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EFFICIENT COMPUTATION OF THE ACCELERATION OF THE CONTACT POINT BETWEEN ROTATING SURFACES AND APPLICATION TO CAM-FOLLOWER MECHANISM

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ABSTRACT−On a rotating contact surface of arbitrary shape, the relative velocity of the contact point sliding between the surfaces is computed with the basic geometries of the rotating surfaces, and the acceleration of the contact point between the contact surfaces is computed by using the relative velocity of the contact point. Thus the equation for the acceleration constraint between the contact surfaces in multibody dynamics is not coupled with the parameters such as the relative velocity of the contact point. In case of the kinematic analysis, the acceleration of the contact point on any specific instant may also be efficiently computed by the present technique because the whole displacement of a full cycle need not be interpolated. Employing a cam-follower mechanism as a verification model, the acceleration of the contact point computed by the present technique is compared with that computed by differentiating the displacement interpolated with a large number of nodal points.

KEY WORDS : Contact, Kinematics, Multibody dynamics, Rotating contact surfaces, Cam mechanism

1. INTRODUCTION

In the automobile engineering the kinematic analysis and dynamic analysis of the mechanism composed of multibodies are very important. The constraints between the bodies are frequently imposed by the contact conditions. When the contact surface deforms elastically, the spring constant may be efficiently computed with only the local deformation, and the contact condition may be rather easily applied without considering the acceleration constraint (e.g., Shin *et al.*, 1998) But, for the problems having the contact constraint between the rigid bodies, the acceleration of the contact point as well as the displacement and velocity of the contact point are very important for the accurate analysis because the solution without the proper acceleration constraints of the contact points may accompany the severe spurious oscillations (e.g., see the numerical solutions of Cardona and Geradin, 1993). Moreover the acceleration analysis of the contact point has many applications in the kinematics of multibodies because the acceleration of any component having contact constraints can be simply computed with the acceleration of the contact point. Thus the various techniques to compute the acceleration of the contact point existing between the rigid bodies have been shown in the literature. For example, in the kinematic analysis of the mechanisms such as cam-follower, the acceleration of the follower which is equivalent to that of the contact point has usually been computed by differentiating the displacement interpolated with the spline functions (Yoon, 1993; Tsay and Huey, 1993; Yan *et al.*, 1996). In such techniques using the spline interpolations, even to compute the acceleration at a specific cam angle, the whole displacement of a full cycle should be interpolated with the spline function. Thus the computation may become inconvenient and uneconomical. Moreover, as the differentiation in a digital computer may accompany a large amount of error, the accuracy of the computed acceleration should be inferior to that of the displacement. In case of dynamic analysis of multibodies including the contact constraint between the rotating rigid surfaces, even though the acceleration of the contact point can be computed with the acceleration constraints (e.g. Haug, 1989; Deo and Walker, 1995), it is usually very complex because the acceleration constraints are obtained by differentiating the velocity constraints and because such accelerations are nonlinearly coupled with the other unknown variables such as relative velocity and the unknown parameters for detecting the contact points. Thus the well-developed numerical techniques of multibody dynamics (e.g., Yoo *et al.*, 2001) generally require more considerations if the contact constraints are involved.

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In this work it is explained that, by using the strategies used for the dynamic contact analysis of gears by the author (Lee, 2001), the acceleration of the contact point sliding between the rotating surfaces of general shapes can be simply and explicitly computed in any kind of the multibody mechanism. In this work the relative velocity of the contact point sliding between the rotating surfaces is explicitly computed with the basic geometries of the rotating surfaces, and then the acceleration of the contact point between the contact surfaces is directly computed by using the given value of the relative velocity of the contact point. Thus, in the model of the multibody dynamics, the constraint equation for computing the acceleration of the contact point becomes simple because the relative velocities of the contact points are decoupled from the constraint equation to compute the acceleration of the contact points. In the multibody dynamics where the constraints should be simultaneously imposed on the equations of motion of the whole dynamic system, the above acceleration constraint may be efficiently imposed to the global equation of motion by the augmented Lagrange multiplier technique (e.g., see Lee, 2001). Also, in the kinematic analysis of various mechanisms, the present technique to compute acceleration of the contact point can be conveniently applied because the interpolation of the displacement of a full rotation is not required to compute the acceleration of a component. For example, the acceleration of the follower in a cam-follower mechanism may be simply obtained by the acceleration of the contact point computed in this work. In the following sections the procedure for computing the acceleration of a contact point sliding between the rotating bodies is explained, and an example problem is solved with a cam-follower model to check the accuracy of the present solution.

2. EXPLICIT EQUATION FOR THE ACCELERATION OF THE CONTACT POINT

In this work the contact between the two-dimensional rigid bodies is assumed and friction is not considered. The motions of the contact surfaces generally involve rigid body rotations as shown in Figure 1, and thus the contact point on each body slides on the surface. When surfaces A and B are constrained so that contact is maintained during the analysis, the contact point C^A on surface A and the contact point C^B on surface B are determined at any instant by computing the nearest pairing points between the two surfaces. The velocity constraint between the bodies that the normal velocity of point C^A should be equal to that of point C^B may be easily applied with the basic velocity variables of bodies A and B (this is simple because the normal velocity of the contact point sliding on the curve is independent from the

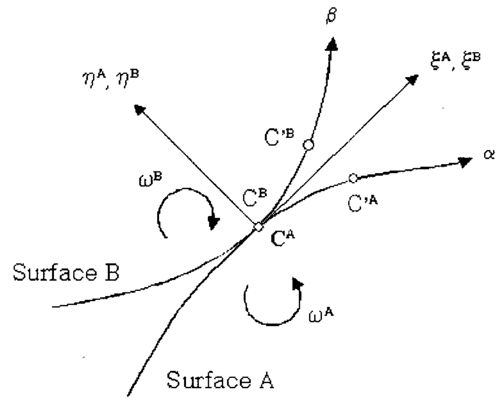

Figure 1. Contact pairing points C^A and C^B, tangential and normal coordinates (ξ^A, η^A) and (ξ^B, η^B), and contact pairing points C'^A and C'^B after time δt.

tangential sliding velocity, and the formulas for the complex objectives may be found in the references (e.g., Chen and Zribi, 2000; Kerr, 1984). However, the acceleration constraint that the normal acceleration on contact point C^A should be equal to that of contact point C^B requires further consideration with additional parameters because the contact points generally slide on the rotating surfaces and the normal accelerations are influenced by such tangential velocities.

In this work a contact point means a point sliding on the surface, and thus a contact point is not fixed to the surface of a body (this definition of the contact point is different from that used in most of the references of the kinematics where the sliding motion of the contact point on the rotating surface is not considered (e.g., Shigley and Uicker, 1995)). When observed from the fixed inertia frame, the position of the contact point sliding on surface A should always coincide with that of the contact point sliding on surface B. Thus the absolute velocity and acceleration of the contact point on surface A should always coincide with those of the contact point on surface B because the both contact points move simultaneously on the common locus observed from the inertia frame. Let (ξ^A, η^A) and (ξ^B, η^B) coordinates of Figure 1 be fixed to the rotating bodies A and B, and assume that the axes of these coordinates instantly agree with the common tangential and normal directions of the contact surfaces on the contact pairing points C^A and C^B. Here α is the parameter which uniquely represents the position of any point on the surface A, and β is the parameter which also uniquely represents the position of any point of the surface B (any convenient angle or length may be employed for parameters α and β as long as every point on the surface can be uniquely represented). Then, as the contact points slide relatively on the surfaces, the absolute normal accelerations a^A and a^B on contact points C^A and

C^B shown in Figure 1 may be written as

$$a^A = \bar{a}^A + 2\omega^A \frac{d}{dt}\xi^A(\alpha) + \frac{d^2}{dt^2}\eta^A(\alpha) \quad (1)$$

$$a^B = \bar{a}^B - 2\omega^B \frac{d}{dt}\xi^B(\beta) + \frac{d^2}{dt^2}\eta^B(\beta) \quad (2)$$

where $\bar{a}^A$ and $\bar{a}^B$ are the normal accelerations of the points C^A and C^B fixed to the surfaces, ω^A denotes the counterclockwise angular velocity of surface A, ω^B denotes the clockwise angular velocity of surface B, and t denotes the time. In equations (1) and (2), the second terms on the right hand sides represent the Coriolis accelerations, and the third terms represent the centripetal accelerations explained in the dynamics text book (e.g., Meriam and Kraige, 2002). These relative accelerations are required for computing the absolute accelerations because the contact points of this work slide relatively on the contact surfaces. In Figure 1, points C'^A and C'^B are assumed to be the contact pairing points after time δt, and the coordinates of the surfaces A and B are assumed to be expressed by using parameters α and β. Then, as the normal directions of the two surfaces on the contact pairing points C'^A and C'^B should coincide after time δt, the following relation is obtained:

$$\left[\frac{d}{d\alpha}\left(\frac{d\eta^A}{d\xi^A}\right)\right]\delta\alpha + \omega^A\delta t = \left[\frac{d}{d\beta}\left(\frac{d\eta^B}{d\xi^B}\right)\right]\delta\beta - \omega^B\delta t \quad (3)$$

In the left hand side of equation (3), the first term denotes the normal direction change of the surface on the contact point due to the geometric curve of the surface A by the movement $\delta\alpha$ of the contact point along the surface A, and the second term denotes the normal direction change due to the rigid-body-rotation of surface A after time δt. In the right hand side of equation (3), the first and the second terms also denote the corresponding normal direction changes on the contact point of surface B with the movement of $\delta\beta$ after time δt. Also, as the tangential velocities of contact points C'^A and C'^B should be identical after time δt, the following relation holds:

$$v^A + \left(\frac{d\xi^A}{d\alpha}\right)\frac{d\alpha}{dt} = v^B + \left(\frac{d\xi^B}{d\beta}\right)\frac{d\beta}{dt} \quad (4)$$

where v^A and v^B denote the tangential velocities of the points fixed to the surfaces on points C^A and C^B. In the above equation, the second term on the left hand side represents the relative tangential velocity of the contact point along surface A, and the second term on the right hand side represents the relative tangential velocity of the contact point along surface B (here, the author would like to emphasize again that the contact point of this work is not the point fixed to the surface but the point sliding on the surface, and the absolute velocity and the acceleration of the contact point sliding on the surface are measured on the fixed coordinate system). Solving equations (3) and (4) simultaneously, $d\alpha/dt$ and $d\beta/dt$ are obtained as

$$\frac{d\alpha}{dt} = \frac{(\omega^A + \omega^B)\frac{d\xi^B}{d\beta} + (v^B - v^A)\frac{d}{d\beta}\left(\frac{d\eta^B}{d\xi^B}\right)}{-\frac{d}{d\alpha}\left(\frac{d\eta^A}{d\xi^A}\right)\frac{d\xi^B}{d\beta} + \frac{d}{d\beta}\left(\frac{d\eta^B}{d\xi^B}\right)\frac{d\xi^A}{d\alpha}} \quad (5)$$

$$\frac{d\beta}{dt} = \frac{(\omega^A + \omega^B)\frac{d\xi^A}{d\alpha} + (v^B - v^A)\frac{d}{d\alpha}\left(\frac{d\eta^A}{d\xi^A}\right)}{-\frac{d}{d\alpha}\left(\frac{d\eta^A}{d\xi^A}\right)\frac{d\xi^B}{d\beta} + \frac{d}{d\beta}\left(\frac{d\eta^B}{d\xi^B}\right)\frac{d\xi^A}{d\alpha}} \quad (6)$$

At the contact points the following relations are derived by the chain rule of differentiation:

$$\frac{d\xi^A}{dt} = \frac{d\alpha}{dt}\frac{d\xi^A}{d\alpha} \quad (7)$$

$$\frac{d^2}{dt^2}(\eta^A(\alpha)) = \frac{d}{dt}\left(\frac{d\alpha}{dt}\frac{d\eta^A}{d\alpha}\right) = \left(\frac{d\alpha}{dt}\right)^2\frac{d^2\eta^A}{d\alpha^2} \quad (8)$$

$$\frac{d\xi^B}{dt} = \frac{d\beta}{dt}\frac{d\xi^B}{d\beta} \quad (9)$$

$$\frac{d^2}{dt^2}(\eta^B(\beta)) = \frac{d}{dt}\left(\frac{d\beta}{dt}\frac{d\eta^B}{d\beta}\right) = \left(\frac{d\beta}{dt}\right)^2\frac{d^2\eta^B}{d\beta^2} \quad (10)$$

The absolute normal acceleration a^A on contact point C^A may be computed by equations (1), (5), (7), (8), and the absolute normal acceleration a^B on contact point C^B may also be computed by equations (2), (6), (9), (10). As the relative velocities $d\alpha/dt$ and $d\beta/dt$ of the contact points are computed by only equations (5) and (6) independently from the accelerations, the normal accelerations a^A and a^B of the contact points may be easily computed by equations (1) and (2) with the given values of $d\alpha/dt$ and $d\beta/dt$. When the contact is maintained, as the absolute normal accelerations of the both contact points observed from the inertia frame should coincide, the following acceleration constraint should be satisfied:

$$a^A = a^B \quad (11)$$

Thus, with the relative velocities $d\alpha/dt$ and $d\beta/dt$ of the contact points computed easily by equations (5) and (6), the acceleration constraint of the two contacting bodies can be simply imposed in multibody dynamics and kinematics (this is compared with the acceleration constraints of the other techniques in the literature where the relative velocities and accelerations of the contact points should be simultaneously solved with the other unknown variables in the acceleration constraint equations).

3. NUMERICAL EXAMPLES

To compare the accuracy and efficiency of the present technique, a cam-follower mechanism shown in Figure 2 is employed here, and the acceleration computed by the present technique is compared with that computed from

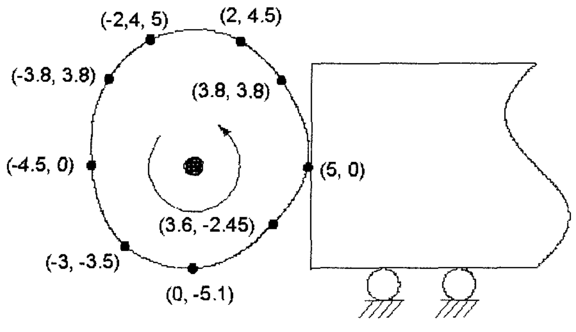

Figure 2. Cam-follower model (unit: cm).

the differentiation of the interpolated displacement (it is worth to note that the acceleration of the follower has been usually computed from the interpolated displacement as shown in the references (e.g., Yoon, 1993; Tsay and Huey, 1993; Yan *et al.*, 1996)). For this purpose the normal acceleration of the contact point on the rotating cam surface is computed by the present technique, and the acceleration of the follower moving on a straight is computed from the differentiation of the interpolated displacement here. The cam contour is expressed by using the cubic spline interpolations with the nine points as shown in Figure 2, and the cam rotates with the speed of 1000 rpm.

With the given cam profile, by computing the right-extreme point of the cam at each instant, the displacement of the follower is easily interpolated as shown in Figure 3. Here the cubic spline interpolation (e.g., Atkinson, 1989; Burden and Faires, 1993) was used for expressing the follower displacement, and the two additional parameters generally required for the cubic spline interpolations besides the nodal displacement are not used here because the end point coincides with the start point in a cyclic motion. After expressing the follower displacement of Figure 3 by cubic spline interpolations with the 1000 nodal points

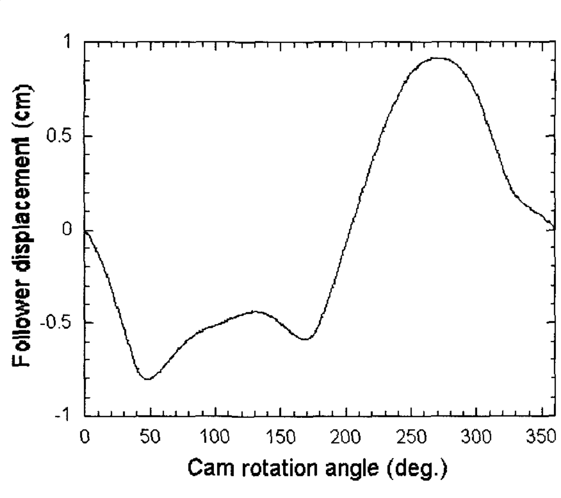

Figure 3. Displacement of the follower.

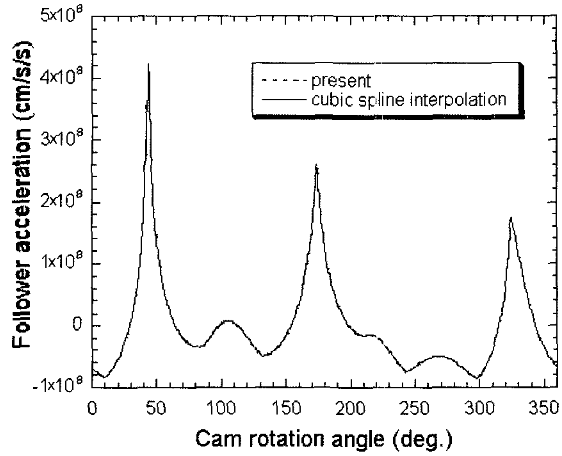

Figure 4. Accelerations of the contact point computed by the present technique and computed by differentiating the displacement interpolated with 1000 points.

(i.e., using the follower displacement computed at each cam rotation of 0.36°), the acceleration of the follower is computed by differentiating the displacement twice and is shown in Figure 4. As the contact point on the follower always shares the same position with the contact point sliding on the cam surface, the absolute accelerations of the both contact points measured from the inertia frame should always coincide. Thus the follower acceleration may be computed by computing the normal acceleration of the contact point sliding on the cam surface. Here the normal acceleration of the contact point sliding on the cam surface is computed by the presented technique of this work and is also shown in Figuee 4. As compared in Figure 4, the acceleration of the contact point computed from the differentiation of the displacement interpolated with 1000 nodal points almost agrees with that computed by the present technique.

On the other hand, the acceleration computed from the differentiation of the interpolated displacement should be subjected to the numerical error due to the interpolation, and the error generally increases as the number of the nodal points for the interpolation reduces. For example, as shown in Figure 5, when the number of the nodal points of the displacement interpolation changes from 1000 to 100, the relative error in computing the acceleration by the interpolated displacement increases drastically (here, the relative error is defined as the ratio of the computed error to the maximum value of the acceleration in a full cycle). Of course the error in computing the acceleration of the contact point further reduces as the number of nodal points for the displacement interpolation increases. For example, when the error is defined as the difference between the acceleration computed by the present technique and that

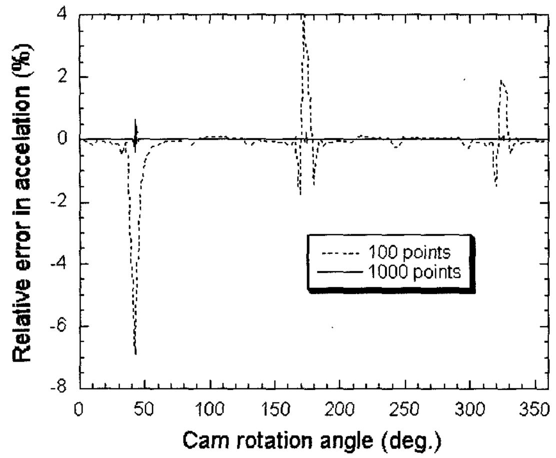

Figure 5. Comparison of the relative errors of the accelerations computed with the different number of nodal points for the displacement interpolations of the follower in a full cam-rotation.

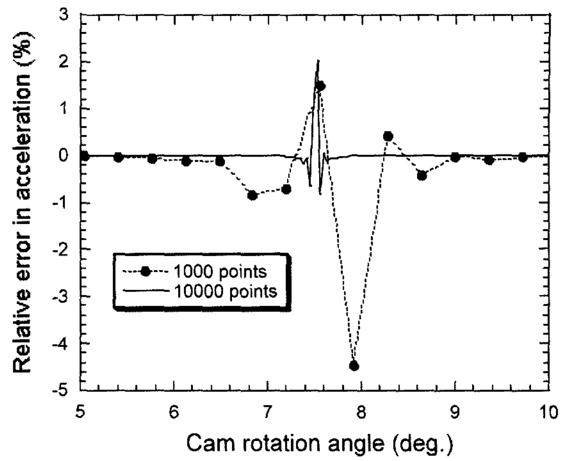

Figure 7. Comparison of the relative errors of the accelerations computed with the different number of nodal points for the displacement interpolations of the follower in a full cam-rotation (revised cam profile).

by the interpolation technique, the maximum relative error of the acceleration reduces to 0.05% if 10,000 nodal points are used for the displacement of a cycle (thus, it may be stated that the acceleration of the contact point computed from the interpolated displacement converges to a simple solution of the present technique if the displacement is interpolated with a number of nodal points).

However, when the cam profile changes more sharply, the computation error in the acceleration obtained by differentiating the interpolated displacement also increases sharply even if the displacement is interpolated with a large number of the nodal points. For example, when

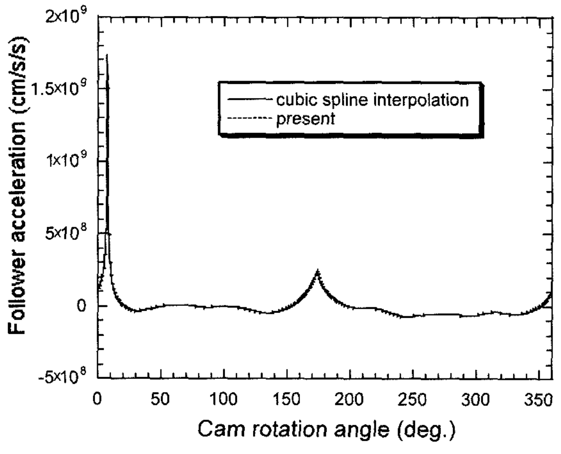

Figure 6. Accelerations of the contact point computed by the present technique and computed by differentiating the displacement interpolated with 1000 points (revised cam profile.

only the coordinates of the right-extreme nodal point of the cam surface in Figure 2 are revised from (5, 0) to (4.2, 0), the peak acceleration increases as shown in Figure 6. Even though the both accelerations of Figure 6 computed by the different techniques seem to almost coincide, as shown in Figure 7 for a cam rotation angle between 5° and 10°, the acceleration computed by differentiating the interpolated displacement contains a relatively large amount of the error even with 10,000 nodal points for the displacement interpolation of a full cycle.

When the follower acceleration is computed by differentiating the interpolated displacement, it generally requires much more computing efforts than the present technique because the whole displacement of the follower associated with a full rotation of the cam should be precisely expressed by a special interpolation function with a number of nodal points. The accuracy of the acceleration obtained by differentiating the displacement generally depends on the interpolating technique and the number of the nodal points employed for the displacement, and a large amount of the computing efforts are required for reducing the numerical error of the computed acceleration. Most of all, in the practical engineering where only a part of the cam profile frequently changes, even to compute the acceleration only at a specific cam rotation angle due to the design change, the whole displacement of a full cycle should be precisely interpolated if the acceleration should be computed from differentiating the interpolated displacement. In contrast, in the present technique, the acceleration of the follower at any specific cam rotation angle can be precisely and simply computed by the acceleration of the contact point sliding on the cam with only the local geometry near the contact point.

4. CONCLUSION

It is shown that, after computing the relative velocity of the contact point sliding on the rotating surface with the basic geometries of the pairing rotating bodies, the acceleration of the contact point between the rigid bodies can be directly computed without differentiating any kind of the displacement constraint or the interpolated displacement. The present technique for the acceleration computation is simple and economical because the relative velocity of the contact point is not coupled with the large system of nonlinear equations in the acceleration constraint and because the whole displacement of a full cycle need not be interpolated. Thus the acceleration of a contact point with good accuracy may be obtained by the present technique without any dependence on the complexity of the geometries of the contact surfaces. As demonstrated in the numerical example of a cam-follower mechanism, the acceleration of the contact point can be accurately and easily computed by the present technique, and may be applied to compute the accelerations of the components in various mechanisms in the kinematic analysis.

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