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Constrained Dynamic Responses of Structures Subjected to Earthquake

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Abstract

Starting from the quadratic optimal control algorithm, this study obtains the relation of the performance index for constrained systems and Gauss's principle. And minimizing a function of the variation in kinetic energy at constrained and unconstrained states with respect to the velocity variation, the dynamic equation is derived and it is shown that the result compares with the generalized inverse method proposed by Udwadia and Kalaba. It is investigated that the responses of a 10-story building are constrained by the installation of a two-bar structure as an application to utilize the derived equations. The structural responses are affected by various factors like the length of each bar, damping, stiffness of the bar structure, and the junction positions of two structures. Under an assumption that the bars have the same mass density, this study determines the junction positions to minimize the total dynamic responses of the structure.

Keywords : Constraint, Control, Generalized Inverse Matrix, Control Force

1. INTRODUCTION

External excitations like earthquake must be a main cause to give the loss of property and life. It is desirable to accommodate the structural design method for alleviating the structural damage on seismic or wind load. And it is sometimes necessary to assure more positive safety by installing active or passive control devices in structures. The control devices influence on the dynamic characteristics of structures and provide the control forces calculated by proper control algorithms. Many control systems have been utilized in the structures to control the dynamic responses.

The control forces executed by active control systems are calculated from the quadratic form of state space and control force. If the dynamic responses are restricted by some given trajectories, the state space is deleted in the quadratic form and the responses are obtained by minimizing the quadratic function of control force only with respect to the control force. In this case, the control force is interpreted as the constraint force to provide the structure for satisfying the given paths. The constraint forces act on the structure by control systems like passive devices or actuators. However, it is not easy to determine the constraint forces. There have been many attempts to explicitly describe the constrained responses of structures after Lagrange in 1797.

Gibbs (1879) and Appell (1911) provided an analytical method through a felicitous choice of quasi-coordinates. This approach is usually amenable to problem-specific situations and is likewise difficult to use, when dealing with systems having several tens of freedom. Kane (1983) introduced a method for constrained systems based on the development of Lagrange equations from D'Alembert's Principle.

Based on Gauss's principle (Gauss, 1829) and fundamental linear algebra (Graybill, 1983), Udwadia and Kalaba (1992) derived the generalized inverse method, which does not require the numerical determination of undetermined multipliers like Lagrange multipliers. Udwadia, Kalaba and Eun (1997) presented an extended D'Alembert's principle and proved the generalized inverse method. In spite of such effort, it is necessary that the validity and uniqueness of the generalized inverse method will be investigated.

There have been few papers to consider the dynamic control of structures with constraints. Gurgoze and Muller (1992) presented a method to determine the optimal positioning of the dampers, actuators and sensors for a linear conservative mechanical system on the basis of an energy criterion. Boutin, Misra and Modi (1999) presented a method to obtain the equations governing the constrained dynamics of the entire systems from equations of motion for individual sub-structures by eliminating the non-working constraint forces.

In this study, comparing the performance index for controlling the constrained motion and Gauss's principle, the weighting matrix in the performance index of control algorithm will be determined. And starting from a function of the variation in kinetic energy at unconstrained and constrained states, and minimizing it with respect to the velocity variation, this study derives the constrained equation of motion. The result is compared with the generalized inverse method. The validity of the derived equations is illustrated by describing the dynamic responses of a 10-story building structure subjected to earthquake and controlled by a two-bar structure. The constrained control by the bar structure is affected by the variables like the mass and mass moment of inertia of each bar, damping, stiffness of the bar structure, and the junction positions of two structures. Under an assumption that the bars have the same mass density, this study determines the optimal junction positions to minimize the total responses.

2. OPTIMAL CONTROL ALGORITHM

The control system based on a quadratic performance index may be defined by

$$\dot{\mathbf{q}} = \mathbf{D}\mathbf{q} + \mathbf{E}\mathbf{f} \tag{1}$$

where $\mathbf{q}$: $n \times 1$ state vector
$\mathbf{f}$: $r \times 1$ control vector, $r < n$
$\mathbf{D}$: $n \times n$ constant matrix
$\mathbf{E}$: $n \times r$ constant matrix

The performance index is given by

$$J = \int_0^\infty \left(\mathbf{q}^{\mathbf{T}} \mathbf{Q} \mathbf{q} + \mathbf{f}^{\mathbf{T}} \mathbf{R} \mathbf{f} \right) dt \tag{2}$$

where $\mathbf{Q}$ is a positive-definite Hermitian or real symmetric matrix, $\mathbf{R}$ is a positive-definite Hermitian or real symmetric matrix, and $\mathbf{f}$ is unconstrained.

Based on the second method of Liapunov, the optimal control system is derived by minimizing the performance index, and the control forces are determined. The control algorithm yields the minimum values of the state vector and the control forces at the unconstrained state.

If the dynamic responses of the system are restricted by constraints, the state vector in the performance index should be deleted because the constrained paths govern the dynamic responses. Replacing the control vector $\mathbf{f}$ by the constraint force vector $\mathbf{F^c}$, the performance index can be written as

$$J = \int_0^\infty \left(\mathbf{F^c}^{\mathbf{T}} \mathbf{R} \mathbf{F^c} \right) dt \tag{3}$$

Minimizing the performance index given by equation (3), the constraint force vector may be derived. This procedure is exactly the same as Gauss's principle. The Gauss's principle is as follows. Assuming that the configuration, $\mathbf{q}(t) = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$, and the velocity, $\dot{\mathbf{q}}(t) = [\dot{q}_1 \quad \dot{q}_2 \quad \cdots \quad \dot{q}_n]^T$, of a constrained system at time t are prescribed, the acceleration of the unconstrained systems, $\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}, t)$, is known. Then the Gauss's principle informs us that the accelerations, $\ddot{\mathbf{q}}(t)$, are such that the Gaussian function, G, defined as

$$G = [\ddot{\mathbf{q}} - \mathbf{a}]^T \mathbf{M} [\ddot{\mathbf{q}} - \mathbf{a}] \tag{4}$$

is minimized over all $\ddot{\mathbf{q}}$ which satisfy the constraints.

The equation of motion at time t of the constrained system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{F^c}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{5}$$

where $\mathbf{M}$ is an $n \times n$ mass matrix. Substituting equation (5) into equation (4), the Gaussian function is modified as

$$G = \mathbf{F^c}^{\mathbf{T}} \mathbf{M}^{-1} \mathbf{F^c} \tag{6}$$

It can be observed that the Gaussian function G is utilized as the same meaning as the performance index of equation (3). However, comparing two equations (3) and (6), it is indicated that the weighting matrix $\mathbf{R}$ in equation (3) must be the matrix $\mathbf{M}^{-1}$ based on the Gauss's principle. Thus, it is alluded that the constrained algorithm can be derived by the minimization of the performance index (3) or the Gaussian function (6).

3. EQUATIONS OF MOTION FOR CONSTRAINED SYSTEMS

The description of the constrained responses depends on the determination of the constraint forces. Minimizing a function of the variation in kinetic energy at unconstrained and constrained states with respect to the velocity variation, the constraint forces and the constrained equation of motion are derived.

The kinetic energy of unconstrained structure to be described by a velocity vector $\dot{\tilde{\mathbf{u}}} = [\dot{\tilde{u}}_1 \quad \dot{\tilde{u}}_2 \quad \cdots \quad \dot{\tilde{u}}_n]^T$ can be written as

$$\tilde{\mathrm{T}} = \frac{1}{2} \dot{\tilde{\mathbf{u}}}^{\mathbf{T}} \mathbf{M} \dot{\tilde{\mathbf{u}}} \tag{7}$$

where $\tilde{\mathrm{T}}$ denotes the kinetic energy of unconstrained structure and $\mathbf{M}$ is the $n \times n$ positive definite mass matrix.

Assume that the structure is subjected to m displacement constraints

$$f_i(\mathbf{u}) = r(t), \qquad i = 1, \quad 2, \quad \cdots, \quad m, \quad m < n \tag{8}$$

where $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_n]^T$ denotes the actual displacements deviated from unconstrained state. Differentiating once the constraints with respect to time t, the constraints can be written in matrix form of

$$\mathbf{A}\dot{\mathbf{u}}(t) = \mathbf{b_1}(t) \tag{9}$$

where $\mathbf{A}$ is a real matrix of $m \times n$ and $\mathbf{b_1}(t)$ is an $m \times 1$ vector. The actual kinetic energy of the structure due to the existence of the constraints is also expressed by

$$\mathrm{T} = \frac{1}{2} \dot{\mathbf{u}}^{\mathbf{T}} \mathbf{M} \dot{\mathbf{u}} \tag{10}$$

Utilizing the velocity variation, $\boldsymbol{\delta}\dot{\mathbf{u}}$, due to the constraints, the following relation between $\dot{\mathbf{u}}$ and $\dot{\tilde{\mathbf{u}}}$ is established that

$$\dot{\mathbf{u}} = \dot{\tilde{\mathbf{u}}} + \delta\dot{\mathbf{u}} \tag{11}$$

Also, let us $\dot{\tilde{\mathbf{u}}} = \mathbf{R}\left(\delta\dot{\tilde{\mathbf{u}}}\right)$, where $\mathbf{R}$ is a positive definite matrix. Substituting equation (11) into equations (7) and (10), and finding the difference of the results, the variation in the kinetic energy can be expressed as

$$\delta \mathrm{T} = \mathrm{T} - \tilde{\mathrm{T}} = \frac{1}{2}\left[\delta\dot{\mathbf{u}} + \mathbf{R}(\delta\dot{\mathbf{u}})\right]^{\mathrm{T}} \mathbf{M}\left[\delta\dot{\mathbf{u}} + \mathbf{R}(\delta\dot{\mathbf{u}})\right] - \frac{1}{2}\dot{\tilde{\mathbf{u}}}^{\mathrm{T}}\mathbf{M}\dot{\tilde{\mathbf{u}}}$$

$$= \frac{1}{2}\left[\mathbf{M}^{1/2}\delta\dot{\mathbf{u}} + \mathbf{M}^{1/2}\mathbf{R}(\delta\dot{\mathbf{u}})\right]^{\mathbf{T}}\left[\mathbf{M}^{1/2}\delta\dot{\mathbf{u}} + \mathbf{M}^{1/2}\mathbf{R}(\delta\dot{\mathbf{u}})\right]$$

$$-\frac{1}{2}\left[\mathbf{M}^{1/2}\mathbf{R}(\delta\dot{\mathbf{u}})\right]^{\mathbf{T}}\left[\mathbf{M}^{1/2}\mathbf{R}(\delta\dot{\mathbf{u}})\right] \tag{12}$$

Extremizing equation (12) with respect to the variation $\delta\dot{\mathbf{u}}$, the result yields

$$\frac{\delta \mathrm{T}}{\delta\dot{\mathbf{u}}} = \mathbf{M}^{1/2}\delta\dot{\mathbf{u}} = 0 \tag{13}$$

The utilization of equation (11) into equation (9) yields

$$\mathbf{A}\left(\dot{\tilde{\mathbf{u}}} + \delta\dot{\mathbf{u}}\right) = \mathbf{b}_1(t) \tag{14}$$

In order to use equation (14) into equation (13), equation (14) is modified as

$$\mathbf{A}\mathbf{M}^{-1/2}\mathbf{M}^{1/2}\left(\dot{\tilde{\mathbf{u}}} + \delta\dot{\mathbf{u}}\right) = \mathbf{b}_1(t) \tag{15}$$

Utilizing the fundamental properties[1] of generalized inverse matrix, the general solution of equation (15) can be derived as

$$\mathbf{M}^{1/2}\left(\dot{\tilde{\mathbf{u}}} + \delta\dot{\mathbf{u}}\right) = \mathbf{Q}^{+}\mathbf{b}_1 + \left[\mathbf{I} - \mathbf{Q}^{+}\mathbf{Q}\right]\mathbf{y} \tag{16}$$

where $\mathbf{Q} = \mathbf{A}\mathbf{M}^{-1/2}$, the vector $\mathbf{y}$ is an arbitrary vector and '+' denotes the generalized inverse matrix.

Utilizing equation (13) into equation (16) and the fundamental relation of $\mathbf{Q}\mathbf{Q}^{+}\mathbf{Q} = \mathbf{Q}$, and solving the result with respect to the vector **y**, we obtain the equation

$$\mathbf{y} = \left[\mathbf{I} - \mathbf{Q}^{+}\mathbf{Q}\right]\left(\mathbf{M}^{1/2}\dot{\tilde{\mathbf{u}}} - \mathbf{Q}^{+}\mathbf{b}_1\right) + \left[\mathbf{I} - \mathbf{Q}^{+}\mathbf{Q}\right]\mathbf{z} \tag{17}$$

where $\mathbf{z}$ is another arbitrary vector.

Substituting equation (17) into equation (16) and arranging the result, it follows that

$$\mathbf{M}^{1/2}\delta\dot{\mathbf{u}} = \left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}\left(\mathbf{b}_1 - \mathbf{A}\dot{\tilde{\mathbf{u}}}\right) \tag{18}$$

Finally, substituting equation (18) into equation (11) and differentiating the result once with respect to time, the equation of motion for constrained structure is written as

$$\ddot{\mathbf{u}} = \ddot{\tilde{\mathbf{u}}} + \mathbf{M}^{-1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A}\ddot{\tilde{\mathbf{u}}}\right) \tag{19}$$

where $\ddot{\tilde{\mathbf{u}}}$ denotes the acceleration at the unconstrained state and $\mathbf{b} = \dot{\mathbf{b}}_1$. From equation (19), it is understood that the variation of acceleration due to the constraints and the constraint forces are expressed, respectively, by

$$\delta\ddot{\mathbf{u}} = \mathbf{M}^{-1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A}\ddot{\tilde{\mathbf{u}}}\right) \tag{20a}$$

$$\mathbf{F}^{\mathrm{c}} = \mathbf{M}^{1/2}\left(\mathbf{A}\mathbf{M}^{-1/2}\right)^{+}\left(\mathbf{b} - \mathbf{A}\ddot{\tilde{\mathbf{u}}}\right) \tag{20b}$$

It is shown that the derived equations coincide with the generalized inverse method proposed by Udwadia and Kalaba. The dynamic responses of constrained structures are explicitly described by equation (19) and the constraint forces are calculated by equation (20b). Thus, it is expected that the proper selection of constraints will obtain desirable responses of structures. In the following, we consider the constrained effects by the installation of a substructure as a kind of passive system.

4. CONSTRAINED CONTROL OF A 10-STORY BUILDING

Dynamic vibration of structures by installation of control systems is controlled by changing their dynamic characteristics through proper control algorithm or dissipating the energy. The structural control by constraints is carried out by the constraint forces to act on the structure and depends on the design of the control devices. Installing a device on the structure for constrained control, the constraint forces are executed on the structure to change the dynamic characteristics of the structure and the dynamic responses are controlled.

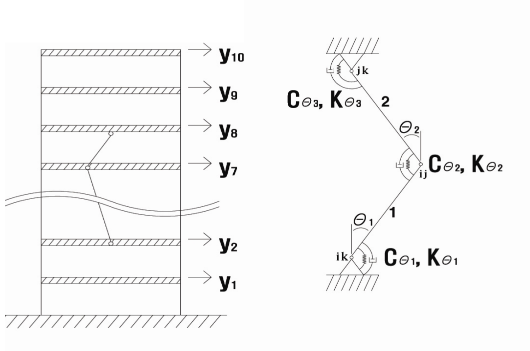

Figure 1. A 10-story shear building

[1] The generalized solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A}$ is $m \times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ and $m \times 1$ vectors, respectively, can be written as

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} + \left[\mathbf{I} - \mathbf{A}^{+}\mathbf{A}\right]\mathbf{d},$$

where $\mathbf{I}$ is $n \times n$ identity matrix and $\mathbf{d}$ is $n \times 1$ arbitrary vector.

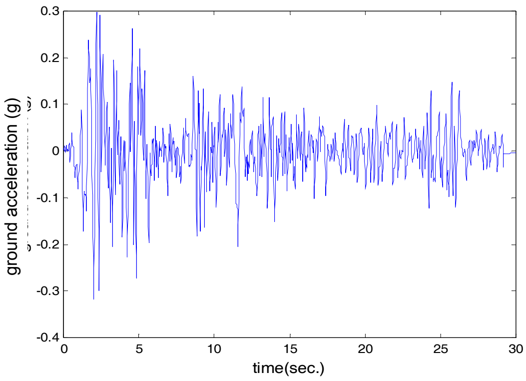

Figure 2. North-south components of earthquake accelerations of El Centro in 1940

This study considered the constrained motion of a 10-story shear building which a two-bar substructure is installed as shown in Fig. 1. The dynamic characteristics of the original structure are changed with the action of the constraint forces such that the same responses at the junction positions of two structures are obtained. The proper selection of constraints will be enough in controlling the responses of the structure.

The equation of motion for the structure subjected to an earthquake is expressed as

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = -\mathbf{M}\{\mathbf{1}\}\ddot{u}_g \tag{21}$$

where **M**, **C** and **K** denote mass, damping and stiffness matrices of the structure, respectively, **{1}** represents vector which all elements are 1, and $\ddot{u}_g$ is horizontal accelerations. The data of north-south components of El Centro earthquake in 1940 was utilized as shown in Fig. 2.

A two-bar structure is installed on the structure and both ends of the bar structure have pin joints. The bar 1 has length l_a, mass m_a, and moment of inertia I_a, and the bar 2 has l_b, m_b, I_b, respectively. Assume that the bars have the same mass per unit length and each bar has the same length. The bar structure shows a rotational spring $K_{\theta i}(i=1,2,3)$ and a dashpot $C_{\theta i}(i=1,2,3)$ at each joint. The structure is a nonlinear system described by $\boldsymbol{\theta} = [\theta_1 \quad \theta_2]^T$. The equations of motion for the bar structure is written as

$$\mathbf{M_s}\ddot{\boldsymbol{\theta}} + \mathbf{H_s}\dot{\boldsymbol{\theta}}^2 + \mathbf{C_s}\dot{\boldsymbol{\theta}} + \mathbf{K_s}\boldsymbol{\theta} = \mathbf{0} \tag{22}$$

where

$$\mathbf{M_s} = \begin{bmatrix} I_a + \left(\frac{1}{4}m_a + m_b\right)l_a^2 & \frac{1}{2}m_b l_a l_b \cos(\theta_2 - \theta_1) \\ \frac{1}{2}m_b l_a l_b \cos(\theta_2 - \theta_1) & I_b + \frac{1}{4}m_b l_b^2 \end{bmatrix}$$

$$\mathbf{H_s} = \begin{bmatrix} 0 & \frac{1}{2}m_b l_a l_b \sin(\theta_2 - \theta_1) \\ -\frac{1}{2}m_b l_a l_b \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$\mathbf{C_s} = \begin{bmatrix} C_{\theta 1} + C_{\theta 2} & -C_{\theta 2} \\ -C_{\theta 2} & C_{\theta 2} + C_{\theta 3} \end{bmatrix},$$

$$\mathbf{K_s} = \begin{bmatrix} K_{\theta 1} + K_{\theta 2} & -K_{\theta 2} \\ -K_{\theta 2} & K_{\theta 2} + K_{\theta 3} \end{bmatrix}.$$

The constraint to indicate the same responses at the junction positions of two structures is expressed as

$$y_{jk} - y_{ik} = l_a \sin\theta_1 + l_b \sin\theta_2,$$
$$ik = 1,2,\cdots,10 \quad jk = 1,2,\cdots 10, \quad ik \neq jk \tag{23}$$

The constraint indicates that the relative responses of the *ik*-th and *jk*-th floors coincide with the relative response of two ends of the bar structure. The joints at the top, middle and bottom of the bat structure were called as *jk*, *ij* and *ik*, respectively.

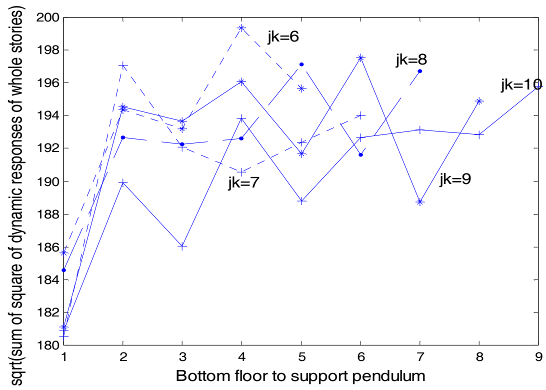

Figure 3. Comparisons of dynamic responses of the structure according to the junction positions of two structures

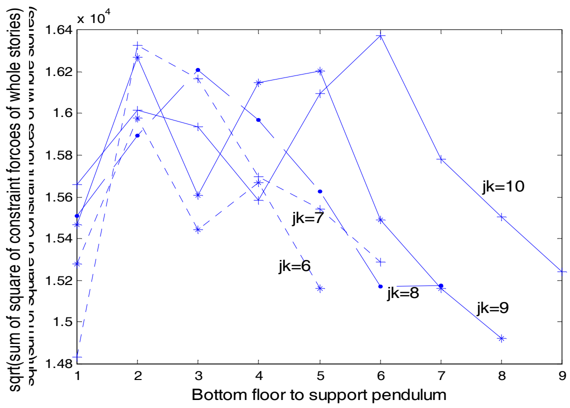

Figure 4. Comparisons of constraint forces of the structure according to the junction positions of two structures

Differentiating equation (23) twice with respect to time *t* and substituting the result, equations (21) and (22) into equation (19), it yields the equations of motion for the structure constrained by the responses at both ends of the two-bar structure.

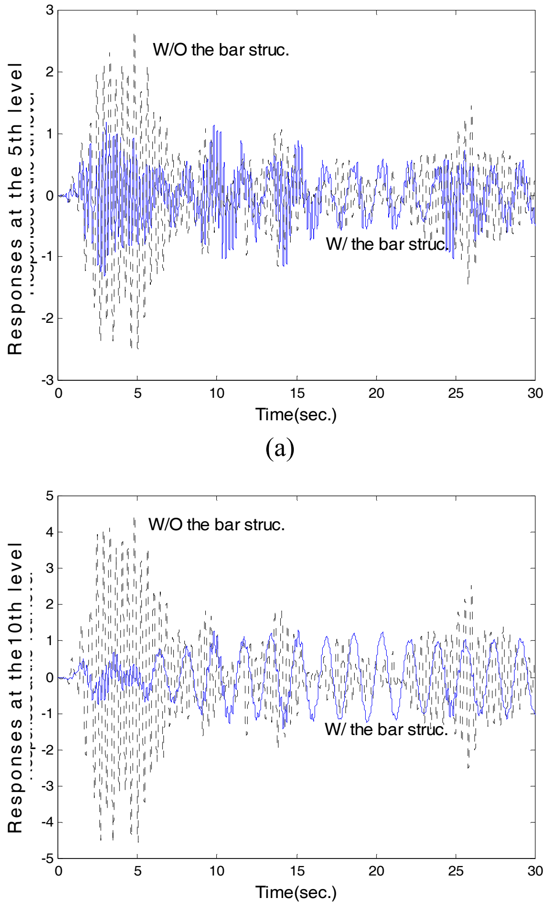

Figure 5. Comparison of the responses at (a) 5th floor, (b) 10th floor; The solid line and dotted line exhibit the responses with and without the bar structure, respectively.

The constraint forces to act on the structure are functions of the length of each bar, damping, stiffness of the bar structure, and the junction positions of two structures. Utilizing the junction positions as a variable, this study numerically calculated the dynamic responses of the structure and the constraint forces at the interconnected positions, *ik* and *jk* during the first 30 seconds expressed as

$$P_1 = \sqrt{\int_0^{30} \mathbf{y}^T \mathbf{y} dt} \, , \tag{24a}$$

$$P_2 = \sqrt{\int_0^{30} \mathbf{F^c}^T \mathbf{F^c} dt} \tag{24b}$$

respectively. The non-dimensionalized physical properties for numerical results are given in Tables 1 and 2.

Table 1. Physical properties of the structure

floor	1	2	3	4	5	6	7	8	9	10
Mass	3	3	3	3	2	3	4	3	4	5
stiffness ($\times 100$)	5	5	5	4	4	4	3	3	3	3

Table 2. Physical properties of the bar structure

joint	*ik*	*ij*	*jk*
stiffness	70	50	60
damping coefficient	0.3	0.5	0.5

The masses of the top and bottom bars are 0.7 and 0.5, respectively. And the initial positions of the bars were selected as $\theta_1 = \pi/10$ and $\theta_2 = -\pi/10$.

Figures 3 and 4 compare the responses and constraint forces, P_1 and P_2, according to the interconnected positions *ik* and *jk* of two structures. From the figures, it is observed that the responses and constraint forces depend on the interconnected positions. The minimum responses of the structure due to the bar structure appeared at the

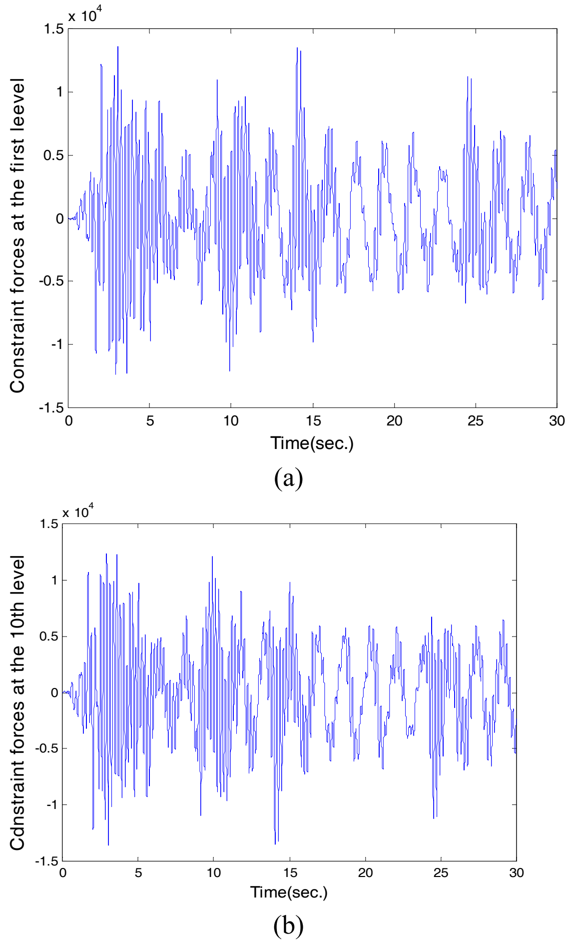

Figure 6. Constraint forces at (a) first floor (b) 10th floor

positions *jk*=10 and *ik*= 1. And the highest constraint forces occurred at the positions *jk*=10 and *ik*= 6. However, comparing the constraint forces to act on the structure when the bottom joint located at the position *ik*=1, the minimum responses were observed when the top joint is located at the position *jk*=10 to show the maximum constraint forces. It is indicated that the magnitude of the dynamic responses of whole stories of the structure is inversely proportional to the magnitude of the constraint forces according to the junction positions of two structures.

Figure 5 compares the responses of the structure with and without the bar structure. It is represented that the presence of the bar structure led to the remarkable reduction of the responses. It can be investigated that the installation of the bar structure should be a control system to utilize the constraint which restricts the dynamic responses. Also, Fig. 6 represents the constraint forces for the constrained responses. Recognizing that the constraint forces act on the interconnected positions only, it is expected that the installation of multiple bar structures will yield less structural responses and will obviate the twist of the structure.

5. CONCLUSIONS

This study considered the constrained control of structures for alleviating the dynamic response. It was verified that the constrained control of structures can reduce the dynamic responses and the structural damages. The results of this study are summarized as follows.

(1) It was observed that the quadratic performance index for constrained control can be utilized as the same meaning as the Gaussian function and the weighting matrix indicates the inverse of mass matrix.

(2) Minimizing the function of the variation in the kinetic energy at the unconstrained and constrained structures with respect to the velocity variation, the equation of motion for constrained structure was derived. The derived result corresponded with the generalized inverse method proposed by Udwadia and Kalaba.

(3) It was also exhibited that the constraint control by the installation of a two-bar structure can reduce and control the dynamic responses of structures. The dynamic responses of structures were inversely proportional to the constraint forces according to the junction positions of two structures.

(4) From the application, it is recognized that the dynamic control by constraints can be widely utilized in the control field of mechanical and structural systems.

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