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TRACE FORMULAS ON FINITE GROUPS

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ABSTRACT. In this paper, we study the right regular representation R_Γ of a finite group G on the vector space consisting of vector valued functions on $\Gamma\backslash G$ with a subgroup Γ of G and give a trace formula using the work of M. -F. Vignéras.

1. Introduction

Following the suggestion of D. Kazhdan, James Arthur proved the so-called *local trace formula* for a reductive group $G(F)$ over a non-archimedean local field F investigating the regular representation of $G(F) \times G(F)$ on the Hilbert space $L^2(G(F))$ (cf. [1]-[4]). Motivated by the work of J. Arthur on the local trace formula, M. -F. Vignéras (cf. [10]) gave a trace formula for the regular representation of $G \times G$ in $L^2(G)$ for a finite group G. In this paper, motivated by the above mentioned work of M. -F. Vignéras, we study the trace formula of the right regular representation R_Γ of a finite group G on the vector space of consisting of all vector valued functions on the coset space $\Gamma\backslash G$ for a subgroup Γ. We derive the trace formula for $R_\Gamma(f)$ using the result of M. -F. Vignéras (cf. [10]). This trace formula simplifies the proofs of the well known results on a finite group.

In this paper, we shall study the right regular representation R of G on the vector space $V[\Gamma\backslash G]$ consisting of all vector valued functions on $\Gamma\backslash G$ with values in V and give a trace formula for $R_\Gamma(f)$ with a function f on G. Using this formula, we derive some well known results.

NOTATION. We denote by $\mathbb{C}$ the complex number field. For a finite set A, we denote by $|A|$ the cardinality of A. For a finite group G, we denote

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by $\hat{G}$ the set of all equivalence classes of irreducible representations of G. For $\lambda \in \hat{G}$, we let d_λ be the degree of λ.

2. Trace formula

Let Γ be a subgroup of a finite group G. Let V be a finite dimensional complex vector space. We let $X_\Gamma = \Gamma\backslash G$ and denote by V_Γ the vector space consisting of all vector valued functions $\varphi : X_\Gamma \longrightarrow V$. We note that G acts on X_Γ transitively by right multiplication. We let R_Γ be the right regular representation of G on V_Γ, namely,

$$(R_\Gamma(g)\varphi)(x) = \varphi(xg), \quad g \in G,\ \varphi \in V_\Gamma \text{ and } x \in X_\Gamma.$$

For any $g \in G$, we set $X_\Gamma^g = \{\, x \in X_\Gamma \mid xg = x \,\}$.

THEOREM 1. *Let G, Γ, V_Γ, X_Γ and X_Γ^g be as above. We let χ_{R_Γ} be the character of the regular representation R_Γ of G. For each $\lambda \in \hat{G}$, we let χ_λ be the character of λ. We assume that $R_\Gamma = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V)\lambda$ is the decomposition of R_Γ into irreducibles. Here $m_\lambda(\Gamma, V)$ denotes the multiplicity of λ in R_Γ. Then*

$$\chi_{R_\Gamma}(g) = \dim_{\mathbb{C}} V \cdot |X_\Gamma^g| \quad \text{for all } g \in G, \tag{1}$$

$$m_\lambda(\Gamma, V) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g \in G} |X_\Gamma^g|\, \chi_\lambda(g^{-1}) \quad \text{for each } \lambda \in \hat{G}, \tag{2}$$

$$|G|^2 = |\Gamma| \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_\lambda\, |X_\Gamma^g|\, \chi_\lambda(g^{-1}). \tag{3}$$

For a function $f \in \mathbb{C}[G]$, we define the endomorphism $R_\Gamma(f)$ of V_Γ by

$$R_\Gamma(f) = \sum_{g \in G} f(g) R_\Gamma(g).$$

Then for a function $f \in \mathbb{C}[G]$,

$$tr\, R_\Gamma(f) = \dim_{\mathbb{C}} V \cdot \sum_{g \in G} f(g)\, |X_\Gamma^g|, \tag{4}$$

and for any $f_1,\, f_2 \in \mathbb{C}[G]$,

$$tr\, R_{\{1\}}(f_1 * f_2) = \sum_{\lambda\in\hat{G}} d_\lambda\, tr\,(\mathcal{F}f_1(\lambda)\mathcal{F}f_2(\lambda)). \tag{5}$$

Here $f_1 * f_2$ *denotes the convolution of* f_1 *and* f_2 *defined by*

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(h) f_2(h^{-1}g), \qquad g \in G,$$

d_λ *is the degree of* λ *and* $\mathcal{F}f(\lambda)$ *is the Fourier transform of* f *defined by*

$$\mathcal{F}f(\lambda) = \sum_{g\in G} f(g)\,\lambda(g), \quad \lambda \in \hat{G}.$$

PROOF. We let $V[X]$ be the set of all functions $\phi : X \longrightarrow V$ with values in V. We describe a basis for the vector space $V[X]$ and its dual basis. If $V = \mathbb{C}$, the vector space $\mathbb{C}[X]$ has a basis $\{\, \delta_x \,|\, x \in X \,\}$, where

$$\delta_x(y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in X$ and $v \in V$, we define the function $\delta_x \otimes v : X \longrightarrow V$ by

$$(\delta_x \otimes v)(y) := \begin{cases} v & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\, v_1, \dots, v_n \,\}$ be a basis for V with $\dim_{\mathbb{C}} V = n$. Then it is easy to see that the set $\{\, \delta_x \otimes v_k \,|\, x \in X,\ 1 \le k \le n \,\}$ forms a basis for $V[X]$. Let V^* be the dual space of V. For $x \in X$ and $v^* \in V^*$, we define the linear functional $\delta_x^* \otimes v^* : V[X] \longrightarrow \mathbb{C}$

$$(\delta_x^* \otimes v^*)\,(\phi) := < \phi(x),\, v^* >, \quad \phi \in V[X].$$

Suppose $\{\, v_1^*, \dots, v_n^* \,\}$ is the dual basis of a basis $\{\, v_1, \dots, v_n \,\}$. Then we see easily that the set $\{\, \delta_x^* \otimes v_k^* \mid x \in X,\ 1 \le k \le n \,\}$ forms a basis for the dual space $V[X]^*$. We also see that for each $g \in G$,

$$< R_\Gamma(g)(\delta_x \otimes v_k),\, (\delta_x^* \otimes v_k^*) >= \begin{cases} 1 & \text{if } xg = x, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$\chi_{R_\Gamma}(g) = \mathrm{tr}\, R_\Gamma(g) = n \cdot |X_\Gamma^g|$ for each $g \in G$. This proves Formula (1).

We define the hermitian inner product $<\ ,\ >$ on the group algebra $\mathbb{C}[G]$ by

$$< f_1, f_2 >= \frac{1}{|G|} \sum_{g\in G} f_1(g)\, \overline{f_2(g)}, \quad f_1, f_2 \in \mathbb{C}[G].$$

Since $\chi_{R_\Gamma} = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V)\, \chi_\lambda$, we have

$$\begin{aligned} & m_\lambda(\Gamma, V) \\ = & < \chi_{R_\Gamma}, \chi_\lambda > \\ & \quad (\text{by Schur orthogonality relation (cf. [6], p. 148)}) \\ = & \frac{1}{|G|} \sum_{g\in G} \chi_{R_\Gamma}(g)\, \chi_\lambda(g^{-1}) = \frac{n}{|G|} \sum_{g\in G} |X_\Gamma^g|\, \chi_\lambda(g^{-1}). \quad (\text{by (1)}) \end{aligned}$$

This proves Formula (2).

We observe that

$$\dim_{\mathbb{C}} V_\Gamma = \frac{|G|}{|\Gamma|} \cdot \dim_{\mathbb{C}} V. \tag{6}$$

Since $R_\Gamma = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V)\lambda$, we see that

$$\dim_{\mathbb{C}} V_\Gamma = \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V) \cdot d_\lambda, \tag{7}$$

where d_λ denotes the degree of $\lambda \in \hat{G}$. By substituting (2) into (7), we get

$$\dim_{\mathbb{C}} V_\Gamma = \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_\lambda\, |X_\Gamma^g|\, \chi_\lambda(g^{-1}). \tag{8}$$

Therefore according to (6) and (8), we obtain Formula (3).

Let $f \in \mathbb{C}[G]$. Then we obtain

$$\begin{aligned} \operatorname{tr} R_\Gamma(f) = \operatorname{tr} \left(\sum_{g\in G} f(g) R_\Gamma(g) \right) &= \sum_{g\in G} f(g) \operatorname{tr}\left(R_\Gamma(g)\right) \\ &= \sum_{g\in G} f(g) \chi_{R_\Gamma}(g) \\ &= \dim_{\mathbb{C}} V \cdot \sum_{g\in G} f(g)\, |X_\Gamma^g|. \quad (\text{by (1)}) \end{aligned}$$

This proves Formula (4).

Finally we shall prove Formula (5). If we take $\Gamma = \{1\}$, then $X_\Gamma^1 = G$ and $X_\Gamma^g = \emptyset$ if $g \neq 1$. Thus by Formula (4), we get

$$\operatorname{tr} R_{\{1\}}(f) = \dim_{\mathbb{C}} V \cdot |G|\, f(1).$$

Therefore

$$f(1) = \frac{\operatorname{tr} R_{\{1\}}(f)}{|G| \dim_{\mathbb{C}} V}. \tag{9}$$

We recall the fact (see [6], Corollary 3.4.5) that for any $f_1, f_2 \in \mathbb{C}[G]$, the following Plancherel formula holds:

$$(f_1 * f_2)(1) = \frac{\operatorname{tr} R(f_1 * f_2)}{|G|} = \frac{1}{|G|} \sum_{\lambda \in \hat{G}} d_\lambda \operatorname{tr}_{V_\lambda} (\mathcal{F} f_1(\lambda) \mathcal{F} f_2(\lambda)), \tag{10}$$

where $\operatorname{tr}_{V_\lambda}(A)$ denotes the trace of an endomorphism $A : V_\lambda \longrightarrow V_\lambda$ with the representation space of λ.

On the other hand, for any $f_1, f_2 \in \mathbb{C}[G]$, we get

$$(f_1 * f_2)(1) = \frac{\operatorname{tr} R_{\{1\}}(f_1 * f_2)}{|G|} \qquad \text{(by (9))}. \tag{11}$$

Hence according to (10) and (11), we obtain Formula (5). □

COROLLARY 2. *(a)* $|G| = \sum_{\lambda \in \hat{G}} d_\lambda^2$.

(b) $|G| = \sum_{g \in G} \sum_{\lambda \in \hat{G}} d_\lambda \chi_\lambda(g^{-1})$.

(c) Let λ_0 *be the trivial representation of* G. *Then*

$$m_{\lambda_0}(\Gamma, V) = \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{g \in G} |X_\Gamma^g|.$$

In particular, $m_{\lambda_0}(\{1\}, \mathbb{C}) = 1$.

(d) For any $f \in \mathbb{C}[G]$,

$$f(1) = \frac{\operatorname{tr} R_{\{1\}}(f)}{|G| \cdot \dim_{\mathbb{C}} V}.$$

(e) For a subgroup Γ *of* G,

$$\sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V)^2 = \frac{(\dim_{\mathbb{C}} V)^2}{|G|} \cdot \sum_{g \in G} |X_\Gamma^g|\, |X_\Gamma^{g^{-1}}|.$$

(f) For each $\lambda \in \hat{G}$, $m_\lambda(\{1\}, V) = d_\lambda \, dim_{\mathbb{C}} V \neq 0$. *That is, each* $\lambda \in \hat{G}$ *occurs in the regular representation* $(R_{\{1\}}, V[G])$ *of* G *with multiplicity* $d_\lambda \cdot dim_{\mathbb{C}} V$.

(g) For each $\lambda \in \hat{G}$,

$$\sum_{g \in G} \chi_\lambda(g) = \frac{|G|}{dim_{\mathbb{C}} V} \cdot m_\lambda(G, V).$$

PROOF. (a) If we take $\Gamma = \{1\}$, we see easily that $X^1_{\{1\}} = G$ and $X^g_{\{1\}} = \emptyset$ if $g \neq 1$. Then we get

$$\begin{aligned} |G|^2 &= 1 \cdot \sum_{\lambda \in \hat{G}} d_\lambda \cdot |X^1_{\{1\}}| \cdot \chi_\lambda(1) \qquad \text{(by (3))} \\ &= |G| \sum_{\lambda \in \hat{G}} d_\lambda^2. \qquad \text{(because } \chi_\lambda(1) = d_\lambda) \end{aligned}$$

This proves Formula (a). We recall that another proof of (a) follows from the fact that the group algebra $\mathbb{C}[G]$ is isomorphic to $\sum_{\lambda \in \hat{G}} \mathrm{End}(V_\lambda)$ as algebras, where V_λ is the representation space of $\lambda \in \hat{G}$ (cf. [9]).

(b) We take $\Gamma = G$. It is easy to see that $X_G = \{\bar{1}\}$ is a point and $X^g_G = X_G$ for all $g \in G$. According to Formula (3), we obtain

$$|G|^2 = |G| \sum_{\lambda \in \hat{G}} \sum_{g \in G} d_\lambda \chi_\lambda(g^{-1}).$$

This proves the statement (b).

(c) It follows from Formula (2).

(d) We take $\Gamma = \{1\}$. Then $X^1_\Gamma = G$ and $X^g_\Gamma = \emptyset$ for $g \neq 1$. From Formula (4), we obtain

$$\mathrm{tr}\, R_{\{1\}}(f) = \dim_{\mathbb{C}} V \cdot |G| \, f(1).$$

(e) By Schur orthogonality relation, $< \chi_{R_\Gamma}, \chi_{R_\Gamma} >= \sum_{\lambda \in \hat{G}} m_\lambda(\Gamma, V)^2$. On the other hand, according to the formula (1),

$$\begin{aligned} < \chi_{R_\Gamma}, \chi_{R_\Gamma} > &= \frac{1}{|G|} \sum_{g \in G} \chi_{R_\Gamma}(g) \chi_{R_\Gamma}(g^{-1}) \\ &= \frac{(\dim_{\mathbb{C}} V)^2}{|G|} \cdot \sum_{g \in G} |X^g_\Gamma| \, |X^{g^{-1}}_\Gamma|. \end{aligned}$$

(f) We take $\Gamma = \{1\}$. Then $X^1_{\{1\}} = G$, $X^g_{\{1\}} = \emptyset$ for $g \neq 1$, and $V_{\{1\}} = V[G]$. Therefore we obtain the desired result from Formula (2).

(g) We take $\Gamma = G$. We see easily that $|X^g_G| = 1$ for all $g \in G$. According to Formula (2), we get

$$m_\lambda(G,V) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g\in G} \chi_\lambda(g^{-1}) = \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g\in G} \chi_\lambda(g).$$

Hence this proves the statement (g). □

THEOREM 3. *For each $f \in \mathbb{C}[G]$, we have the following trace formula*

$$tr\, R_\Gamma(f) = \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{g\in G} |X^g_\Gamma|\, |Z_g|\, f(C_g), \tag{12}$$

where Z_g is the centralizer of g in G, C_g is the conjugacy class of g and $f(C_g) = \sum_{h\in C_g} f(h)$.

PROOF. For $f \in \mathbb{C}[G]$ and $\lambda \in \hat{G}$, we define

$$\lambda(f) := \sum_{g\in G} f(g)\lambda(g).$$

Investigating the spectral decomposition of the regular representation R of $G \times G$ on $\mathbb{C}[G]$ defined by

$$(R(g_1,g_2)F)(g) = F(g_1^{-1} g g_2), \quad g, g_1, g_2 \in G,\ F \in \mathbb{C}[G].$$

M. -F. Vignéras (cf. [10], p.284) obtained the following trace formula

$$|Z_g| F(C_g) = \sum_{\pi\in\hat{G}} \chi_\pi(g^{-1}) \operatorname{tr} \pi(F) \tag{13}$$

for any $g \in G$ and $F \in \mathbb{C}[G]$.

Let $R_\Gamma = \sum_{\lambda\in\hat{G}} m_\lambda(\Gamma,V)\lambda$ be the decomposition of R_Γ into irreducibles. If $f \in \mathbb{C}[G]$,

$$\begin{aligned}
\operatorname{tr} R_\Gamma(f) &= \sum_{\lambda\in\hat{G}} m_\lambda(\Gamma,V) \operatorname{tr} \lambda(f) \\
&= \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{\lambda\in\hat{G}} \sum_{g\in G} |X^g_\Gamma|\, \chi_\lambda(g^{-1}) \operatorname{tr} \lambda(f) \qquad \text{(by (2))} \\
&= \frac{\dim_{\mathbb{C}} V}{|G|} \sum_{g\in G} |X^g_\Gamma| \left(\sum_{\lambda\in\hat{G}} \chi_\lambda(g^{-1}) \operatorname{tr} \lambda(f) \right) \\
&= \frac{\dim_{\mathbb{C}} V}{|G|} \cdot \sum_{g\in G} |X^g_\Gamma|\, |Z_g|\, f(C_g).
\end{aligned}$$

The last equality follows from Formula (13). □

COROLLARY 4. *Let* Γ *be a subgroup of* G. *Then for any* $f \in \mathbb{C}[G]$, *we have the following identity*

$$|G| \sum_{g \in G} f(g) |X_\Gamma^g| = \sum_{g \in G} |X_\Gamma^g| |Z_g| f(C_g), \tag{13}$$

where Z_g *is the centralizer of* g *in* G, C_g *is the conjugacy class of* g *and* $f(C_g) = \sum_{h \in C_g} f(h)$.

PROOF. The proof follows immediately from Formula (4) and the trace formula (12). □

REMARK 5. The trace formula (12) is similar to the trace formula on the adele group. For the trace formula on the adele group, we refer to [1]-[5], [7], and [8].

REMARK 6. If $\Gamma \neq \{1\}$, the multiplicity $m_\lambda(\Gamma, V)$ of some $\lambda \in \hat{G}$ may be zero. It is natural to ask when $m_\lambda(\Gamma, V)$ is not zero. Namely, which $\lambda \in \hat{G}$ does occur in the regular representation (R_Γ, V_Γ) of G?

References

[1] J. Arthur, *The trace formula and Hecke operators: Number theory, trace formulas and discrete groups (Oslo, 1987)*, Academic Press (1989), 11–27.

[2] ———, *Towards a local trace formula: algebraic analysis, geometry and number theory (Baltimore, MD, 1988)*, Johns Hopkins Univ. Press, Baltimore, MD (1989), 1–23.

[3] ———, *Some problems in local harmonic analysis : harmonic Analysis on reductive groups (Brunswick, ME, 1989)*, Prog. Math. Birkhäuser, Boston vol 101 (1991), 57–78.

[4] ———, *A local trace formula*, IHES Publication Math. **73** (1991), 5–96.

[5] S. Gelbart, *Lectures on Arthur-Selberg Trace Formula*, American Math. Soc., Providence, 1996.

[6] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge University Press, 1998.

[7] A. W. Knapp, *Theoretical Aspect of the Trace Formula for* $GL(2)$, *Proc. Symosia in Pure Math.*, American Math. Soc. **61** (1997), 355–405.

[8] A. W. Knapp and J. D. Rogawski, *Applications of the Trace Formula, Proc. Symosia in Pure Math.*, American Math. Soc. **61** (1997), 413–431.

[9] J. -P. Serre, *Linear representations of finite groups*, Springer-Verlag, 1977.

[10] M. -F. Vignéras, *An elementary introduction to the local trace formula of J. Arthur. The case of finite groups*, Jber. d. Dt. Math.-Verein., Jubiläumstagung 1990, B.G. Teubner Stuttgart (1992), 281–296.

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