

COMMON FIXED POINTS OF COMPATIBLE MAPS OF TYPE (β) ON FUZZY METRIC SPACES

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ABSTRACT. In this paper we prove a common fixed point theorem for compatible maps of type (β) on fuzzy metric spaces with arbitrary continuous t-norm.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [28]. Deng [4], Erceg [6], Kaleva and Seikkala [15] and Kramosil and Michalek [18] have introduced the concepts of fuzzy metric spaces in different ways. George and Veeramani [8] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [18] in order to get the Hausdorff topology.

Grebiec [9] extended the fixed point theorems of Banach [1] and Edelstein [5] to fuzzy metric spaces in the sense of Kramosil and Michalek [18] whose study is useful in the field of fixed point theorems of contractive type maps. Since then Fang [7] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize and extend some main results of [1, 5, 10-12, 23].

Sessa [24] defined a generalization of commutativity, which called weak commutativity. Further Jungck [14] introduced more generalized commutativity, so called compatibility. Following Grabiec [9], Kramosil and Michalek [18] and Mishra et al. [19] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric spaces which generalize, extend and fuzzify several fixed point theorems for contractive-type maps on metric spaces and other spaces.

Received June 10, 2005.

2000 Mathematics Subject Classification: Primary 54H25; Secondary 47H10.

Key words and phrases: fuzzy metric spaces, common fixed point, compatible maps of type (β) .

Pathak et al. [20] introduced the concept of compatible maps of type (P) in metric spaces, which is equivalent to the concept of compatible maps under some conditions and proved common fixed point theorems in metric spaces. Cho et al. [3] introduced the notion of compatible maps of type (β) in fuzzy metric spaces.

Many authors have studied the fixed point theory in fuzzy metric spaces. The most interesting references are [7, 9-11, 19, 21, 25].

In this paper, we prove common fixed point theorems for four maps satisfying some conditions in fuzzy metric spaces in the sense of George and Veeramani [8]. We also give an example to illustrate our main theorem.

2. Preliminaries

Now, we give some definitions.

DEFINITION 1. (Schweizer and Sklar [22]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *continuous t-norm* if $([0, 1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

DEFINITION 2. (George and Veeramani [8]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

- (fm-1): $M(x, y, t) > 0$,
- (fm-2): $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
- (fm-3): $M(x, y, t) = M(y, x, t)$,
- (fm-4): $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (fm-5): $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous.

DEFINITION 3. (Grabiec [9]). Let $(X, M, *)$ be an FM-space:

- (1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point x in X (denoted by $\lim_{n \rightarrow \infty} x_n = x$) if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$.
- (2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1$$

for all $t > 0$ and $p > 0$.

- (3) An FM-space in which every Cauchy sequence is convergent is said to be complete.

REMARK 1. Since $*$ is continuous, it follows from (fm-4) that the limit of sequence in FM-space is uniquely determined.

Throughout this paper $(X, M, *)$ will denote the fuzzy metric space with the following condition:

(fm-6): $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

LEMMA 1. (Cho [2] and Mishra et al. [19]). *Let $\{y_n\}$ be a sequence in an FM-space $(X, M, *)$ with the condition (fm-6). If there is a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \dots$, then $\{y_n\}$ is a Cauchy sequence in X .

3. Compatible maps of type (β)

In this section, we give the concept of compatible maps of type (β) in FM-spaces and some properties of these maps.

The notion of compatible maps of type (β) in FM-space $(X, M, *)$ was first introduced by Cho et al. [3]. The condition “ $a * a \geq a$ for all $a \in [0, 1]$ ” in properties given by Cho et al. [3] did not play an essential role in proof of our main results. So, we give the properties of compatible maps of type (β) in fuzzy metric spaces with arbitrary continuous t-norms. More details we refer to reader [10,16,17].

DEFINITION 4. (Mishra et al. [19]). Let A and B be maps from an FM-space $(X, M, *)$ into itself. The maps A and B are said to be compatible if

$$\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1$$

for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$$

for some $z \in X$.

DEFINITION 5. (Cho et al. [3]). Let A and B be maps from an FM-space $(X, M, *)$ into itself. The maps A and B are said to be compatible of type (β) if

$$\lim_{n \rightarrow \infty} M(AAx_n, BBx_n, t) = 1$$

for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$$

for some $z \in X$.

REMARK 2. In [14, 20], we can find the equivalent formulations of Definition 4 and 5 and their examples in metric spaces. Such maps are independent of each other and more general than commuting and weakly commuting maps [13, 24].

PROPOSITION 1. *Let $(X, M, *)$ be an FM-space and A, B be continuous maps from X into itself. Then A and B are compatible if and only if they are compatible of type (β) .*

PROPOSITION 2. *Let $(X, M, *)$ be an FM-space and A, B be maps from X into itself. If A and B are compatible of type (β) and $Az = Bz$ for some $z \in X$, then $ABz = BBz = BAz = AAz$.*

PROPOSITION 3. *Let $(X, M, *)$ be an FM-space and A, B be compatible maps of type (β) from X into itself. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$$

for some $z \in X$. Then we have the following:

- (i) $\lim_{n \rightarrow \infty} BBx_n = Az$ if A is continuous at z ,
- (ii) $\lim_{n \rightarrow \infty} AAx_n = Bz$ if B is continuous at z ,
- (iii) $ABz = BAz$ and $Az = Bz$ if A and B are continuous at z .

EXAMPLE 1. Let the set $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$. Clearly $(X, M, *)$ is a fuzzy metric space where $*$ is defined by $a * b = ab$. Define $A, B : X \rightarrow X$ by $Ax = 1$ for $x \in [0, 1]$, $Ax = 1 + x$ for $x \in (1, \infty)$, and $Bx = 1 + x$ for $x \in [0, 1]$, $Bx = 1$ for $x \in [1, \infty)$. Then A and B both are discontinuous at $x = 1$. Consider the sequence $\{x_n\}$ in X defined by $x_n = 1/n$, $n = 1, 2, \dots$. Then we have $\lim_n Ax_n = \lim_n Bx_n = 1$. Further, $\lim_n M(ABx_n, BAx_n, t) \neq 1$ and $\lim_n M(AAx_n, BBx_n, t) = 1$. Therefore A and B are compatible of type (β) but they are not compatible.

EXAMPLE 2. Let the set $X = \mathbb{R}$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$. Clearly $(X, M, *)$ is a fuzzy metric space where $*$ is defined by $a * b = ab$. Define $A, B : X \rightarrow X$ by $Ax = 1/x^3$ for $x \neq 0$, $Ax = 1$ for $x = 0$, and $Bx = 1/x^2$ for $x \neq 0$, $Bx = 2$ for $x = 0$. Then A and B both are discontinuous at $x = 0$. Consider the sequence $\{x_n\}$ in X defined by $x_n = n$, $n = 1, 2, \dots$. Then we have $\lim_n Ax_n = \lim_n Bx_n = 0$. Further, $\lim_n M(ABx_n, BAx_n, t) = 1$ and $\lim_n M(AAx_n, BBx_n, t) = 0$.

Therefore A and B are compatible but they are not compatible of type (β) .

4. Main results

THEOREM 1. *Let $(X, M, *)$ be a complete FM-space and let P, S, T and Q be maps from X into itself such that*

- (1) $PT(X) \cup QS(X) \subset ST(X)$,
- (2) *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} & M^2(Px, Qy, kt) * [M(Sx, Px, kt)M(Ty, Qy, kt)] * M^2(Ty, Qy, kt) \\ & + aM(Ty, Qy, kt)M(Sx, Qy, 2kt) \\ \geq & [pM(Sx, Px, t) + qM(Sx, Ty, t)]M(Sx, Qy, 2kt) \end{aligned}$$

for all x, y in X and $t > 0$ where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

- (3) S and T are continuous and $ST = TS$,
- (4) *the pairs P, S and Q, T are compatible of type (β) .*

Then P, S, T and Q have unique common fixed point in X .

PROOF. Let x_0 be an arbitrary point of X . By (1), we can construct a sequence $\{x_n\}$ in X as follows:

$$PTx_{2n} = STx_{2n+1}, QSt_{2n+1} = STx_{2n+2}, n = 0, 1, 2, \dots$$

Indeed, such a sequence was first introduced in [26, 27].

Now, let $z_n = STx_n$. Then, by (2), we have

$$\begin{aligned} & M^2(PTx_{2n}, QSt_{2n+1}, kt) * [M(STx_{2n}, PTx_{2n}, kt) \\ & M(TSx_{2n+1}, QSt_{2n+1}, kt)] * M^2(TSx_{2n+1}, QSt_{2n+1}, kt) \\ & + aM(TSx_{2n+1}, QSt_{2n+1}, kt)M(STx_{2n}, QSt_{2n+1}, 2kt) \\ \geq & [pM(STx_{2n}, PTx_{2n}, t) + qM(STx_{2n}, TSx_{2n+1}, t)] \\ & M(STx_{2n}, QSt_{2n+1}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M^2(STx_{2n+1}, STx_{2n+2}, kt) * [M(z_{2n}, STx_{2n+1}, kt) \\ & M(z_{2n+1}, STx_{2n+2}, kt)] * M^2(z_{2n+1}, STx_{2n+2}, kt) \\ & + aM(z_{2n+1}, STx_{2n+2}, kt)M(z_{2n}, STx_{2n+2}, 2kt) \\ \geq & [pM(z_{2n}, STx_{2n+1}, t) + qM(z_{2n}, z_{2n+1}, t)]M(z_{2n}, STx_{2n+2}, 2kt) \end{aligned}$$

then

$$\begin{aligned} & M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)M(z_{2n+1}, z_{2n+2}, kt)] \\ & * M^2(z_{2n+1}, z_{2n+2}, kt) + aM(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \\ \geq & [pM(z_{2n}, z_{2n+1}, t) + qM(z_{2n}, z_{2n+1}, t)] M(z_{2n}, z_{2n+2}, 2kt) \end{aligned}$$

so

$$\begin{aligned} & M^2(z_{2n+1}, z_{2n+2}, kt) * [M(z_{2n}, z_{2n+1}, kt)M(z_{2n+1}, z_{2n+2}, kt)] \\ & + aM(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \\ \geq & [p + q] M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M(z_{2n+1}, z_{2n+2}, kt) [M(z_{2n}, z_{2n+1}, kt) * M(z_{2n+1}, z_{2n+2}, kt)] \\ & + aM(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \\ \geq & [p + q] M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \\ & + aM(z_{2n+1}, z_{2n+2}, kt)M(z_{2n}, z_{2n+2}, 2kt) \\ \geq & [p + q] M(z_{2n}, z_{2n+1}, t)M(z_{2n}, z_{2n+2}, 2kt). \end{aligned}$$

Thus, it follows that

$$M(z_{2n+1}, z_{2n+2}, kt) \geq M(z_{2n}, z_{2n+1}, t)$$

$0 < k < 1$ and for all $t > 0$.

Similarly, we also have

$$M(z_{2n+2}, z_{2n+3}, kt) \geq M(z_{2n+1}, z_{2n+2}, t)$$

$0 < k < 1$ and for all $t > 0$.

In general, for $m = 1, 2, \dots$, we have

$$M(z_{m+1}, z_{m+2}, kt) \geq M(z_m, z_{m+1}, t)$$

$0 < k < 1$ and for all $t > 0$. Hence, by Lemma 1, $\{z_n\}$ is a Cauchy sequence in X . Since $(X, M, *)$ is complete, it converges to a point z in X . Since $\{PTx_{2n}\}$ and $\{Qsx_{2n+1}\}$ are subsequences of $\{z_n\}$, $PTx_{2n} \rightarrow z$ and $Qsx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$.

Let $y_n = Tx_n$ and $w_n = Sx_n$ for $n = 1, 2, \dots$. Then, we have $Py_{2n} \rightarrow z$, $Sy_{2n} \rightarrow z$, $Tw_{2n+1} \rightarrow z$ and $Qw_{2n+1} \rightarrow z$,

$$M(PPy_{2n}, SSy_{2n}, t) \rightarrow 1 \text{ and } M(QQw_{2n+1}, TTW_{2n+1}, t) \rightarrow 1$$

as $n \rightarrow \infty$. Moreover, by the continuity of T and Proposition 3, we have

$$TQw_{2n+1} \rightarrow Tz \text{ and } QQw_{2n+1} \rightarrow Tz$$

as $n \rightarrow \infty$. Now, taking $x = y_{2n}$ and $y = Qw_{2n+1}$ in (2), we have

$$\begin{aligned} & M^2(Py_{2n}, QQw_{2n+1}, kt) * [M(Sy_{2n}, Py_{2n}, kt) \\ & M(TQw_{2n+1}, QQw_{2n+1}, kt)] * M^2(TQw_{2n+1}, QQw_{2n+1}, kt) \\ & + aM(TQw_{2n+1}, QQw_{2n+1}, kt)M(Sy_{2n}, QQw_{2n+1}, 2kt) \\ \geq & [pM(Sy_{2n}, Py_{2n}, t) \\ & + qM(Sy_{2n}, TQw_{2n+1}, t)]M(Sy_{2n}, QQw_{2n+1}, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M^2(z, Tz, kt) * [M(z, z, kt)M(Tz, Tz, kt)] * M^2(Tz, Tz, kt) \\ & + aM(Tz, Tz, kt)M(z, Tz, 2kt) \\ \geq & [pM(z, z, t) + qM(z, Tz, t)]M(z, Tz, 2kt) \end{aligned}$$

then, it follows that

$$M^2(z, Tz, kt) + aM(z, Tz, 2kt) \geq [p + qM(z, Tz, t)]M(z, Tz, 2kt)$$

and since $M(x, y, \cdot)$ is non-decreasing for all x, y in X , we have

$$\begin{aligned} & M(z, Tz, 2kt)M(z, Tz, t) + aM(z, Tz, 2kt) \\ \geq & [p + qM(z, Tz, t)]M(z, Tz, 2kt) \end{aligned}$$

thus

$$M(z, Tz, t) + a \geq p + qM(z, Tz, t)$$

and

$$M(z, Tz, t) \geq \frac{p-a}{1-q} = 1$$

for all $t > 0$ so $z = Tz$. Similarly, we have $z = Sz$.

Now, taking $x = y_{2n}$ and $y = z$ in (2), we have

$$\begin{aligned} & M^2(Py_{2n}, Qz, kt) * [M(Sy_{2n}, Py_{2n}, kt)M(Tz, Qz, kt)] \\ & * M^2(Tz, Qz, kt) + aM(Tz, Qz, kt)M(Sy_{2n}, Qz, 2kt) \\ \geq & [pM(Sy_{2n}, Py_{2n}, t) + qM(Sy_{2n}, Tz, t)]M(Sy_{2n}, Qz, 2kt) \end{aligned}$$

and

$$\begin{aligned} & M^2(z, Qz, kt) * [M(z, z, kt)M(z, Qz, kt)] * M^2(z, Qz, kt) \\ & + aM(z, Qz, kt)M(z, Qz, 2kt) \\ \geq & [pM(z, z, t) + qM(z, z, t)]M(z, Qz, 2kt) \end{aligned}$$

then

$$\begin{aligned} & M^2(z, Qz, kt) * M(z, Qz, kt) + aM(z, Qz, kt)M(z, Qz, 2kt) \\ \geq & (p + q) M(z, Qz, 2kt) \end{aligned}$$

so

$$\begin{aligned} & M(z, Qz, kt) [M(z, Qz, kt) * 1] + aM(z, Qz, kt)M(z, Qz, 2kt) \\ \geq & (p + q) M(z, Qz, 2kt) \end{aligned}$$

and since $M(x, y, \cdot)$ is non-decreasing for all x, y in X , we have

$$\begin{aligned} & M(z, Qz, 2kt)M(z, Qz, kt) + aM(z, Qz, kt)M(z, Qz, 2kt) \\ \geq & (p + q) M(z, Qz, 2kt). \end{aligned}$$

Thus it follows that

$$M(z, Qz, kt) + aM(z, Qz, kt) \geq p + q$$

and

$$M(z, Qz, kt) \geq \frac{p + q}{1 + a} = 1$$

$0 < k < 1$ and for all $t > 0$ so $z = Qz$. Similarly, we have $z = Pz$. Therefore, z is a common fixed point of P, Q, S and T .

Let v be second common fixed point of P, Q, S and T . Then using inequality (2), we have

$$\begin{aligned} & M^2(Pz, Qv, kt) * [M(Sz, Pz, kt)M(Tv, Qv, kt)] * M^2(Tv, Qv, kt) \\ & + aM(Tv, Qv, kt)M(Sz, Qv, 2kt) \\ \geq & [pM(Sz, Pz, t) + qM(Sz, Tv, t)] M(Sz, Qv, 2kt) \end{aligned}$$

so

$$M^2(z, v, kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)] M(z, v, 2kt)$$

and

$$M(z, v, t)M(z, v, 2kt) + aM(z, v, 2kt) \geq [p + qM(z, v, t)] M(z, v, 2kt).$$

Thus, it follows that

$$M(z, v, t) \geq \frac{p - a}{1 - q} = 1$$

for all $t > 0$ so $z = v$. Hence P, S, T and Q have unique common fixed point. \square

If we put $a = 0$ in Theorem 1, we have the following result:

COROLLARY 1. Let $(X, M, *)$ be a complete FM-space and let P, S, T and Q be maps from X into itself such that the conditions (1), (3) and (4) of the Theorem 1 hold and there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} & M^2(Px, Qy, kt) * [M(Sx, Px, kt)M(Ty, Qy, kt)] * M^2(Ty, Qy, kt) \\ & \geq [pM(Sx, Px, t) + qM(Sx, Ty, t)] M(Sx, Qy, 2kt) \end{aligned}$$

for all x, y in X and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$. Then P, S, T and Q have unique common fixed point in X .

If we put $S = T$ in Theorem 1, we have the following result:

COROLLARY 2. Let $(X, M, *)$ be a complete FM-space and let P, S and Q be maps from X into itself such that

- (1) $P(X) \cup Q(X) \subset S(X)$,
- (2) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} & M^2(Px, Qy, kt) * [M(Sx, Px, kt)M(Sy, Qy, kt)] * M^2(Sy, Qy, kt) \\ & + aM(Sy, Qy, kt)M(Sx, Qy, 2kt) \\ & \geq [pM(Sx, Px, t) + qM(Sx, Sy, t)] M(Sx, Qy, 2kt) \end{aligned}$$

for all x, y in X and $t > 0$ where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

- (3) S is continuous,
- (4) the pairs P, S and Q, S are compatible of type (β) .

Then P, S and Q have unique common fixed point in X .

If we put $S = T$ and $P = Q$ in Theorem 1, we have the following result:

COROLLARY 3. Let $(X, M, *)$ be a complete FM-space and let S and P be maps from X into itself such that

- (1) $P(X) \subset S(X)$,
- (2) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} & M^2(Px, Py, kt) * [M(Sx, Px, kt)M(Sy, Py, kt)] * M^2(Sy, Py, kt) \\ & + aM(Sy, Py, kt)M(Sx, Py, 2kt) \\ & \geq [pM(Sx, Px, t) + qM(Sx, Sy, t)] M(Sx, Py, 2kt) \end{aligned}$$

for all x, y in X and $t > 0$ where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

- (3) S is continuous,
- (4) P and S are compatible of type (β) .

Then P and S have unique common fixed point in X .

If we put $S = T = I_X$ (the identity map on X) in Theorem 1, we have the following result:

COROLLARY 4. *Let $(X, M, *)$ be a complete FM-space and let P, Q be maps from X into itself. If there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} & M^2(Px, Qy, kt) * [M(x, Px, kt)M(y, Qy, kt)] * M^2(y, Qy, kt) \\ & + aM(y, Qy, kt)M(x, Qy, 2kt) \\ \geq & [pM(x, Px, t) + qM(x, y, t)] M(x, Qy, 2kt) \end{aligned}$$

for all x, y in X and $t > 0$ where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$. Then P and Q have unique common fixed point in X .

The following example illustrates our main Theorem.

EXAMPLE 3. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t > 0$ define $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$. Clearly $(X, M, *)$ is a complete fuzzy metric space where $*$ is defined by $a * b = ab$. Let P, S, T and Q be maps from X into itself defined as

$$Px = \frac{x}{4}, Sx = \frac{x}{2}, Tx = x, Qx = 0$$

for all $x \in X$. Then

$$PT(X) \cup QS(X) = \left\{ \frac{1}{4n} : n \in \mathbb{N} \right\} \cup \{0\} \subset \left\{ \frac{1}{2n} : n \in \mathbb{N} \right\} \cup \{0\} = ST(X).$$

Clearly $ST = TS$ and S, T are continuous. If we take $k = \frac{1}{2}$ and $t = 1$, we see that the condition (2) of the main Theorem is also satisfied. Moreover, the maps P and S are compatible of type (β) if $\lim_{n \rightarrow \infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Sx_n = 0$ for some $0 \in X$. Similarly, the maps Q and T are also compatible of type (β) . Thus, all the conditions of main Theorem are satisfied and 0 is the unique common fixed point of P, S, T and Q .

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