 + 2^{-s}e_3 + 3^{-s}e_4 + \cdots\Big), \\ f_3 &= \frac{1}{\sqrt{2}}\Big(2^{-s}e_1 + e_2 + 2^{-s}e_3 + 3^{-s}e_4 + \cdots\Big), \\ f_4 &= \frac{1}{\sqrt{3}}\Big(3^{-s}e_1 + 2^{-s}e_2 + e_3 + 2^{-s}e_4 + 3^{-s}e_5 + \cdots\Big), \\ f_5 &= \frac{1}{\sqrt{3}}\Big(3^{-s}e_1 + 2^{-s}e_2 + e_3 + 2^{-s}e_4 + 3^{-s}e_5 + \cdots\Big), \\ f_6 &= \frac{1}{\sqrt{3}}\Big(3^{-s}e_1 + 2^{-s}e_2 + e_3 + 2^{-s}e_4 + 3^{-s}e_5 + \cdots\Big) \\ &\cdots \end{aligned}$$

and $I_i = \{\frac{i(i-1)}{2} + 1, \frac{i(i-1)}{2} + 2, \ldots, \frac{i(i-1)}{2} + i\}$ for each $i \in \mathbb{N}$. Then for every $i, j \in \mathbb{N}$ and $k \in I_i$,

$$|\langle f_k, e_j\rangle| = \frac{1}{\sqrt{i}}(1 + |i-j|)^{-s},$$

and

$$\sum_{i=1}^{\infty} |\langle f_i, e_j\rangle|^2 = (j^{-s})^2 + ((j-1)^{-s})^2 + \cdots$$
$$+ (2^{-s})^2 + 1 + (2^{-s})^2 + (3^{-s})^2 + \cdots \geq 1.$$

Let $C_1 = 1$ and $C_2 = 1$. Then,

$$\sum_{i=1}^{\infty} |\langle f_i, e_j\rangle|^2 \geq C_2^2$$

and

$$\sqrt{2}C_2 = \sqrt{2} > \frac{7}{5} = \frac{6+1}{6-1} \geq \Big(\frac{s+1}{s-1}\Big)C_1.$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

REMARK. In [5], Gröchenig gave more general definition of localization than in [7]. Essentially, his definition allows only bounded number of elements to be localized at each vector in the given Riesz basis. But, our definition sets no bound for the number of localized elements. In

Example 4.1 and Example 4.2, the number of localized elements f_i for each g_j is unbounded. So Gröchenig's definition is not applicable to these examples.

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