Hjorth모형과 Dhillon모형에 대한 재생함수 추정

남경현*† · 장석주* · 김도훈**

* 경기대학교, 응용정보통계학과
** 호남대학교, 경영학과

Approximation of the Renewal Function for Hjorth Model and Dhillon Model

Kyung H. Nam*† · Seog-Ju Chang** · Do Hoon Kim*

* Department of Applied Information Statistics, Kyonggi University
** Department of Business Administration, Honam University

Key Words : Renewal Function, Approximation, Hjorth Model, Dhillon Model

Abstract

This paper applies approximation of the renewal function for Hjorth model and Dhillon model which show the trend change in its aging properties. We obtain the renewal function for Hjorth model and Dhillon model by a numerical solution of an approximate integral. We observe the influence of each parameter in these models. The results of the computation are described and their corresponding graphs are provided.

1. Introduction

A renewal process is defined by a sequence $X_1, X_2, X_3, \cdots$ of nonnegative random variables where the interevent times are independent and identically distributed. These processes have proved to be a powerful tool in stochastic modeling in a wide variety of applications (Karlin, 1958 ; Ross, 1970). Also a renewal process is used to model the successive repairs or replacements of a failed items in reliability theory (Barlow and Proschan, 1965 ; 1975), and is used to model the successive purchases of a new item following the expiration of a free-replacement warranty (Blischke and Scheuer 1975 ; 1981).

Even though a renewal process is considered useful, easy and reliable methods of solution to renewal function for renewal processes associated with general life distributions are rather limited. The problem is more complicated by the fact that when the renewal density is highly skewed with an extremely long upper tail, the limiting value of the renewal density is approached too slowly for it to be of any practical significance.

Feller (1941), Weiss (1962), Smith and Leadbetter (1963), Lomnicki (1966) and Jaquette (1972) use the Laplace transformation method to approximate the renewal function.

Bartholomew (1963), Ozbaykal (1971) and Deligonul (1985) present other approximation methods by employing special forms of renewal function.

In addition, a variety of computational procedures for renewal function have been considered. Baxter et al. (1982), using the cubic spline method developed by Cleroux and Mc-

† 교신저자 knam@kyonggi.ac.kr

Conalogue (1976), produce comprehensive tables for the renewal function, the variance function, and the integral of the renewal function for Gamma, Weibull, inverse Gaussian, lognormal, and truncated normal distribution to a wide range of values of shape parameter of each distribution.

In this paper, we obtain an approximation of renewal function for Hjorth model and Dhillon model which exhibit various failure shapes. As a result, we examine the influence of the renewal function as property of each parameter of Hjorth model and Dhillon model. Also, we observe relationship between change of approximation of renewal function and change of cumulative failure rate according to the change of a specific parameter values in these models. We describe the results of computation for the renewal function and provide graphs of the renewal function for each parameter.

2. Renewal Theory

2.1 Renewal function

Let X_i denote the nonnegative random variable that is presented interevent time of component or system, and F is the distribution function of X_i. Suppose $S_n = X_1 + X_2 + \cdots + X_n$, $n \geq 1$, then S_n is the random time at which n-th replacement occurs. The renewal counting function $N(t)$ is the number of renewals in $(0, t]$, that is, $N(t) = sup\{n \mid S_n \leq t\}$. Now, we define the renewal function $M(t)$ to be the expected value of $N(t)$. It can be verified that $M(t)$ can be defined in terms of an integral equation

$$M(t) = F(t) + \int_0^t M(t-x)dF(x). \qquad (1)$$

Recognizing that equation (1) is a non-decreasing function and is called the renewal function with $F(t)$.

2.2 Approximations of the renewal function

The renewal function plays an important role in the analysis of reliability theory and warranty policy. Despite its wide use, only a few classes of life distribution function have its analytical expression for renewal function. Therefore, we need an approximation method to obtain the corresponding renewal function with $F(t)$. There are three approximations to $M(t)$ have been proposed.

Bartholomew (1963) proposed an approximation which is

$$M_b(t) = F(t) + \gamma \int_0^t \frac{F^2(x)}{F_e(x)} dx, \qquad (2)$$

where

$$F_e(t) = \gamma \int_0^t [1 - F(x)]dx, \qquad t \geq 0$$

with $\gamma = 1/\mu$, being the expected value of interevent time. Ozbaykal (1971) proposed an approximation given by

$$M_o(t) = \gamma t - F_e(t) + \gamma \int_0^t [1 - F_e(t)]dx. \qquad (3)$$

And Deligonul (1985) proposed

$$M_{de}(t) = \gamma t - F_e(t) + \int_0^t [1 - F_e(t-x)] \left[f(x) + \gamma \frac{F^2(x)}{F_e(x)} \right] dx. \qquad (4)$$

Deligonul provides some numerical comparisons, involving IFR and DFR distribution, showing that $M_{de}(t)$ is superior to both $M_b(t)$ and $M_o(t)$ by comparing the different approximations with exact $M(t)$.

In this paper, we will use approximation $M_{de}(t)$ in order to obtain the renewal function for Hjorth model and Dhillon model.

3. Approximations for Hjorth model and Dhillon model

3.1 Hjorth model

Hjorth (1980) suggests a three-parameter life distribution obtained by generalizing the Rayleigh distribution which itself is generalization of the exponential distribution at least as flexible as the Weibull family and with capacity to also describe bathtub-shaped failure rates. The Hjorth model includes distribution with increasing, constant, decreasing, and bathtub-shaped failure rate depending upon the values of the parameter. The reliability function is given by

$$R(t)=1-F(t)=\frac{\exp(-\delta t^2/2)}{(1+\beta t)^{\theta/\beta}}, \tag{5}$$

when $\delta,\ \beta,\ \theta>0$.

Also their corresponding failure rate is given by

$$\lambda(t)=\delta t+\frac{\theta}{(1+\beta t)}, \qquad t\geq 0. \tag{6}$$

Henceforth, the reliability function given in equation (5) is referred to as Hjorth model.

Special cases of Hjorth model are shown in the following table.

<Table 1> Failure rate behavior for selected parameter combination of Hjorth model

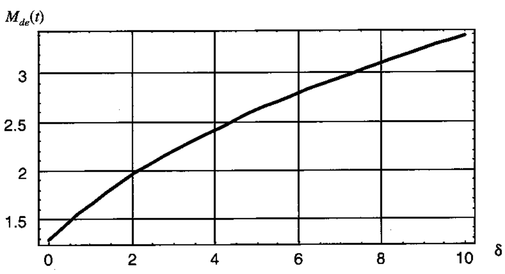
	Failure rate behavior or distribution
$\theta=0$	the Rayleigh distribution
$\delta=\beta=0$	constant failure rate (CFR)
$\delta=0$	decreasing failure rate (DFR)
$\delta\geq\theta\beta$	increasing failure rate (IFR)
$0<\delta<\theta\beta$	bathtub-shaped failure rate (DIFR)

For the special cases of Hjorth model mentioned above, we are only interested in the case for DIFR. Also, we observe the turning point of the renewal function in graphical point of view.

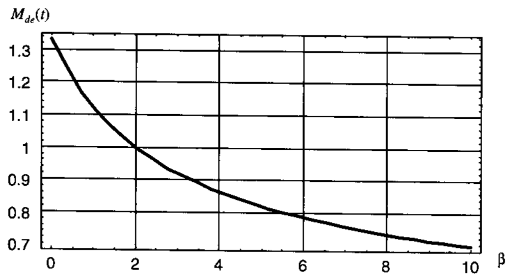
The results listed in <Table 2> show both computation results of the renewal function and cumulative failure rate for several combinations of $\delta,\ \beta,\ \theta$. As the values of δ and θ increase, computation results of the renewal function increase. On the other hand, as the values of β increase, its computation results decrease gradually. Also, these results are accordance with the change point of cumulative failure rate.

<Table 2> Computation results of renewal function $M_{de}(t)$ and cumulative failure rate in Hjorth model

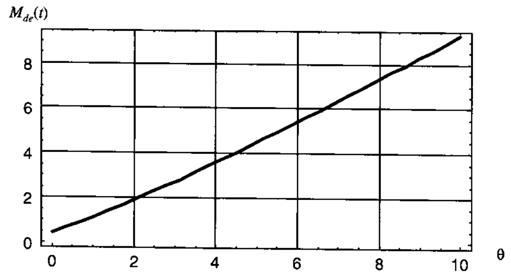
δ	β	θ	Approximation $M_{de}(t)$ of renewal function (Cumulative failure rate)				
		t	0.5	1	1.5	2	5
1	1	2	0.957 (0.936)	1.888 (1.886)	2.847 (2.958)	3.813 (4.197)	9.613 (16.08)
1	1	5	2.292 (2.152)	4.495 (3.966)	6.677 (5.707)	8.880 (7.493)	22.037 (21.46)
1	1	10	4.668 (4.179)	9.247 (7.432)	13.822 (10.28)	18.398 (12.98)	45.849 (30.42)
1	2	1	0.471 (0.472)	0.999 (1.049)	1.585 (1.819)	2.188 (2.805)	5.760 (13.69)
1	5	1	0.379 (0.376)	0.819 (0.858)	1.336 (1.553)	1.875 (2.476)	5.020 (13.15)
1	10	1	0.307 (0.304)	0.697 (0.739)	1.177 (1.402)	1.679 (2.305)	4.555 (12.89)
0.5	2	2	0.796 (0.756)	1.483 (1.349)	2.165 (1.949)	2.854 (2.609)	7.056 (8.648)
1	2	2	0.846 (0.818)	1.652 (1.599)	2.482 (2.511)	3.327 (3.609)	8.401 (14.89)
2	2	2	0.942 (0.943)	1.946 (2.099)	2.987 (3.636)	4.029 (5.609)	10.256 (27.39)



<Figure 1> Renewal function for the value of δ in Hjorth model with $\beta=2$, $\theta=2$, and $t=1$



<Figure 2> Renewal function for the value of β in Hjorth model with $\delta=1$, $\theta=1$, and $t=1$



<Figure 3> Renewal function for the value of θ in Hjorth model with $\delta=1$, $\beta=1$, and $t=1$

We also provide three figures <Figure 1>, <Figure 2>, and <Figure 3> for the special value of each parameter when $t=1$.

3.2 Dhillon Model

Dhillon (1979 ; 1981) suggests a model which can describe increasing, decreasing, and decreasing initially, then increasing failure rate depending on the conditions on the parameters. The reliability function and failure rate of Dhillon's model are defined by

$$R(t)=\exp\{-k\lambda t^{c}-(1-k)[\exp(\beta t^{b}-1)]\}, \quad (7)$$

and

$$\lambda(t)=k\lambda c t^{c-1}+(1-k)t^{b-1}b\beta\exp(\beta t^{b}) \quad (8)$$

for $b, c, \beta, \lambda > 0$, $0 \le k \le 1$, and for $t \ge 0$.

This model contains five parameters with two shape parameter b and c with two scale parameters λ and β. The parameter k is sort of weight parameter. As special cases of distribution defined in (7), the well-known distribution such as Weibull, exponential, extreme value, Makeham, and DIFR distribution are obtained as follows.

<Table 3> Failure rate behavior for selected parameter combination of Dhillon model

	Failure rate behavior or distribution
$c=1$, $b=1$	Makeham distribution
$k=1$, $b=1$	Extreme value
$k=1$	Weibull
$c=0.5$, $b=1$	bathtub-shaped failure rate(DIFR)

Without loss of generality, we will fix scale parameter λ and β to be 1. Thus (7) and (8) are reduced to

$$R(t)=\exp\{-kt^{c}-(1-k)[\exp(t^{b}-1)]\}, \quad (9)$$

and

$$\lambda(t)=kct^{c-1}+(1-k)t^{b-1}b\exp(t^{b}). \quad (10)$$

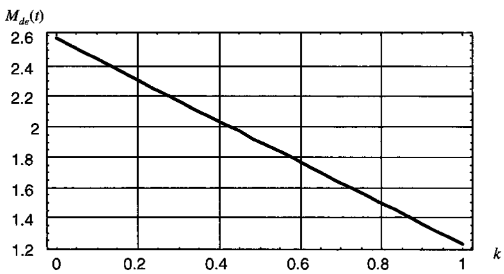
We consider only DIFR case as in the Dhillon model.

The results listed in <Table 4> show computation results of renewal function $M_{de}(t)$ and cumulative failure rate for several combinations of k, b, c. <Table 4> shows that for fixed b and c, as the values of k increase the renewal function $M_{de}(t)$ decrease for all t. In the case of $0<t<1$, as the values of b increase, the renewal function $M_{de}(t)$ decrease for fixed k and c. On the other hand, in the case of $t>1$, as the values of b increase, the renewal function $M_{de}(t)$ increase, and in the case of $t=1$, the renewal function $M_{de}(t)$ constant for fixed k and c. These results are equal to the case of parameter c. Also, these results are accordance with the change point of cumulative failure rate.

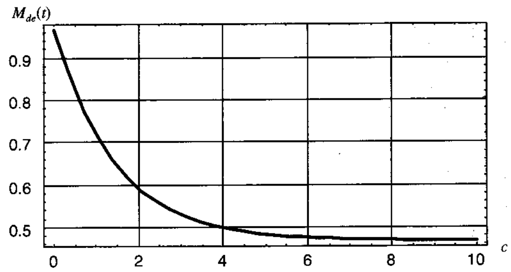
We also provide figures <Figure 4>, <Figure 5>, and <Figure 6> which show that same trends as in the <Table 4>.

<Table 4> Computation results of renewal function $M_{de}(t)$ and cumulative failure rate in Dhillon model

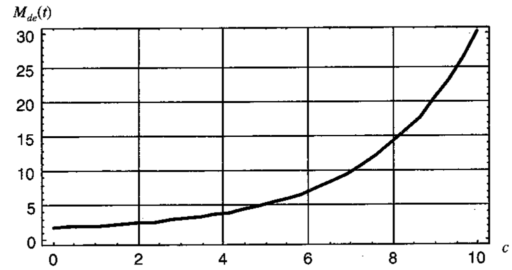
δ	β	θ	Approximation $M_{de}(t)$ of renewal function (Cumulative failure rate) t = 0.5	1	1.5	2	5
.0	.5	.5	0.912 (1.028)	1.647 (1.718)	2.445 (2.403)	3.3398 (3.113)	11.767 (8.357)
.2	.5	.5	0.889 (0.964)	1.584 (1.575)	2.309 (2.168)	3.125 (2.773)	10.708 (7.132)
.5	.5	.5	0.821 (0.868)	1.368 (1.359)	1.903 (1.814)	2.484 (2.264)	7.531 (5.296)
.7	.5	.5	0.775 (0.803)	1.234 (1.216)	1.631 (1.578)	2.056 (1.924)	5.413 (4.072)
.5	.1	.7	0.8946 (1.079)	1.234 (1.359)	1.479 (1.581)	1.713 (1.773)	2.831 (2.661)
.5	.2	.7	0.847 (1.002)	1.234 (1.359)	1.517 (1.643)	1.783 (1.889)	3.052 (3.029)
.5	.3	.7	0.807 (0.934)	1.234 (1.359)	1.558 (1.711)	1.865 (2.025)	3.377 (3.571)
.5	.7	.7	0.686 (0.733)	1.234 (1.359)	1.762 (2.051)	2.359 (2.850)	8.421 (11.98)
5	.5	.2	0.902 (0.949)	1.368 (1.359)	1.832 (1.744)	2.351 (2.131)	7.103 (4.868)
.5	.5	.3	0.873 (0.920)	1.368 (1.359)	1.855 (1.766)	2.392 (2.172)	7.223 (4.989)
.5	.5	.5	0.821 (0.868)	1.368 (1.359)	1.903 (1.814)	2.484 (2.264)	7.531 (5.296)



<Figure 4> Renewal function for the value of k in Dhillon model with b=0.5, c=0.5, t=1.5



<Figure 5> Renewal function for the value of c in Dhillon model with k=0.5, b=0.5, t=0.5



<Figure 6> Renewal function for the value of c in Dhillon model with k=0.5, b=0.5, t=1.5

4. Conclusion

In this paper, we provided an approxi- mation of the renewal function for Hjorth model and Dhillon model.

From this result, we found there was a trend for renewal function based on each parameter values. Also, we observed there was a relation-ship between renewal function and cumulative failure rate. We may derive an approximation of renewal function for some other life distribution through further research.

Reference

[1] Barlow, R. E. and Proschan, F.(1965), *Mathematical Theory of Reliability*, New York : John Wiley.

[2] ______(1975), *Statistical Theory of Reliability and Life Testing*, New York : Holt, Rinehart and Winston.

[3] Bartholomew, D. J.(1963), "An Approximation Solution of the Integral Equation of Renewal Theory", *Journal of Royal Statistical Society*, Vol. 25B, pp. 432-441.

[4] Baxter, L. A., Scheuer, L. M., McConalogue, D. J., and Blischke, W. R.(1982), "On the Tabulation of the Renewal Function", *Technometrics*, Vol. 24, pp. 151-156.

[5] Blischke, W. R. and Scheuer, E. M.(1975), "Calculating of the Cost of Warranty Policies as a Function of Estimated Life Distributions", *Naval Research Logistics Quarterly*, Vol. 22, pp. 681-696.

[6] ______(1981), "Application of Renewal Theory in Analysis of the Free- Replacement Warranty", *Naval Research Logistics Quarterly*, Vol. 28, pp. 193-205.

[7] Cleroux, R. and McConalogue, D. J.(1976), "A Numerical Algorithm for Recursively-Defined Convolution Integrals Involving Distribution Functions", *Management Science*, Vol. 22, pp. 1138-1146.

[8] Dhillon, B. S.(1979), "A Hazard Rate Model", *IEEE Transactions on Reliability*, Vol. 28, p. 150.

[9] ______(1981), "Life Distribution", *IEEE Transactions on Reliability*, Vol. 30, pp. 457-460.

[10] Deligonul, Z. S.(1985), "An Approximation Solution of the Integral Equation of Renewal Theory", *Journal of Applied Probability*, Vol. 22, pp. 926-931.

[11] Feller, W.(1941), "On the Integral Equation of Renewal Theory", *Annual of Mathematical Statistics*, Vol. 12, pp. 243-267.

[12] Hjorth, U.(1980), "A Reliability Distribution with Increasing, Decreasing, Constant and Bathtub-Shaped Failure Rates", *Technometrics*, Vol. 22, pp. 99-107.

[13] Jaquette, D. L.(1972), "Approximations to the Renewal Function $m(t)$", *Operations Research*, Vol. 20, pp. 722-727.

[14] Karlin, S,(1958), "The Application of Renewal Theory to the Study of Inventory Policies", in Studies in the Mathematical Theory of Inventory and Production, eds. K. J. Arrow, Karlin, S., and Scarf, H., Stanford : Stanford University Press, pp. 270-297.

[15] Lomnicki, Z. A.(1966), "A Note on the Weibull Renewal Process", *Biometrika*, Vol. 53, pp. 375-381.

[16] Ozbaykal, T.(1971), Bounds and Approximations for the Renewal Function, Unpublished M.S. Thesis, Dept. of Operations Research, Naval Post-graduate School, Monterrey, CA.

[17] Ross, S. M.(1970), *Applied Probability Models with Optimization Applications*, San Francisco : Holden-Day.

[18] Smith, W. L. and Leadbetter, M. R.(1963), "On the Renewal Function for the Weibull Distribution", *Technometrics*, Vol. 5, pp. 393-396.

[19] Weiss, G. H.(1962), "Laguerre Expansion for Successive Generations of a Renewal Process", *Journal of Research National Bur. Standards*, Vol. 66B, pp. 165-168.