경영정보학연구
제16권 제1호
2006년 3월

순위가 있는 가중치 평균 방법에서 일정한 수준의 결합력을 갖는 가중치 함수의 성질 및 다기준의사결정 문제에의 활용*

안 병 석**

The Ordered Weighted Averaging (OWA) Operator Weighting Functions with Constant Value of Orness and Application to the Multiple Criteria Decision Making Problems

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Actual type of aggregation performed by an ordered weighted averaging (OWA) operator heavily depends upon the weighting vector. A number of approaches have been suggested for obtaining the associated weights. In this paper, we present analytic forms of OWA operator weighting functions, each of which has such properties as rank-based weights and constant value of *orness*, irrespective of number of objectives aggregated. Specifically, we propose four analytic forms of OWA weighting functions that can be positioned at 0.25, 0.334, 0.667, and 0.75 on the *orness* scale. The merits for using these weights over other weighting schemes can be mentioned in a couple of ways. Firstly, we can efficiently utilize the analytic forms of weighting functions without solving complicated mathematical programs once the degree of *orness* is specified *a priori* by decision maker. Secondly, combined with well-known OWA operator weights such as *max*, *min*, and *average*, any weighting vectors, having a desired value of *orness* and being independent of the number of objectives, can be generated. This can be accomplished by convex combinations of predetermined weighting functions having constant values of *orness*. Finally, in terms of a measure of dispersion, newly generated weighting vectors show just a few discrepancies with weights generated by maximum entropy OWA.

Keywords : Ordered Weighted Averaging, Analytic Forms of Weights, Orness, Dispersion

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* This research was financially supported by Hansung University in the year of 2006.
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Ⅰ. Introduction

Decision making involves choosing some course of action among various alternatives. In almost all decision making problems, there are several criteria for judging alternatives. A multiple criteria decision making method largely consists of two phases: 1) decision problem construction and information specification, and 2) aggregation and exploitation [Ahn, 2003; Ahn 2005]. Among others, synthesizing judgments is an important part of multiple criteria decision making methods. Yager [1988] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating multiple inputs that lie between the *max* and *min* operators. As the term 'ordered' implies, the OWA operator pursues a nonlinear aggregation of objects considered, different from the existent multicriteria aggregation methods such as, for instance, multiattribute utility theory (MAUT) [Winterfeldt and Edwards, 1986; Keeney and Raiffa, 1976] and simple weighted sum [Hwang and Lin, 1986]. In the short time since its first appearance, the OWA operators have been used in an astonishingly wide range of applications in the fields including neural networks [Yager, 1995; Yager, 1992], database systems [Yager, 1987], fuzzy logic controllers [Yager, 1991; Yager and Filev, 1992], group decision making [Herrera *et al.*, 1995; Herrera *et al.*, 1996a: Herrera *et al.*, 1996b] and so on. The main reason for this is their great flexibility to model a wide variety of aggregators, as their nature is defined by a weighting vector, and not by a single parameter [Salido and Murakami, 2003]. By appropriately selecting the weighting vector, it is possible to model different kinds of relations among the criteria aggregated. Recently, Xu and Da [2003] presented a survey of the main aggregation operators that encompass a broad range of existing operators (more than 20 aggregators). It is clear that actual type of aggregation performed by an OWA operator depends upon the weighting vector, which plays key role in aggregation process. Filev and Yager [1998] presented a way of obtaining weights associated with the OWA aggregation in the situation where we have observed data on the arguments and the aggregated value.

Another appealing point was the introduction of the concept of *orness* and the definition of an *orness* measure that could establish how 'orlike' a certain operator is, based on the values of its weighting function. Thus the measure can be interpreted as the mode of decision making circumstances by conferring the semantic meaning to the weights used in aggregation process. If an aggregated value is close to the maximum of the ordered objects, the aggregation pursues the 'orlike' aggregation. If an aggregated value is close to the minimum of the ordered objects, on the other hand, the aggregation pursues the 'andlike' aggregation. This concept perfectly coincides with the traditional decision making theory in which *max* decision principle denotes the optimistic decision context and *min* decision principle denotes the pessimistic decision context.

On the other hand, Yager [1988], based on a measure of entropy, proposed a measure of dispersion which gauges the degree of utilization of information in the sense that each of weighting vectors considered can be different to each other by degree of dispersion though they have the same degree of *orness*. One of the first approaches, suggested by O'Hagan [1990], determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of *orness*, algorithmically based on the solution of a constrained optimization problem. The resulting weights are called maximum entropy OWA (MEOWA) weights for a given degree of *orness* and analytic forms and property for these weights are further investigated by several researchers [Filev and Yager, 1995; Fuller and Majlender, 2001]. Instead of maximizing the degree of dispersion, Fuller and Majlender [2003] presented a method of deriving the minimal variability weighting vector for any level of *orness*, using Kuhn-Tucker second-order sufficiency conditions for optimality.

In this paper, we present analytic forms of OWA operator weighting functions, each of which has properties such as rank-based weights and constant degree of *orness*, irrespective of number of objectives considered. Specifically, the four analytic forms of OWA operator weights can be positioned at 0.25, 0.334, 0.667, 0.75 respectively on the *orness* scale. The merits of using these kinds of analytic forms of weights can be outlined in a couple of ways. First, we can efficiently utilize them without solving complicated mathematical programs once a value of *orness* is specified a priori by the decision maker. Second, combined with already well-known OWA operator weights such as *max* (*orness* = 1), *min* (*orness* = 0), and *average* (*orness* = 0.5), any weighting vectors, having constant degree of *orness*, can be constructed as well. This can be accomplished by convex combinations of the weighting vectors already known to have constant values of *orness*. Finally, the proposed OWA operator weights with constant level of *orness* display just a few discrepancies in terms of dispersion with weights derived by MEOWA method. Further, this statement applies to the weights generated from predetermined weighting vectors with constant values of *orness*.

The paper is organized as follows: in Section Ⅱ, we will briefly review Yager's definition for the OWA operators and their *orness* measure. The analytical forms of operator weights functions with constant level of *orness* and their properties are investigated in Section Ⅲ. In Section Ⅳ, possible applications to the multiple criteria decision making problems are discussed, followed by concluding remarks in Section Ⅴ.

Ⅱ. The OWA operators and their orness measure

An OWA operator [Yager, 1988] of dimension n is a mapping $f: R^n \rightarrow R$ that has an associated weighting n vector $W = [w_i, w_2, \cdots, w_n]^T$, such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, and where the function value $f(a_1, a_2, \cdots, a_n)$ determines the aggregated value of arguments the $a_1, a_2, \cdots, a_n$ in such a manner that

$$f(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where b_i is the ith largest element of the collection of the n aggregated objects $a_1, a_2, \cdots, a_n$, thus satisfying the relation

$$Min_i[a_i] \le f(a_1, a_2, \cdots, a_n) \le Max_i[a_i].$$

The fundamental aspect of the OWA operator is the re-ordering step, in particular, an argument a_i is not associated with a particular weight u_i, but rather a weight w_i is associated with a particular ordered position, *i* of the argument $a_1, a_2, \cdots, a_n$, thus yielding a nonlinear aggregation. Its generality lies in the fact that by selecting appropriate weights, different aggregation can be implemented. Specifically, by appropriately selecting the weights in *W* we can emphasize different arguments based upon their positions in the ordering. Thus, if we place most of the weights near the top of *W*, we can emphasize the higher scores, while placing the weights near the bottom of *W* emphasizes the lower scores in the aggregation [Yager, 1988].

Example. Assume $W = [0.4, 0.3, 0.2, 0.1]^T$. Then, $f(0.7, 1.0, 0.3, 0.6) = (0.4)(1) + (0.3)(0.7) + (0.2)(0.6) + (0.1)(0.3) = 0.76$.

Yager introduced two characterizing measures associated with weighting vector *W* of the OWA operator. The first one, the measure of *orness* of the aggregation, is defined as

$$orness(W) = \Omega = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$$

and it characterizes the degree to which the aggregation is like an *or* operation.

Example. Assume $W = [0.4, 0.3, 0.2, 0.1]^T$ then

$$orness(W) = (1/3)(3(0.4) + 2(0.3) + 1(0.2)) = 0.666.$$

If we consider the special cases of OWA operators,

$W^* = [1, 0, 0, \cdots, 0]^T$ (maximum operator),
$W_* = [0, 0, 0, \cdots, 1]^T$ (minimum operator),
$W_{Ave} = [1/n, 1/n, 1/n, \cdots, 1/n]^T$ (average operator),

then it can easily be shown that

(1) $orness(W^*) = 1$,
(2) $orness(W_*) = 0$,
(3) $orness(W_{Ave}) = 0.5$.

A measure of *andness* for an OWA operator with weights *W* can also defined as

$$andness(W) = 1 - orness(W) = 1 - \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i.$$

The OWA operators with many of the weights near the top will be an 'orlike' operator ($orness(W) \geq 0.5$), while those operators with most of the weights at the bottom will be 'andlike' operators ($orness(W) \leq 0.5$). As to the semantics of the OWA's measure of *orness*, Yager suggests that, based on Hurwicz's model [Chernoff and Moses, 1959], the measure of *orness* can be interpreted as a measure of optimism of the decision making, while the measure of andness is a measure of pessimism. Another measure, the dispersion of weights and a way of its determination while satisfying the prescribed ornsss will be discussed in the next section.

Ⅲ. Analytic forms of operator weights with constant level of orness and their properties

In what follows, we present four analytic OWA weights functions via some theorems and corollaries.

Theorem 1. The OWA weights function in (1), applying the *orness* measure of aggregation, results in the constant value of *orness* (i.e., optimism) $3/4$, irrespective of the number of objectives.

$$w_i = \frac{1}{n}\sum_{j=i}^{n}\frac{1}{j} \qquad (1)$$

Proof. From a substitution of the OWA weights function into the definition of *orness* measure and some manipulations, then we can obtain the following result,

$$\begin{aligned} orness(W) &= \frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}\right) \\ &= \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}\left(n\cdot\sum_{j=i}^{n}\frac{1}{j}\right)-\sum_{i=1}^{n}\left(i\cdot\sum_{j=i}^{n}\frac{1}{j}\right)\right) \\ &= \frac{1}{n(n-1)}\left(n\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{1}{j}-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j}{i}\right) \\ &= \frac{1}{n(n-1)}\left(n\cdot n-\sum_{i=1}^{n}\frac{1}{i}\cdot\frac{i(i+1)}{2}\right) \\ &= \frac{1}{n(n-1)}\left(n^2-\frac{n^2+3n}{4}\right)=\frac{3}{4}. \end{aligned}$$

Corollary 1. The *orness* value of OWA weights function in (2) results in the constant value of *orness* (i.e., optimism) $1/4$, irrespective of the number of objectives.

$$w_i = \frac{1}{n}\sum_{j=1}^{i}\frac{1}{(n-j+1)}. \qquad (2)$$

Proof. Substituting the OWA weights function in (2) into the definition of the *orness* measure, then, we obtain the following result.

$$\begin{aligned} orness(W) &= \frac{1}{(n-1)}\sum_{i=1}^{n}\left((n-i)\cdot\frac{1}{n}\sum_{j=1}^{i}\frac{1}{(n-j+1)}\right) \\ &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\left((n-i)\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}\right) \\ &= \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}n\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}-\sum_{i=1}^{n}i\cdot\sum_{k=n-i+1}^{n}\frac{1}{k}\right) \\ &= \frac{1}{n(n-1)}\left(n\cdot\sum_{i=1}^{n}\sum_{k=n-i+1}^{n}\frac{1}{k}-\sum_{i=1}^{n}\sum_{k=n-i+1}^{n}\frac{k}{i}\right) \\ &= \frac{1}{n(n-1)}\left(n^2-\sum_{i=1}^{n}\left(\frac{1}{i}\cdot\frac{i}{2}(2n-i+1)\right)\right)=\frac{1}{4}. \end{aligned}$$

In multiattribute value theory (MAVT), the OWA weights $W(3/4)=[w_1(3/4),\cdots w_n(3/4)]$, having a constant value of orness $3/4$, are called centroid (center of mass) weights in efforts to seek to identify a single set of weights that is representative of all the possible weight combinations that are admissible, consistent with the established linear inequality constraints on the weights $w_1 \ge w_2 \ge \cdots \ge w_n$ [Winterfeldt and Edwards, 1986; Stillwell *et al.*, 1981]. Edwards and Barron [1986] give a straightforward formula for determining the centroid point for the case where all criteria are ranked simply in the form $w_1 \ge w_2 \ge \cdots \ge w_n$. The points should be the centroid of the feasible region of admissible sets of weight values and specifically they assign weights as follow, where $w_1(3/4)$ is the weight of the most important objective, $w_2(3/4)$ the weight of the second most important objective, and so on. For n objectives,

$$\begin{aligned} w_1(3/4) &= (1+1/2+1/3+\cdots+1/n)/n \\ w_2(3/4) &= (0+1/2+1/3+\cdots+1/n)/n \\ &\vdots \\ w_n(3/4) &= (0+0+0+\cdots+1/n)/n. \end{aligned}$$

Further, they suggest that by eliciting rank orders of importance over all criteria and using rank-ordered centroid weights, one has nearly the same accuracy as is found with more complex methods. The detailed weights of $W(3/4)$ by indicated number of objectives from 2 to 10 are shown in <Table 1> below.

Theorem 2. The *orness* value of OWA weights function in (3) results in the constant value of *orness* (i.e., optimism) $2/3$, irrespective of the number of objectives.

$$w_i = \frac{n-i+1}{\sum_{j=1}^{n}(n-j+1)} = \frac{2(n+1-i)}{n(n+1)}. \quad (3)$$

Proof. Substituting the OWA weights function in (3) into the definition of the *orness* measure, then, we obtain the following result.

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\cdot\frac{2(n+1-i)}{n(n+1)}\right)$$
$$= \frac{2}{(n-1)n(n+1)}\sum_{i=1}^{n}((n^2+n)-(2n+1)\cdot i + i^2)$$

Using $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,

$$orness(W) = \frac{2}{(n-1)n(n+1)}\left(n(n^2+n)-(2n+1)\cdot\frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}\right)=\frac{2}{3}$$

The detailed weights of $W(2/3)=[w_1(2/3), \cdots, w_n(2/3)]$ by indicated number of objectives from 2 to 10 are shown in <Table 2>.

<Table 1> The OWA weights $W(3/4)$ for indicated number of objectives

Rank	Number of objectives								
	2	3	4	5	6	7	8	9	10
1	.7500	.6111	.5208	.4567	.4083	.3704	.3397	.3143	.2929
2	.2500	.2778	.2708	.2567	.2417	.2276	.2147	.2032	.1929
3		.1111	.1458	.1567	.1583	.1561	.1522	.1477	.1429
4			.0625	.0900	.1028	.1085	.1106	.1106	.1096
5				.0400	.0611	.0728	.0793	.0828	.0846
6					.0278	.0442	.0543	.0606	.0646
7						.0204	.0335	.0421	.0479
8							.0156	.0262	.0336
9								.0123	.0211
10									.0100

<Table 2> The OWA weights $W(2/3)$ for indicated number of objectives

Rank	Number of objectives								
	2	3	4	5	6	7	8	9	10
1	.6667	.5000	.4000	.3333	.2857	.2500	.2222	.2000	.1818
2	.3333	.3333	.3000	.2667	.2381	.2143	.1944	.1778	.1636
3		.1667	.2000	.2000	.1905	.1786	.1667	.1556	.1455
4			.1000	.1333	.1429	.1429	.1389	.1333	.1273
5				.0667	.0952	.1071	.1111	.1110	.1091
6					.0476	.0714	.0833	.0889	.0909
7						.0357	.0556	.0667	.0727
8							.0278	.0444	.0545
9								.0222	.0364
10									.0182

Corollary 2. The *orness* value of OWA weights function in (4) results in the constant value of *orness* (i.e., optimism) $1/3$, irrespective of the number of objectives.

$$w_j = \frac{i}{\sum_{j=1}^{n}(n-j+1)} = \frac{2i}{n(n+1)}. \quad (4)$$

Proof. Substituting the OWA weights function in (4) into the definition of the *orness* measure, then, we obtain the following result.

$$\begin{aligned} orness(W) &= \frac{1}{n-1}\sum_{i=1}^{n}\left((n-i)\cdot\frac{2i}{n(n+1)}\right) \\ &= \frac{2}{(n-1)n(n+1)}\left(n\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2\right) \\ &= \frac{2}{(n-1)n(n+1)}\left(n\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right) = \frac{1}{3}. \end{aligned}$$

The formulas for generating the OWA weights and their function values are outlined in <Table 3>.

Example. Assume the objective arguments to be aggregated are (0.7, 1.0, 0.3, 0.6) and an orness we want to set is 0.75. Then the formula in (1) results in weights, $W=[0.52, 0.27, 0.15, 0.06]^T$. Hence, a function value is

$$\begin{aligned} f(0.7, 1.0, 0.3, 0.6) &= (0.52)(1) + (0.27)(0.7) \\ &\quad + (0.15)(0.6) + (0.06)(0.3) \\ &= 0.82. \end{aligned}$$

With the same objective arguments, let us assume that an orness we want to set is 0.67. Then the formula in (3) results in weights, $W=[0.4, 0.3, 0.2, 0.1]^T$. Hence, a function value is

$$\begin{aligned} f(0.7, 1.0, 0.3, 0.6) &= (0.4)(1) + (0.3)(0.7) \\ &\quad + (0.2)(0.6) + (0.1)(0.3) \\ &= 0.76. \end{aligned}$$

<Table 3> Formulas for specifying weights of OWA operators

Weighting function	Orness	OWA aggregation: $f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i b_i$
$w_1 = 1,\ w_j = 0,\ j \neq 1$	1	b_1
$w_i = \frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}$	3/4	$\frac{1}{n}\left(b_1 + \frac{1}{2}(b_1+b_2) + \cdots + \frac{1}{n-1}\sum_{i=1}^{n-1} b_i + \frac{1}{n}\sum_{i=1}^{n} b_i\right)$
$w_i = \frac{2(n+1-i)}{n(n+1)}$	2/3	$\frac{2}{n(n+1)}\left((n+1)\sum_{i=1}^{n} b_i - \sum_{i=2}^{n} i \cdot b_i\right)$
$w_i = \frac{1}{n}$	1/2	$\frac{1}{n}\left(\sum_{i=1}^{n} b_i\right)$
$w_i = \frac{2i}{n(n+1)}$	1/3	$\frac{2}{n(n+1)}\left(\sum_{i=1}^{n} i \cdot b_i\right)$
$w_i = \frac{1}{n}\sum_{j=1}^{i}\frac{1}{(n-j+1)}$	1/4	$\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n} b_i + \frac{1}{n-1}\sum_{i=2}^{n} b_i + \cdots + b_n\right)$
$w_n = 1, w_j = 0,\ j \neq n$	0	b_n

Theorem 3. If w_i are any collection of OWA weights having the property that $w_i \geq w_j$ for $i < j$, then $0.5 < \Omega \leq 1$. If w_i are any collection of OWA weights having the property that $w_i \leq w_j$ for $i < j$, then $0 \leq \Omega < 0.5$.

Proof. See the paper by Filev and Yager [1998].

Corollary 3. The weights $W(2/3)$ and $W(3/4)$, of which orness belongs to the interval $0.5 < \Omega \leq 1$, maintain the relation $w_i \geq w_j$ for $i < j$. The weights $W(1/4)$ and $W(1/3)$, of which orness belongs to the interval $0 \leq \Omega < 0.5$, also maintain the relation $w_i \leq w_j$ for $i < j$.

Proof. For the weights $W(3/4)$, it holds that $w_i(3/4) > w_j(3/4)$ for $i < j$ due to the fact $1/n\sum_{k=i}^{n} 1/k > 1/n\sum_{k=j}^{n} 1/k$ and for the weights $W(2/3)$, it holds that $w_i(2/3) > w_j(2/3)$ for $i < j$ due to the fact $2(n+1-i)/n(n+1) > 2(n+1-j)/n(n+1)$. This can be proved for the weights $W(1/4)$ and $W(1/3)$ in a similar manner.

As a simple extension of the OWA operator weights shown in (3) and (4), we can construct a family of OWA weights functions that have following general forms

$$w_i = \frac{(n-i+1)^k}{\sum_{j=1}^{n}(n-j+1)^k}$$

$$(\text{or } w_{n-i+1} = \frac{i^k}{\sum_{j=1}^{n}(n-j+1)^k}) \text{ for } k \geq 2.$$

It is obvious that $w_i \geq w_j$ (or $w_i \leq w_j$) for $i < j$, $k \geq 2$ and $w_1 + \cdots + w_n = 1$, thus satisfying rank-based OWA weights condition. The *orness* of the OWA weights function for $k = 2$, for instance, can be derived in a simple form such as

$$orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\frac{(n-i+1)^2}{\sum_{j=1}^{n}(n-j+1)^2}$$

$$= \frac{3n+2}{2(2n+1)},$$

after some computations and using the fact $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Though the *orness* of the OWA weights function for $k = 2$ is not in the form of a constant value of *orness* as is in the formulas (1)-(4), the *orness* converges at the number 3/4 as the number of objectives increases, thus yielding $\lim_{n\to\infty}\frac{3n+2}{2(2n+1)} = \frac{3}{4}$. The *orness* of the weights are denoted below for various numbers of objectives.

$n=2$	$n=6$	$n=10$	$n=13$	$n=15$
.800	.769	.762	.759	.758
$n=20$	$n=25$	$n=30$	$n=40$	$n=50$
.756	.755	.754	.753	.752

From the weights shown in <Table 1> and <Table 2>, it is conceived that the $W(2/3)$ weights are much more flatter that $W(3/4)$ weights. To prove this conjecture, we define

$$Q_k = \sum_{j=1}^{k} w_j$$

where $Q_n = 1$ and $Q_k \geq Q_{k-1}$.

Theorem 4. The $W(2/3)$ weights are flatter than $W(3/4)$ weights. In other words, $Q_k(3/4) > Q_k(2/3)$ for $k = 2, 3, \cdots, n$ where $Q_k(3/4) = \sum_{j=1}^{k} w_j(3/4)$ and $Q_k(2/3) = \sum_{j=1}^{k} w_j(2/3)$.

Proof. For $k \geq 2$,

$$Q_k(3/4) = \sum_{j=1}^{k}\left(\frac{1}{n}\sum_{p=j}^{n}\frac{1}{p}\right) = \frac{1}{n}\left(\sum_{p=1}^{k}k \cdot \frac{1}{k} + \sum_{p=k+1}^{n}\frac{1}{p}\right)$$
$$> \frac{1}{n}\left(k + k\frac{n-k}{n}\right) > \frac{1}{n}\left(k + k\frac{n-k}{n+1}\right)$$
$$= \frac{1}{n}\left(\frac{k+2nk-k^2}{n+1}\right).$$

On the other hand,

$$Q_k(2/3) = \sum_{j=1}^{k}\frac{2(n+1-j)}{n(n+1)} = \frac{1}{n}\left(\frac{k+2nk-k^2}{n+1}\right).$$

Thus, $Q_k(3/4) > Q_k(2/3)$ holds.

We can also prove a relation $Q_k(1/3) > Q_k(1/4)$ for $0 \leq \Omega < 0.5$ and $k = 2,3,\cdots,n$, analogously. Yager [1988] mentioned that assuming W and W' are two weighting functions such that for each k, $Q_k \geq Q_k'$, then $orness(W) \geq oreness(W')$. It can be easily seen that this statement holds in our cases, thus yielding

$$orness(W(3/4)) = 3/4 > orness(W(2/3)) = 2/3(Q_k(3/4) > Q_k(2/3)),$$
$$orness(W(1/3)) = 1/3 > orness(W(1/4)) = 1/4(Q_k(1/3) > Q_k(1/4)).$$

A choice between the weights $W(3/4)$ and $W(2/3)$ depends in part on one's belief about the steepness of the true weights guiding a decision-maker's preferences and decision situation considered. The greater the concentration of values in the first few objectives, that is 'orlike', the more attractive the $W(3/4)$ weights method.

In what follows, we shall investigate the properties of proposed OWA weights in terms of a measure of dispersion, which is defined as

$$disp(W) = -\sum_{i=1}^{n}w_i \ln w_i.$$

This measure can be used to gauge the degree to which the information about the individual aggregates is used in the aggregation process. We note that since this dispersion is really a measure of entropy and thus the following properties are valid

1) if $w_i = 1$ for some i then the dispersion is minimum and $disp(W) = 0$
2) the dispersion is maximum if $w_i = 1/n$ and $disp(W) = \ln n$.

O'Hagan [1990] determines a special class of OWA operators having a maximal entropy of the OWA weights for some prescribed level of *orness*. This approach is based on the solution of the following mathematical programming problem:

$$\text{maximize } disp(W) = -\sum_{i=1}^{n}w_i \ln w_i$$
$$\text{subject to } orness(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \alpha,$$
$$0 \leq \alpha \leq 1,$$
$$w_1 + w_2 + \cdots + w_n = 1,\ 0 \leq w_i \leq 1, \quad (5)$$

where α is a desired value of *orness*.

One interesting point to be noted is that if a weighting vector W is optimal for problem (5) under some prescribed value of *orness* α, then its reverse, denoted by W^R, and defined as $w_i^R = w_{n-i+1}$ is also optimal for problem (5) under value of *orness* $(1-\alpha)$ and $disp(W^R) = disp(W)$. It is easy to show that the proposed weighting functions $W(k)$ for $k = 1/4, 1/3, 2/3, 3/4$, though they are not weights based on the maximal entropy, satisfy these properties. Then, our concern is that the proposed weights functions $W(k)$ for $k = 1/4, 1/3, 2/3, 3/4$ lie in what certain level of dispersion, compared to the weights computed by maximal entropy method. This consideration can be set forth by each comparison of MEOWA weights and the weights of proposed weights functions for prescribed values of *orness* 0.75 and 0.667. In <Table 4> and <Table 5>, entropy values are shown when *orness* is fixed at 0.75 and 0.667 respectively. As can be seen in the two tables, small differences in entropy values can be found only at third decimal places of the numbers. This fact holds true for the MEOWA weights and the proposed weights when *orness* is fixed 0.25 and 0.334 respectively. The other concern we want to address falls into a case that the decision maker wants to make an aggregation at some other value of *orness* except four constant values of *orness*. This consideration is depicted in <Figure 1> in which solid arrows signify the identified weights functions with constant value of *orness* and the dashed arrows signify alreacy well-known weights functions. A way of determining other OWA operator weights is described in Theorem 5 below.

Theorem 5. The convex combinations of any two OWA weights functions with constant value of *orness* results in OWA weights which have also constant value of *orness* irrespective of the number of objectives.

Proof. When a desired value of *orness* is k, let us denote newly generated OWA weights as $W^{New}(k)$. The weights $W^{New}(k)$ can be constructed by a convex combination of $W(k')$ and $W(k'')$ that are already identified weights with constant value of *orness* k', k'' respectively, that is

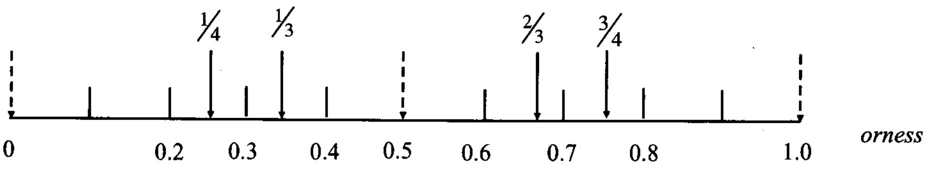
$$w_i^{new} = \beta w_i(k') + (1-\beta) w_i(k'') \text{ for } \beta \in [0,1].$$

Then we can always find $\beta \in [0,1]$ that satisfies $\beta \cdot k' + (1-\beta) \cdot k'' = k$ and applying this newly generated OWA weights into the *orness* measure of aggregation, we obtain

$$\begin{aligned} orness(W^{New}) &= \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i^{new} \\ &= \frac{1}{n-1}\sum_{i=1}^{n}(n-i)(\beta w_i(k) \\ &\quad + (1-\beta)w_i(k')) \\ &= \frac{\beta}{n-1}\sum_{i=1}^{n}(n-i)w_i(k) \\ &\quad + \frac{1-\beta}{n-1}\sum_{i=1}^{n}(n-i)w_i(k') \\ &= \beta \cdot k + (1-\beta) \cdot (k') = k. \end{aligned}$$

Example. Suppose that we want to generate OWA operator weights which have *orness* 0.7 from the OWA weights functions $W(2/3)$ and $W(3/4)$, then we simply solve a following equation

$$\beta W(2/3) + (1-\beta) W(3/4) = 0.7.$$
$$\beta \cdot \frac{2}{3} + (1-\beta) \cdot \frac{3}{4} = 0.7,$$



<Figure 1> Identified weighting functions with constant value of orness

which results in $\beta = 0.6$. Thus if n = 3, newly generated OWA operator weights with *orness* =0.7 becomes

$$W^{New}(0.7) = [(0.6)(0.5)+(0.4)(0.611),(0.6)(0.333)+(0.4)(0.278),(0.6)(0.167)+(0.4)(0.111)] = [0.544,0.311,0.145].$$

The *orness* of the newly generated weights is, of course,

$$orness(W^{New}(0.7)) = [(0.5)((2)(0.544)+(1)(0.311))= 0.7$$

One interesting point to be noted is that a convex combination with $\beta = 0.5$ of the OWA weights functions (3) and (4) exactly coincides with the OWA weights $W_{Ave} = [1/n, 1/n, \cdots, 1/n]$, which is stated in Corollary 4.

<Table 4> Individual weights and entropy values of MEOWA and proposed weighting method with *orness* = 0.75

orness = 0.75																						
	MEOWA weights											Weights with constant level of *orness*										
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E
2	.750	.250									.562	.750	.250									.562
3	.616	.268	.116								.901	.611	.278	.111								.901
4	.526	.268	.137	.069							1.148	.521	.271	.146	.063							1.148
5	.459	.261	.148	.084	.048						1.344	.457	.257	.157	.090	.040						1.343
6	.408	.250	.154	.094	.058	.036					1.507	.408	.242	.158	.103	.061	.028					1.505
7	.367	.239	.156	.101	.066	.043	.028				1.646	.370	.228	.156	.109	.073	.044	.020				1.644
8	.334	.228	.155	.106	.072	.049	.033	.023			1.768	.340	.215	.152	.111	.079	.054	.034	.016			1.765
9	.306	.217	.153	.108	.077	.054	.038	.027	.019		1.876	.314	.203	.148	.111	.083	.061	.042	.026	.012		1.873
10	.283	.206	.150	.110	.080	.058	.043	.031	.023	.017	1.973	.293	.193	.143	.110	.085	.065	.048	.034	.021	.010	1.970

<Table 5> Individual weights and entropy values of MEOWA and proposed weighting method with *orness* = 0.667

orness = 0.667																						
	MEOWA weights											Weights with constant level of *orness*										
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E
2	.667	.333									.637	.667	.333									.637
3	.514	.305	.181								1.013	.500	.333	.167								1.011
4	.421	.277	.182	.120							1.284	.400	.300	.200	.100							1.280
5	.358	.252	.177	.125	.088						1.495	.333	.267	.200	.133	.067						1.490
6	.311	.230	.170	.126	.093	.069					1.670	.286	.238	.191	.143	.095	.048					1.662
7	.276	.212	.162	.125	.096	.074	.056				1.817	.250	.214	.179	.143	.107	.071	.036				1.809
8	.247	.195	.154	.122	.097	.076	.060	.048			1.946	.222	.194	.167	.139	.111	.083	.056	.028			1.937
9	.224	.181	.147	.119	.096	.078	.063	.051	.041		2.060	.200	.178	.156	.133	.111	.089	.067	.044	.022		2.050
10	.205	.169	.140	.115	.095	.078	.065	.053	.044	.036	2.162	.182	.164	.146	.127	.109	.091	.073	.055	.036	.018	2.151

Corollary 4. A convex combination of $W(2/3)$ and $W(1/3)$ with $\beta = 0.5$ results in the well-known OWA weights $W_{Ave} = [1/n, 1/n, \cdots, 1/n]$.

Proof. $w_i^{New} = \frac{1}{2} w_i(2/3) + \frac{1}{2} w_i(1/3)$

$$= \frac{2(n+1-i)}{2n(n+1)} + \frac{2i}{2n(n+1)}.$$

$$= \frac{1}{n} = W_{Ave}$$

The OWA operator weights with a desired value of *orness* can be constructed by using known end points which encompass the desired point of *orness*. The difficult is, however, that there exist many alternatives to be chosen for the end points. For instance, if we want to generate new OWA operator weights with *orness*=0.7, then we can make lots of combinations by differing the parameter β which makes the weights locate at the corresponding *orness*. The pairs of end points such as $(2/3, 3/4)$,$(1/2, 3/4)$, $(1/3, 3/4)$, and $(1/4, 3/4)$ considering $\beta = 0.6, 0.2, 0.12, 0.1$ can be chosen as the options. Our consideration is that which of them is the most appropriate to use in the aggregation process. As one of criteria that can be considered, let us suppose that we select newly generated weights that result in maximum entropy. We investigate the difference of dispersion among the newly generated weights, and the difference of dispersion between MEOWA weights and newly generated weights at fixed level of *orness* 0.7 in <Table 6> and <Table 7>. In this case, only small differences can be found only at third decimal places of the numbers in all comparisons of consideration. Further we found that these results also apply to any other desired values of *orness* between 0.25 and 0.75 since different parameter β, depending on the different end points, smoothes the differences of weights and thus finally make the weights close to each other.

<Table 6> Individual weights and entropy values of newly generated weights using endpoints when *orness* is fixed at 0.7

	End point (2/3, 3/4)											End point (1/2, 3/4)											
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	E[a]
2	.700	.300									.611	.700	.300									.611	.611
3	.544	.311	.144								.974	.556	.289	.156								.975	.975
4	.448	.288	.178	.085							1.235	.467	.267	.167	.100							1.237	1.237
5	.383	.263	.183	.116	.056						1.441	.405	.245	.165	.112	.072						1.443	1.443
6	.335	.240	.178	.127	.082	.040					1.610	.360	.227	.160	.116	.082	.056					1.613	1.614
7	.298	.220	.170	.129	.093	.061	.030				1.754	.325	.211	.153	.115	.087	.064	.045				1.757	1.759
8	.269	.203	.161	.128	.098	.072	.047	.023			1.880	.297	.197	.147	.113	.088	.068	.052	.037			1.883	1.885
9	.246	.188	.152	.124	.100	.078	.057	.037	.018		.1991	.274	.185	.140	.111	.088	.071	.056	.043	.032		1.995	1.997
10	.226	.175	.144	.120	.099	.080	.063	.046	.030	.015	2.092	.254	.174	.134	.108	.088	.072	.058	.047	.037	.028	2.096	2.098

Note) [a] maximum entropy of OWA operator weights when orness is fixed at 0.7

<Table 7> Individual weights and entropy values of newly generated weights using endpoints when *orness* is fixed at 0.7

n	End point (1/3, 3/4)											End point (1/4, 3/4)											
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E	E
2	.700	.300									.611	.700	.300									.611	.611
3	.558	.284	.158								.975	.561	.278	.161								.974	.975
4	.470	.262	.164	.103							1.237	.475	.258	.158	.108							1.236	1.237
5	.410	.242	.162	.111	.075						1.442	.415	.240	.157	.107	.082						1.441	1.443
6	.365	.224	.156	.113	.082	.059					1.612	.370	.224	.153	.108	.079	.066					1.611	1.614
7	.330	.209	.150	.113	.085	.065	.048				1.757	.335	.209	.148	.109	.081	.063	.055				1.755	1.759
8	.302	.196	.144	.111	.086	.068	.053	.040			1.882	.307	.197	.142	.107	.082	.064	.052	.048			1.880	1.885
9	.279	.184	.138	.108	.086	.069	.056	.044	.035		1.994	.284	.186	.137	.106	.083	.066	.053	.044	.043		1.991	1.997
10	.260	.174	.132	.105	.085	.070	.057	.047	.038	.031	2.094	.265	.176	.132	.103	.083	.067	.054	.045	.038	.038	2.092	2.098

Ⅳ. Discussions

Assume we have a decision environment in which we have a payoff matrix

$$\begin{array}{cccccc} & C_1 & C_2 & \cdots & C_n \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}$$

In the above matrix, a set $A=\{A_1, A_2,\cdots, A_m\}$ corresponds to a set of alternatives and a set $C=\{C_1, C_2,\cdots, C_n\}$ corresponds to the possible state of nature. In the above, a_{ij} indicates the payoff for selecting alternative A_i when the state of nature is C_j. In the decision making under uncertainty environment, knowledge of the state of nature is unavailable. A number of approaches have been suggested for addressing this problem, based upon the concept of a decision attitude associated with the decision maker. The two classic attitudes that have been suggested are the pessimistic and optimistic. Assuming a pessimistic attitude, one calculates

$$H_P(A_i)=\text{Min}_j\ a_{ij}.$$

Then we select, as the optimal alternatiave, the A^* such that

$$H_P(A^*)=\text{Max}_i\ H_P(A_i).$$

Thus we see that

$$H_P(A^*)=\text{Max}_i\ \text{Min}_j\ a_{ij}.$$

This approach is the so called *maximin* strategy, which is widely used in game theory. With the same line of reasoning, *maximax* strategy is applied when we assume an optimistic attitude, thus yielding

$$H_O(A^*)=\text{Max}_i\ \text{Max}_j\ a_{ij}.$$

A strategy lying between these two extremes, called the Hurwicz criteria, has been suggested. This strategy involves calculating for some $0\leq \alpha \leq 1$

<Table 8> Semantic meanings for the orness values derived by the formulas

Decision strategy	Order weights (n=2)	Order weights (n=3)	Orness
Optimistic	(1, 0)	(1, 0, 0)	1
Moderately Optimistic	(0.750, 0.250)	(0.611, 0.278, 0.111)	0.75
Rather Optimistic	(0.667, 0.333)	(0.500, 0.333, 0.167)	0.67
Neutral	(0.50, 0.50)	(0.333, 0.333, 0.333)	0.5
Rather Pessimistic	(0.333, 0.667)	(0.167, 0.333, 0.500)	0.34
Moderately Pessimistic	(0.250, 0.750)	(0.111, 0.278, 0.611)	0.25
Pessimistic	(0, 1)	(0, 0, 1)	0

$$H_H(A_i) = \alpha\ H_P(A_i) + (1-\alpha)\ H_O(A_i).$$

The concept of OWA operators can provide a unifying and generalizing formulation for this problem. In particular, we associate with our decision process on OWA function F and its associated weighting vector W. Then for each A_i we calculate

$$H_w(A_i) = F(a_{i1},\ a_{i2},\ \cdots,\ a_{in}).$$

The optimal selection becomes the A^* such that $H_w(A^*)$ is maximal.

We note here that when $W=W_*$ we get the maximin strategy and when $W=W^*$ we get the maximax. The introduction of the OWA framework provides a more general approach to the solution of the decision making under uncertainty problem by allowing for a whole spectrum of potential ways of aggregating the payoffs for the individual alternatives. In this framework, a useful unifying concept can be obtained with the aid of the measure of orness associated with an OWA operator. In particular, the measure of orness can be used as a measure of the decision maker's degree of optimism. The closer an OWA operator is to the pure or, the more optimistic s/he is about obtaining the best solution.

These statements can be outlined such as

- If $W=W^*$ (*orness* = 1), $\max_{i=1,m}[\max_{j=1,n}]$: maximax strategy (i.e., optimistic)
- If $W=W_*$ (*orness* = 0), $\max_{i=1,n}[\min_{j=1,n}]$: maximin strategy (i.e., pessimistic)
- If $w_i=1/n$ (*orness* = 1/2), La Place criteria

If we confer semantic meanings for the orness values other than orness values introduced above, it is possible to aggregate, taking into account decision maker's modes as in <Table 8>, multiple values of alternatives.

V. Concluding remarks

Since the OWA operator was introduced, numerous research efforts have been exerted to determine the OWA operator weights. To this end, we present four analytic forms of OWA operator weighting functions of which the *orness* is set to constant value irrespective of the number of objectives. Further, based on the four analytic weighting functions and well-known OWA operator weights with max, min, average, any new weighting vectors can be constructed on the *orness* scale [0, 1] whose *orness* is also constant irrespective of the number of objectives. Further, the degree of utilizing information in aggregation process is very close to that of weights by MEOWA method. Thus in a situation where a priori degree of optimism is specified from decision maker, we only have to apply the analytic weights function or generate OWA weights that have a desired value of *orness* to perform a multicriteria aggregation.

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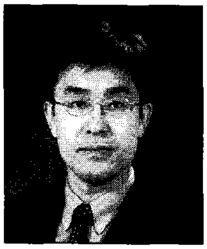
◆ 저자소개 ◆



안병석 (Ahn, Byeong Seok)
현재 한성대학교 경영학부에서 부교수로 재직 중이다. 연세대학교 상경대학 응용통계학과를 졸업하고(1987), 한국과학기술원에서 경영과학분야로 석사(1990), 경영정보공학분야로 박사학위를 취득하였으며(1998), 2003년 LG연암재단의 지원을 받아 미국 UCI 경영대학원의 운영 및 의사결정전공분야(Operation and Decision Technology)에서 방문교수로 1년간 재직한 바 있다. 주요 관심 연구분야로는 다기준 의사결정 이론 및 시스템, 소프트 컴퓨팅 분야를 들 수 있다.

이 논문은 2005년 8월 18일 접수하여 1차 수정을 거쳐 2005년 12월 6일 게재확정되었습니다.