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DIVISIBLE ENVELOPES OF THE FORM $T \hookrightarrow S^{-1}T$

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ABSTRACT. In this article we show that if R is a Noetherian ring with the global dimension at most one, then every S-torsion free R-module T has a divisible envelope of the form $T \hookrightarrow S^{-1}T$.

1. Preliminaries

Throughout this paper, R denotes a commutative ring with identity and all modules are unitary. We denote by S the set of all nonzero-divisors of R. For any R-module M, the injective envelope of M is denoted by $E(M)$.

We say that an R-module D is *divisible* if $D = sD$ for all $s \in S$ and we let $\mathcal{D}$ be the class of all divisible modules.

We first recall the definitions of envelopes and covers.

DEFINITION 1.1. ([5]) Let $\mathcal{X}$ be a class of R-modules which is closed under isomorphisms, direct summands, and finite direct sums. An $\mathcal{X}$-*envelope* of a module M is a linear map $\phi : M \to X$ with $X \in \mathcal{X}$ such that the following two conditions hold ;

(1) $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to 0$ is exact for any $X' \in \mathcal{X}$.
(2) Any $f : X \to X$ with $f \circ \phi = \phi$ is an automorphism of X.

If $\phi : M \to X$ satisfies (1), and perhaps not (2), ϕ is called an $\mathcal{X}$-*preenvelope* of M.

In particular, if $\mathcal{X} = \mathcal{D}$, $\mathcal{X}$-(pre)envelopes are called *divisible* (pre)*envelopes.*

Dually an $\mathcal{X}$-*precover* of M is a linear map $\psi : X \to M$ with $X \in \mathcal{X}$ such that $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$ is exact and if an $\mathcal{X}$-precover $\psi : X \to M$ of M satisfies that any $f : X \to X$ with $\psi \circ f = \psi$ is an automorphism of X, then $\psi : X \to M$ is called an $\mathcal{X}$-*cover* of M.

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Note that an $\mathcal{X}$-envelope is unique up to isomorphism if it exists. And any divisible preenvelope is an injection since every injective module is divisible.

We recall that for a class $\mathcal{X}$ of modules, $\mathcal{X}^{\perp}$ consists of all modules N such that $Ext^1_R(X, N) = 0$ for all $X \in \mathcal{X}$ and ${}^{\perp}\mathcal{X}$ consists of all M such that $\mathrm{Ext}^1_R(M, X) = 0$ for all $X \in \mathcal{X}$ (see [5, p.29]).

LEMMA 1.2. (Wakamatsu, [5, Lemma 2.1.2]) *Let $\mathcal{X}$ be closed under extensions. If $\phi : M \to X$ is an $\mathcal{X}$-envelope of M, then* $\operatorname{coker}\phi \in {}^{\perp}\mathcal{X}$.

Conversely, if $\phi : M \to X$ is an injection with $X \in \mathcal{X}$ and $E = \operatorname{coker}\phi \in {}^{\perp}\mathcal{X}$, then for any $X' \in \mathcal{X}$, $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to \mathrm{Ext}^1_R(E, X') = 0$ is exact. So ϕ is an $\mathcal{X}$-preenvelope of M. Such a preenvelope is called a *special $\mathcal{X}$-preenvelope* of M.

REMARK 1.3. Let $\phi : M \to D$ be a divisible envelope. For R-linear maps $f : M \to M$ and $g : D \to D$, if $\phi \circ f = g \circ \phi$ and if f is an isomorphism, then g is also an isomorphism. So if $\bar{S} \subset R$ is a multiplicatively closed set, $\bar{S}^{-1}M = M$ implies $\bar{S}^{-1}D = D$.

DEFINITION 1.4. ([5]) A module M over a Gorenstein ring R is said to be *Gorenstein injective* if there is an exact sequence

$$\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \xrightarrow{d^0} E^1 \longrightarrow E^2 \longrightarrow \cdots$$

of injective R-modules such that $M = kerd^0$ and $\mathrm{Hom}_R(E, -)$ leaves the sequence exact whenever E is injective.

If R is a Gorenstein ring with Krull dimension at most one, it is known that an R-module M is Gorenstein injective if and only if it is divisible ([3]).

DEFINITION 1.5. For an R-module M, the submodule $t(M) = \{x \in M | sx = 0 \text{ for some } s \in S\}$ is called the *S-torsion submodule* of M. If $t(M) = 0$, then M is said to be *S-torsion free* and if $t(M) = M$, then M is called an *S-torsion* module.

It is easy to prove that the injective envelope of any S-torsion free module is S-torsion free. So it is natural to ask if the corresponding result for divisible envelopes (when they exist). We do not know the answer to this question but we do have the following result.

PROPOSITION 1.6. ([3]) *If every S-torsion free R-module has a divisible envelope which is S-torsion free, then every R-module has a divisible envelope.*

2. Main results

Motivated the Proposition 1.6 we now consider the existence of divisible envelopes of S-torsion free modules. We also note that if R is a Gorenstein ring with Krull dimension at most one, then $R \hookrightarrow K = S^{-1}R$ is a divisible envelope.

LEMMA 2.1. *If $\phi : T \hookrightarrow S^{-1}T$ is a divisible preenvelope of an S-torsion free R-module T, then ϕ is a divisible envelope.*

PROOF. If $f : S^{-1}T \to S^{-1}T$ is any R-linear map with $f \circ \phi = \phi$, then for any $a/s \in S^{-1}T$, $sf(a/s) = f(a/1) = f \circ \phi(a) = \phi(a) = a/1$. Since $S^{-1}T$ is S-torsion free and divisible, $f(a/s) = a/s$. So f is the identity map on $S^{-1}T$. Thus ϕ is a divisible envelope. □

For any flat module F over a Gorenstein ring with Krull dimension at most one, $F \hookrightarrow S^{-1}F$ is a divisible envelope. But it is not true that for every S-torsion free R-module T, $T \hookrightarrow S^{-1}T$ is a divisible envelope even if R is a Gorenstein domain with Krull dimension one ([3]).

Now we consider the problem : When is it true that for an S-torsion free R-module T, $T \hookrightarrow S^{-1}T$ is a divisible envelope?

If R is a Dedekind domain, every divisible R-module is injective. Since T is essential in $S^{-1}T$, $T \hookrightarrow S^{-1}T$ is the injective envelope, and so it is a divisible envelope.

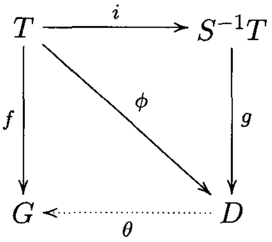
THEOREM 2.2. *If R is a Noetherian ring with the global dimension, gl.dim(R), at most one, then for any S-torsion free R-module T, $T \hookrightarrow S^{-1}T$ is a divisible envelope.*

PROOF. Let $G = S^{-1}T/T$. Then $0 \to T \to S^{-1}T \to G \to 0$ is an exact sequence. Since $\mathrm{gl.dim}(R) \le 1$, R is a Gorenstein ring with $\mathrm{K.dim}(R) \le 1$ and $\mathrm{inj.dim}_R(G) \le 1$. So for any divisible R-module D, D is Gorenstein injective, and thus $\mathrm{Ext}^1_R(G, D) = 0$. Then $\mathrm{Hom}_R(S^{-1}T, D) \to \mathrm{Hom}_R(T, D) \to \mathrm{Ext}^1_R(G, D) = 0$ is exact. So $T \hookrightarrow S^{-1}T$ is a divisible preenvelope. By Lemma 2.1, $T \hookrightarrow S^{-1}T$ is a divisible envelope. □

THEOREM 2.3. *Let T be an S-torsion free R-module. If $\phi : T \to D$ is a divisible envelope and D is S-torsion free, then $D \cong S^{-1}T$.*

PROOF. Since D is S-torsion free and divisible, the map $g : S^{-1}T \to D$ defined by $g(a/s) = \phi(a)/s$ for $a \in T, s \in S$ is a well-defined R-linear map. For any $a \in T$, $g \circ i(a) = g(a/1) = \phi(a)/1 = \phi(a)$, where

$i : T \to S^{-1}T$ is the natural map. So $g \circ i = \phi$. Let G be any divisible R-module and $f : T \to G$ be any R-linear map.



Since $\phi : T \to D$ is a divisible envelope, there exists $\theta : D \to G$ such that $\theta \circ \phi = f$. So $(\theta \circ g) \circ i = \theta \circ \phi = f$. Thus $T \hookrightarrow S^{-1}T$ is a divisible preenvelope. Then it is a divisible envelope by Lemma 2.1. Hence by the uniqueness of envelope, $D \cong S^{-1}T$. □

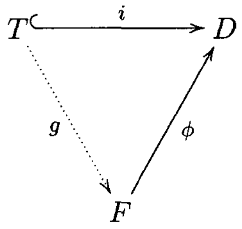
There are some examples of rings which are not Gorenstein but with $R \hookrightarrow K = S^{-1}R$ a divisible envelope.

EXAMPLE 2.4. For an integral domain R, if there exists a sequence $s_1, s_2, s_3, \dots$ of elements of S such that $s_1 | s_2, s_2 | s_3, \dots$ and such that any $s \in S$ divides s_n for some $n \geq 1$, then $R \subset R_1 \subset R_2 \subset \cdots$ and $\bigcup R_n = K$, where $R_n = \{\frac{r}{s_n} | r \in R\}$ for each n. We want to prove that $R \hookrightarrow K$ is a divisible envelope. Let $\phi : R \to D$ be any linear map with D divisible. We argue that ϕ can be extended to a linear map $\phi_1 : R_1 \to D$. Let $\phi(1) = x$. Choose $x_1, x_2, x_3, \dots \in D$ so that $s_1 x_1 = x, s_1 x_2 = x_1, s_1 x_3 = x_2, \dots$. Then it is easy to check that there is a well defined linear map $\phi_1 : R_1 \to D$ such that $\phi_1(\frac{r}{{s_1}^k}) = \phi(r) x_k$ for all $k \geq 1$. In a similar manner there is a linear map $\phi_2 : R_2 \to D$ that extends ϕ_1. So continuing this procedure we get a linear map $\psi : K \to D$ which extends ϕ. Hence $R \hookrightarrow K$ is a divisible preenvelope and it is an envelope by Lemma 2.1.

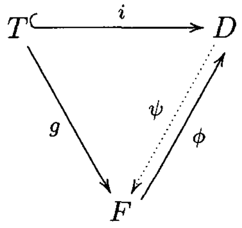
For an example of such a ring R, let $R = \kappa[[x^3, x^4, x^5]]$, where κ is a field and let $s_k = {x_3}^k$ for $k \geq 1$. It is clear that our conditions are satisfied. Also R is not Gorenstein ([2, p.554]).

THEOREM 2.5. *Let R be a commutative ring with identity such that every R-module has a divisible envelope. Then the S-torsion free cover of every divisible module is divisible if and only if the divisible envelope of every S-torsion free module is S-torsion free.*

PROOF. Let T be an S-torsion free R-module and $i : T \hookrightarrow D$ is a divisible envelope. If $\phi : F \to D$ is an S-torsion free cover of D, then there exists $g : T \to F$ such that $\phi \circ g = i$.

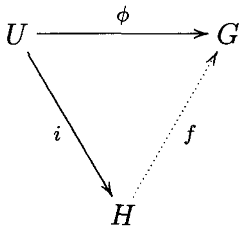


Since $i : T \to D$ is a divisible envelope and F is divisible by the hypothesis, there exists $\psi : D \to F$ such that $\psi \circ i = g$.

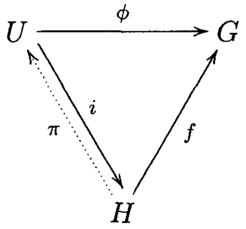


So $\phi \circ \psi \circ i = \phi \circ g = i$. But i is a divisible envelope, $\phi \circ \psi$ is an isomorphism. Thus D is isomorphic to a direct summand of an S-torsion free R-module F. Hence D is S-torsion free.

Conversely, let G be a divisible R-module and $\phi : U \to G$ an S-torsion free cover. If $i : U \to H$ is a divisible envelope of U, then H is S-torsion free by the hypothesis. Since G is divisible,



there exists $f : H \to G$ such that $f \circ i = \phi$. So for the map f,



there exists $\pi : H \to U$ such that $\phi \circ \pi = f$. So $\phi \circ \pi \circ i = f \circ i = \phi$. But ϕ is a cover, so $\pi \circ i$ is an isomorphism. Thus U is a homomorphic image of a divisible module $S^{-1}U$. Hence U is divisible. □

REMARK 2.6. It would be interesting to find necessary and sufficient conditions on a commutative ring with identity that guarantees that $T \hookrightarrow S^{-1}T$ is a divisible envelope for every S-torsion free R-module T.

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