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Variation of Global Coherence on Propagation in Coherent Mode Representation

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The variation of global coherence on propagation plane by plane is examined in the framework of coherent mode representation. It is explained through concrete examples that the global coherence may in general be enhanced, may be reduced, or may not change. When the mode functions form a complete set and the corresponding eigenvalues are in.nitely degenerate, there necessarily develops a certain amount of global coherence on propagation, which is the essence of van Cittert-Zernike theorem. The propagation generates a certain pattern of the eigenvalue spectrum from the initial flat one and this is shown to be related to the non-unitarity of the propagation kernel.

Keywords: Global coherence, Coherent mode representation, van Cittert-Zernike theorem

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I. INTRODUCTION

Every electromagnetic field found in nature has some fluctuations associated with it and one can deduce their existence from suitable experiments for correlations between the fluctuations at two or more space-time points. In the optical coherence theory for the scalar field, the simplest space-time correlation function of a fluctuating field is called the mutual coherence function defined by

$$\Gamma(\vec{r}_1, \vec{r}_2; t_1, t_2) = \langle V^*(\vec{r}_1, t_1) V(\vec{r}_2, t_2) \rangle, \tag{1.1}$$

where $V(\vec{r}, t)$ represents the complex analytic signal associated with the field and the angular brackets denote the ensemble average. Sometimes, the mutual coherence function depends on t_1 and t_2 only through the difference $\tau = t_2 - t_1$ and then one can introduce the cross spectral density defined by [1]

$$W(\vec{r}_1, \vec{r}_2; \omega) = \int \Gamma(\vec{r}_1, \vec{r}_2; \tau) e^{i\omega\tau} d\tau. \tag{1.2}$$

This cross spectral density is the space-frequency correlation function and allows the frequency by frequency analysis which is often more desirable.

The complex degree of spectral coherence, which is the normalized version of the cross spectral density, is defined by [2]

$$\mu(\vec{r}_1, \vec{r}_2; \omega) = \frac{W(\vec{r}_1, \vec{r}_2; \omega)}{\sqrt{W(\vec{r}_1, \vec{r}_1; \omega) W(\vec{r}_2, \vec{r}_2; \omega)}}. \tag{1.3}$$

The complex degree of spectral coherence reflects the *local* coherence properties of field in the sense that it shows the correlation between two spatial points. According to Schwarz's inequality, the modulus of μ takes on values between zero and unity. When $|\mu(\vec{r}_1, \vec{r}_2; \omega)| = 0$ the field is completely incoherent, when $|\mu(\vec{r}_1, \vec{r}_2; \omega)| = 1$, the field is completely coherent, and when $0 < |\mu(\vec{r}_1, \vec{r}_2; \omega)| \le 1$, the field is partially coherent between two points $\vec{r}_1$ and $\vec{r}_2$.

While μ gives a local description of coherence, there is another scheme to reveal the coherence properties of the field from a different point of view [3]. This new scheme is based on the so-called mode functions defined *globally* on the entire domain of the field. This new scheme is often called the coherent mode representation of the field. When a cross spectral density becomes a Hilbert-Schmidt kernel, then the cross spectral density can be written as a factorized form [1]

$$W(\vec{r}_1, \vec{r}_2; \omega) = \sum_n \lambda_n(\omega) \phi_n^*(\vec{r}_1; \omega) \phi_n(\vec{r}_2; \omega), \tag{1.4}$$

where λ_n and ϕ_n are the eigenvalues and eigenfunctions of the integral equation

$$\int_D W(\vec{r}_1,\vec{r}_2;\omega)\phi_n(\vec{r}_1;\omega)d^3r_1 = \lambda_n(\omega)\phi_n(\vec{r}_2;\omega). \quad (1.5)$$

The eigenfunctions are mutually orthogonal

$$\int_D \phi_n^*(\vec{r};\omega)\phi_m(\vec{r};\omega)d^3r = \delta_{nm}, \quad (1.6)$$

and the eigenvalues are positive, $\lambda_n(\omega) > 0$.

Some concrete examples of the coherent mode representation have been carried out mainly for Gaussian Schell model sources [4] and for Bessel-correlated fields [5]. According to the above analyses, the coherent mode representation seems to reflect the global coherence properties of fields. It turns out that the relevant parameter β for global coherence is the ratio of the coherence length l_c to the linear dimension L of the source or field, i.e., $\beta = l_c/L$. The coherence length l_c is determined by μ. When β is small, there are many modes contributing to the cross spectral density and the field is rather incoherent. However, when β is large, relatively few modes contribute to the cross spectral density. The entropic measure of global coherence has been defined and applied to Gaussian Schell model fields recently [3].

As the field develops according to the wave equation, so the state of coherence of light may be appreciably changed in the process of propagation. More specifically, even if the light originates from a completely incoherent source, the field at points sufficiently far from the source may be highly correlated. This is one of the central theorems in the optical coherence theory, formulated by van Cittert [6] and later, in a more general form, by Zernike [7]. The original form of the van Cittert-Zernike theorem is expressed in terms of the mutual intensity $J(\vec{r}_1,\vec{r}_2) \equiv \Gamma(\vec{r}_1,\vec{r}_2,0)$. The mutual intensity has the following propagation law: [8]

$$J(\vec{r}_1,\vec{r}_2) = \left(\frac{\bar{k}}{2\pi}\right)^2 \int_D\int_D J(\vec{r}_1',\vec{r}_2')\frac{e^{i\bar{k}(R_2-R_1)}}{R_1R_2}d^2r_1'd^2r_2', \quad (1.7)$$

where $\bar{k}$ is the central wave number, $J(\vec{r}_1{}',\vec{r}_2{}')$ is the mutual intensity of the source, $R_i = |\vec{r}_i - \vec{r}_i{}'|$ (i = 1, 2) are distances from source points to field points, and D indicates the source domain.

When the source is completely incoherent, we then have $J(\vec{r}_1{}',\vec{r}_2{}') = I(\vec{r}_1{}')\delta(\vec{r}_1{}' - \vec{r}_2{}')$ and Eq. (1.7) reduces to

$$J(\vec{r}_1,\vec{r}_2) = \left(\frac{\bar{k}}{2\pi}\right)^2 \int_D I(\vec{r}_1')\frac{e^{i\bar{k}(R_2-R_1)}}{R_1R_2}d^2r_1', \quad (1.8)$$

where $R_i = |\vec{r}_i - \vec{r}_i{}'|$ (i = 1, 2) are now distances from the same source point. When one calculates the complex degree of coherence defined by

$$j(\vec{r}_1,\vec{r}_2) = \frac{J(\vec{r}_1,\vec{r}_2)}{\sqrt{J(\vec{r}_1,\vec{r}_1)J(\vec{r}_2,\vec{r}_2)}}, \quad (1.9)$$

then one can recognize that the field is more than delta-correlated and hence the field correlation is generated on propagation. We can also formulate the van Cittert-Zernike theorem in terms of the cross spectral density and then the complex degree of coherence may be replaced by the complex degree of spectral coherence defined by Eq. (1.3).

In this paper, we investigate how the global coherence of the field may change on propagation. We consider the wave propagation plane by plane for a given field correlation in the input plane, and examine the possible variation in the framework of the coherent mode representation. The entropic measure will be used to quantify the global coherence of the field in the plane. We will also explore the essence of van Cittert-Zernike theorem in the coherent mode representation. In the coherent mode representation, the global properties of the field may be determined by the structure of eigenmodes defined in Eq. (1.5), in particular, by the effective number of different eigenmodes.

II. ENTROPIC MEASURE OF GLOBAL COHERENCE

We introduced the cross spectral density as the Fourier transform of the mutual coherence function as in Eq. (1.2) and hence, the cross spectral density possesses the two generic properties, hermiticity and non-negative definiteness, i.e.,

$$W(\vec{r}_2,\vec{r}_1,\omega) = W^*(\vec{r}_1,\vec{r}_2,\omega), \quad (2.1)$$

$$\int_D\int_D f^*(\vec{r}_1)W(\vec{r}_1,\vec{r}_2,\omega)f(\vec{r}_1)d^3r_1d^3r_2 \geq 0, \quad (2.2)$$

where D is the domain of the field and $f(\vec{r})$ is a square-integrable function. Furthermore, the cross spectral density often satisfies the inequality

$$\int_D\int_D |W(\vec{r}_1,\vec{r}_2,\omega)|^2 d^3r_1d^3r_2 < \infty, \quad (2.3)$$

and then, the cross spectral density becomes a Hilbert-Schmidt kernel.

With the above three conditions, according to Mercer's expansion theorem, [9] the cross spectral density can then be written as the factorized form

$$W(\vec{r}_1,\vec{r}_2,\omega)=\sum_n \lambda_n(\omega)\phi_n^*(\vec{r}_1,\omega)\phi_n(\vec{r}_2,\omega), \quad (2.4)$$

where λ_n and ϕ_n are the eigenvalues and the eigenfunctions of the integral equation

$$\int_D W(\vec{r}_1,\vec{r}_2,\omega)\phi_n(\vec{r}_1,\omega)d^3r_1=\lambda_n(\omega)\phi_n(\vec{r}_2,\omega). \quad (2.5)$$

The eigenfunctions are mutually orthogonal, i.e.,

$$\int_D \phi_n^*(\vec{r},\omega)\phi_m(\vec{r},\omega)d^3r=\delta_{nm}, \quad (2.6)$$

and the eigenvalues are positive,

$$\lambda_n(\omega)>0. \quad (2.7)$$

Each term under the summation sign in Eq. (2.4) represents a completely coherent constituent, and hence, each eigenfunction may be regarded as an independent natural mode of the field. For this reason, this representation is called the coherent mode representation of the field.

Pursuing an analogy between the cross spectral density and the density operator in quantum mechanics, one may introduce a quantitative measure of global coherence. For this purpose, we normalize Eq. (2.4) and define $\omega(\vec{r}_1,\vec{r}_2,\omega)$ as

$$w(\vec{r}_1,\vec{r}_2,\omega)=W(\vec{r}_1,\vec{r}_2,\omega)/\int_D W(\vec{r},\vec{r},\omega)d^3r. \quad (2.8)$$

Then, $\omega(\vec{r}_1,\vec{r}_2,\omega)$ can be expressed as

$$w(\vec{r}_1,\vec{r}_2,\omega)=\sum_n \Lambda_n(\omega)\phi_n^*(\vec{r}_1,\omega)\phi_n(\vec{r}_2,\omega), \quad (2.9)$$

where

$$\Lambda_n=\lambda_n/\sum_n \lambda_n. \quad (2.10)$$

The quantity $\sum_n \lambda_n$ appearing in the denominator of Eq. (2.10) represents the total intensity and is assumed to be finite. We have already mentioned that $\{\phi_n\}$ are regarded as the natural modes of the field and $\{\Lambda_n\}$ represent the relative contributions of the modes to the field. These observations lead us to an understanding of the coherent mode representation in terms of entropy. As the number of modes increases, we have less knowledge of the field, and the field becomes more random as a whole.

We may define the measure of global coherence exactly in the same way as the entropy of a quantum system is defined, i.e., [3,10]

$$S_W=-\sum_n \Lambda_n \log\Lambda_n. \quad (2.11)$$

The entropic measure defined above is invariant under unitary transformations, and its value cannot be negative. Two extreme cases deserve to be mentioned. When the field is fully coherent for every pair of points, there is only one mode, and the entropic measure of this completely coherent field becomes zero. The other extreme is the case in which a huge number of mode functions make equal contributions to the cross spectral density. The limit where the number of mode functions goes to infinity corresponds to a completely incoherent field. For this case, the entropic measure approaches infinity. For these two cases, the local and the global measures go parallel to each other; however, for partially coherent fields, there is no apparent connection between these two concepts.

III. WAVE PROPAGATION IN COHERENT-MODE REPRESENTATION

Let us consider a fluctuating scalar wave field, represented by a stationary ensemble, that propagates from the plane $z=0$ into the half-space $z>0$. Let $W^{(0)}(\vec{\rho}_1{}', \vec{\rho}_2{}';\omega)$ be the cross spectral density of the field, at frequency ω, in the input plane. Here $\vec{\rho}_1{}'$ and $\vec{\rho}_2{}'$ are the position vectors of two source points S_1 and S_2, referred to some origin in the input plane. We write the coherent mode representation of the cross spectral density $W^{(0)}(\vec{\rho}_1{}', \vec{\rho}_2{}';\omega)$ as

$$W^{(0)}(\vec{\rho}_1',\vec{\rho}_2';\omega)=\sum_n \lambda_n^{(0)}(\omega)\phi_n^{(0)*}(\vec{\rho}_1';\omega)\phi_n^{(0)}(\vec{\rho}_2';\omega), \quad (3.1)$$

where λ_n are the eigenvalues and $\phi_n^{(0)}$ are the eigenfunctions of the integral equation

$$\int W^{(0)}(\vec{\rho}_1',\vec{\rho}_2';\omega)\phi_n^{(0)}(\vec{\rho}_1';\omega)d^2\rho_1'=\lambda_n^{(0)}(\omega)\phi_n^{(0)}(\vec{\rho}_2';\omega). \quad (3.2)$$

The eigenvalues are positive and the eigenfunctions are orthonormal to one another, i.e.,

$$\int \phi_n^{(0)*}(\vec{\rho}';\omega)\phi_m^{(0)}(\vec{\rho}';\omega)d^2\rho'=\delta_{nm}. \quad (3.3)$$

Let us now consider the field in the plane $z=L$ (>0).

We denote the cross spectral density of the field in this plane by $W^{(L)}(\vec{\rho}_1, \vec{\rho}_2; \omega)$. Here $\vec{\rho}_1$ and $\vec{\rho}_2$ are the position vectors of two points P_1 and P_2, referred to the point at which this plane intersects the z axis. It has been shown that $W^{(L)}(\vec{\rho}_1, \vec{\rho}_2; \omega)$ can be expanded in the form

$$W^{(L)}(\vec{\rho}_1,\vec{\rho}_2;\omega) = \sum_n \lambda_n^{(0)}(\omega)\psi_n^{(L)*}(\vec{\rho}_1;\omega)\psi_n^{(L)}(\vec{\rho}_2;\omega), \quad (3.4)$$

where $\lambda_n^{(0)}$ are the same constants as in Eq. (3.1), i.e., the eigenvalues of the integral equation (3.2), and $\psi_n^{(L)}$ is the field in the plane $z = L$ developed from the mode $\phi_n^{(0)}$. Using the angular spectrum representation, one can obtain the explicit expression for $\psi_n^{(L)}$ as [11]

$$\psi_n^{(L)}(\vec{\rho};\omega) = \frac{1}{2\pi}\int \tilde{\phi}_n^{(0)}(\vec{f};\omega)e^{i(\vec{f}\cdot\vec{\rho}+f_z L)}d^2 f, \quad (3.5)$$

where f_z is given by

$$f_z = \begin{cases} \sqrt{k^2 - |\vec{f}|^2}, & \text{for } |\vec{f}|^2 \le k \\ i\sqrt{|\vec{f}|^2 - k^2}, & \text{for } |\vec{f}|^2 > k \end{cases} \quad (3.6)$$

and $\tilde{\phi}_n^{(0)}$ is the two-dimensional Fourier transform of the mode $\phi_n^{(0)}$, i.e.,

$$\tilde{\phi}_n^{(0)}(\vec{f};\omega) = \frac{1}{2\pi}\int \phi_n^{(0)}(\vec{\rho}';\omega)e^{-i\vec{f}\cdot\vec{\rho}'}d^2\rho'. \quad (3.7)$$

From Eq. (3.5), one can notice that the mapping from $\phi_n^{(0)}$ to $\psi_n^{(L)}$ is not in general unitary and hence $\psi_n^{(L)}$ do not generally satisfy orthnormality. For this reason, Eq. (3.4) is not a coherent mode representation, although it looks like one. However, when the cross spectral density $W^{(0)}(\vec{\rho}_1{}', \vec{\rho}_2{}'; \omega)$ of the field in the plane $z = 0$ is band limited with respect to each of $\vec{\rho}_1{}'$ and $\vec{\rho}_2{}'$ to circles of radii k about the origin in the corresponding wave vector planes, i.e.,

$$\tilde{W}^{(0)}(\vec{f}_1,\vec{f}_2;\omega) = 0, \quad \text{for } |\vec{f}_1| > k \text{ or } |\vec{f}_2| > k, \quad (3.8)$$

the orthonormality of $\psi_n^{(L)}$ is preserved and Eq. (3.4) becomes a true coherent mode representation. Moreover, the eigenvalues $\lambda_n^{(0)}$ also remain same and they acquire the significance of invariants upon propagation.

Suppose we are given a finite number of coherent modes, say $\phi_n^{(0)}$ ($n = 1, 2, ..., N$), in the input plane which we designate $z = 0$. The most general cross spectral density that we may construct with $\{\phi_n^{(0)}\}$ is written as

$$W^{(0)}(\vec{\rho}_1',\vec{\rho}_2';\omega) = \sum_{n=1}^{N} \lambda_n^{(0)}(\omega)\phi_n^{(0)*}(\vec{\rho}_1';\omega)\phi_n^{(0)}(\vec{\rho}_2';\omega). \quad (3.9)$$

The entropic measure for the cross spectral density given by Eq. (3.9) can be found to be

$$S_W = -\sum_{n=0}^{N} \Lambda_n^{(0)} \log \Lambda_n^{(0)}, \quad (3.10)$$

where

$$\Lambda_n^{(0)} = \lambda_n^{(0)} / \sum_{n=0}^{N} \lambda_n^{(0)}. \quad (3.11)$$

Notice that the quantity $\sum_{n=0}^{N} \lambda_n^{(0)}$ is nothing but the total intensity in the input plane. On using the entropic measure, we can distinguish quantitatively various cross spectral densities based on the coherent mode representation. In particular, there are two extreme cases that are worthy to be mentioned in terms of coherence.

When the eigenvalues $\lambda_n^{(0)}$ are all zero except a particular one, which we denote by $\lambda^{(0)}$, the cross spectral density is given by

$$W^{(0)}(\vec{\rho}_1',\vec{\rho}_2';\omega) = \lambda^{(0)}(\omega)\phi^{(0)*}(\vec{\rho}_1';\omega)\phi^{(0)}(\vec{\rho}_2';\omega). \quad (3.12)$$

The corresponding entropic measure vanishes. The field becomes deterministic and hence fully coherent in the plane $z = 0$. We can easily notice that the corresponding field in the plane $z = L$ also becomes fully coherent and the cross spectral density of the field in the plane $z = L$ is given by

$$W^{(L)}(\vec{\rho}_1,\vec{\rho}_2;\omega) = \lambda^{(0)}(\omega)\psi^{(L)*}(\vec{\rho}_1;\omega)\psi^{(L)}(\vec{\rho}_2;\omega), \quad (3.13)$$

where

$$\psi^{(L)}(\vec{\rho};\omega) = \frac{1}{2\pi}\int \tilde{\phi}^{(0)}(\vec{f};\omega)e^{i(\vec{f}\cdot\vec{\rho}+f_z L)}d^2 f, \quad (3.14)$$

where f_z is defined in Eq. (3.6). Evidently, the entropic measure in the plane $z = L$ also vanishes and the global coherence does not diminish on propagation.

The other extreme is the case in which all the eigenvalues take on the same value. The entropic measure then attains the largest value. We term this case the degenerate case. For this degenerate case we can write the cross spectral density in the input plane as

$$W^{(0)}(\vec{\rho}_1',\vec{\rho}_2';\omega) = \lambda^{(0)}(\omega)\sum_{n=1}^{N} \phi_n^{(0)*}(\vec{\rho}_1';\omega)\phi_n^{(0)}(\vec{\rho}_2';\omega), \quad (3.15)$$

where all the modes have the same eigenvalue $\lambda^{(0)}(\omega)$. The normalized eigenvalue is then given by

$$\Lambda^{(0)}(\omega) = \frac{1}{N}, \tag{3.16}$$

and the entropic measure for this degenerate case can be obtained to be

$$S_W = \log N. \tag{3.17}$$

Now, let us consider the cross spectral density of the field in the plane $z = L$

$$W^{(L)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lambda^{(0)}(\omega) \sum_{n=1}^{N} \psi_n^{(L)*}(\vec{\rho}_1; \omega) \psi_n^{(L)}(\vec{\rho}_2; \omega). \tag{3.18}$$

when the field in the input plane is given by Eq. (3.9). $\psi_n^{(L)}$ can be obtained from $\phi_n^{(0)}$ as

$$\psi_n^{(L)}(\vec{\rho}; \omega) = \frac{1}{2\pi} \int \tilde{\phi}_n^{(0)}(\vec{f}; \omega) e^{i(\vec{f}\cdot\vec{\rho} + f_z L)} d^2 f, \tag{3.19}$$

where f_z is given by Eq. (3.6). We have already mentioned that Eq. (3.18) is not a coherent mode representation, except the case for which the input field is band limited. We formally write the coherent mode representation of the field in the plane $z = L$ as

$$W^{(L)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \sum_{n=1}^{N} \gamma_n(\omega) \chi_n^*(\vec{\rho}_1; \omega) \chi_n(\vec{\rho}_2; \omega). \tag{3.20}$$

One can notice that, even though Eq. (3.18) is not in general regarded as a coherent mode representation, as the formal expression of Eq. (3.20) implies, the number of coherent modes in the plane $z = L$ can not exceed N. In other words, the number of modes in the output plane may be at most equal to the number of modes in the input plane. Consequently, when the field is degenerate in the input plane, the entropic measure can not increase but may decrease on propagation.

IV. VARIATION OF GLOBAL COHERENCE ON PROPAGATION

Now, let us consider the cross spectral density of the field in the plane $z = Z$, where $kZ >> 1$,

$$W^{(Z)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \sum_{n=1}^{N} \lambda_n^{(0)}(\omega) \psi_n^{(Z)*}(\vec{\rho}_1; \omega) \psi_n^{(Z)}(\vec{\rho}_2; \omega). \tag{4.1}$$

when the field in the input plane is given by Eq. (3.9). Since all the evanescent waves disappear as $kZ \to \infty$, $\psi_n^{(Z)}$ can be obtained from $\phi_n^{(0)}$ as

$$\psi_n^{(Z)}(\vec{\rho}; \omega) = \frac{1}{2\pi} \int_< \tilde{\phi}_n^{(0)}(\vec{f}; \omega) e^{i(\vec{f}\cdot\vec{\rho} + f_z Z)} d^2 f, \tag{4.2}$$

where the symbol $<$ under the integral designates the integral domain $|\vec{f}| < k$ and hence f_z is now positive. We have already mentioned that Eq. (4.1) is not a coherent mode representation, except the case for which the input field is band limited.

When each of the input modes $\phi_n^{(0)}$ satisfies

$$\tilde{\phi}_n^{(0)}(\vec{f}; \omega) = 0, \quad \text{for } |\vec{f}| > k, \tag{4.3}$$

the orthonormality of output modes $\psi_n^{(Z)}$ is preserved, and then, Eq. (4.1) becomes a true coherent mode representation. In particular, the eigenvalues remain same and the entropic measure of the output field takes on the same value given by Eq. (3.10). In the sense of entropic measure, the global coherence of the band limited field does not change upon propagation. One can notice that, for the band limited input field, the total intensity of the field also remains same. When the input field is not band limited, we may then expect some variation of the global coherence on propagation. The entropic measure may either increase or decrease when the input field is properly prepared. Let us consider the increasing case first.

Suppose the coherent mode representation of the field in the plane $z = 0$ is given by

$$\begin{aligned} W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega) &= \lambda^{(0)}(\omega) \sum_{n=1}^{N-1} \phi_n^{(0)*}(\vec{\rho}_1; \omega) \phi_n^{(0)}(\vec{\rho}_2; \omega) \\ &+ \lambda_N^{(0)}(\omega) \phi_N^{(0)*}(\vec{\rho}_1; \omega) \phi_N^{(0)}(\vec{\rho}_2; \omega) \end{aligned} \tag{4.4}$$

The first $(N-1)$ modes are assumed to be band limited and the Nth mode is not. We impose a further constraint

$$\int_< \left|\tilde{\phi}_N^{(0)}(\vec{f}; \omega)\right|^2 d^2 f \Big/ \int \left|\tilde{\phi}_N^{(0)}(\vec{f}; \omega)\right|^2 d^2 f = \frac{\lambda^{(0)}}{\lambda_N^{(0)}}. \tag{4.5}$$

According to Eq. (4.2), we have

$$\psi_n^{(Z)}(\vec{\rho}; \omega) = \frac{1}{2\pi} \int \tilde{\phi}_n^{(0)}(\vec{f}; \omega) e^{i(\vec{f}\cdot\vec{\rho} + f_z Z)} d^2 f. \tag{4.6}$$

for $n = 1, 2, ..., (N-1)$ and

$$\psi_N^{(Z)}(\vec{\rho}; \omega) = \frac{1}{2\pi} \int_< \tilde{\phi}_N^{(0)}(\vec{f}; \omega) e^{i(\vec{f}\cdot\vec{\rho} + f_z Z)} d^2 f. \tag{4.7}$$

While $\psi_n^{(Z)}$ $(n = 1, 2, ..., N)$ are orthogonal to one another, only $\psi_n^{(Z)}$ $(n = 1, 2, ..., N-1)$ are normalized. To get the coherent mode representation in the plane $z = Z$, we need to normalize $\psi_N^{(Z)}$ and, because of the constraint given by Eq. (4.5), we obtain the coherent mode representation of the field in the plane $z = Z$ in a fully degenerate form as

$$W^{(Z)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lambda^{(0)}(\omega) \sum_{n=1}^{N-1} \psi_n^{(Z)*}(\vec{\rho}_1; \omega)\psi_n^{(Z)}(\vec{\rho}_2; \omega) + \lambda^{(0)}(\omega)\Psi_N^{(Z)*}(\vec{\rho}_1; \omega)\Psi_N^{(Z)}(\vec{\rho}_2; \omega), \quad (4.8)$$

where

$$\Psi_N^{(Z)}(\vec{\rho}; \omega) = \sqrt{\frac{\lambda_N^{(0)}}{\lambda^{(0)}}}\psi_N^{(Z)}(\vec{\rho}; \omega). \quad (4.9)$$

For this particular example, the field in the input plane is partially degenerate and the output field becomes fully degenerate. The entropic measure increases upon propagation and hence the global coherence is reduced.

The converse example can also be easily constructed. Now, let us start a fully degenerate field in the plane $z = 0$, i.e.,

$$W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lambda^{(0)}(\omega) \sum_{n=1}^{N} \phi_n^{(0)*}(\vec{\rho}_1; \omega)\phi_n^{(0)}(\vec{\rho}_2; \omega). \quad (4.10)$$

We assume that the first $(N-1)$ modes are band limited for convenience and that the last mode $\phi_N^{(0)}$ satisfies

$$\tilde{\phi}_N^{(0)}(\vec{f}; \omega) = 0, \quad \text{for } |\vec{f}| \le k. \quad (4.11)$$

This condition implies the converse of band limitedness and it is related to the non-radiating condition. Because of this condition, we immediately obtain the field in the plane $z = Z$ as

$$W^{(Z)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lambda^{(0)}(\omega) \sum_{n=1}^{N-1} \psi_n^{(Z)*}(\vec{\rho}_1; \omega)\psi_n^{(Z)}(\vec{\rho}_2; \omega), \quad (4.12)$$

and we recognize the entropic measure decreases from the initial value of log N to the value of log $(N-1)$ in the plane $z = Z$. The reduction of the entropic measure implies the enhancement of global coherence on propagation.

V. VAN CITTERT-ZERNIKE THEOREM IN COHERENT MODE REPRESENTATION

Now, let us consider a fully degenerate field whose mode functions form a complete set in the sense that any square integrable function can be expressed in terms of these mode functions. Formally, the cross spectral density in the plane $z = 0$ then acquires the coherent mode representation as

$$W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lambda^{(0)}(\omega) \sum_{n=1}^{\infty} \phi_n^{(0)*}(\vec{\rho}_1; \omega)\phi_n^{(0)}(\vec{\rho}_2; \omega). \quad (5.1)$$

We are now dealing with an infinite number of mode functions in general, and hence we need to examine whether the Hilbert-Schmidt condition is satisfied, which is required for the coherent mode representation to exist. For a logical convenience, we take a limiting procedure approaching to Eq. (5.1), i.e.,

$$W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lim_{N\to\infty} \lambda^{(0)}(\omega) \sum_{n=1}^{N} \phi_n^{(0)*}(\vec{\rho}_1; \omega)\phi_n^{(0)}(\vec{\rho}_2; \omega). \quad (5.2)$$

The Hilbert-Schmidt condition is now expressed by the inequality

$$\int\int |W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega)|^2 d^2\rho_1 d^2\rho_2 < \infty. \quad (5.3)$$

When we use the hermiticity property of the cross spectral density, Eq. (2.1), the above inequality may be rewritten as

$$\int\int W^{(0)}(\vec{\rho}_1, \vec{\rho}_2; \omega)W^{(0)}(\vec{\rho}_2, \vec{\rho}_1; \omega) d^2\rho_1 d^2\rho_2 < \infty, \quad (5.4)$$

and, on substituting Eq. (5.1) into Eq. (5.4), we arrive at

$$\lim_{N\to\infty} [\lambda^{(0)}(\omega)]^2 N < \infty. \quad (5.5)$$

The inequality in Eq. (5.5) suggests that $\lambda^{(0)}$ is at most proportional to $N^{-1/2}$.

The completeness of the mode functions may be expressed as

$$\sum_n \phi_n^{(0)*}(\vec{\rho}_1; \omega)\phi_n^{(0)}(\vec{\rho}_2; \omega) = \delta(\vec{\rho}_1 - \vec{\rho}_2), \quad (5.6)$$

and hence the limiting procedure in Eq. (5.2) leads to a completely incoherent field. The cross spectral density of field in the plane $z = Z$ may be written as

$$W^{(Z)}(\vec{\rho}_1, \vec{\rho}_2; \omega) = \lim_{N\to\infty} \lambda^{(0)}(\omega) \sum_{n=0}^{N} \psi_n^{*}(\vec{\rho}_1; \omega)\psi_n(\vec{\rho}_2; \omega), \quad (5.7)$$

where

$$\psi_n^{(Z)}(\vec{\rho};\omega) = \frac{1}{2\pi}\int_< \tilde{\phi}_n^{(0)}(\vec{f};\omega)e^{i(\vec{f}\cdot\vec{\rho}+f_z Z)}d^2 f. \qquad (5.8)$$

As we mentioned before, Eq. (5.7) is not a coherent mode representation because the transformation from $\{\phi_n^{(0)}\}$ to $\{\psi_n^{(0)}\}$ is not unitary in general. Interestingly enough, this non-unitarity of the propagation kernel makes the van Cittert-Zernike theorem hold.

The transformation becomes unitary only when each of the mode functions is band limited. If the field is finitely degenerate and the corresponding cross spectral density has only a finite number of modes, the band limitedness condition may be satisfied by all the mode functions, however, when the mode functions form a complete set, it is no longer possible that all the mode functions are band limited. The completeness of the mode functions implies that any square integrable function can be expressed as a linear combination of the mode functions. We can find a square integrable function which is not band limited and hence not all mode functions are band limited.

The spectrum of the eigenvalues of the coherent mode representation experiences a change from a flat shape to a non-flat pattern due to the non-unitary kernel. The non-unitary propagation kernel breaks the orthonormality of mode functions of the source and in the process of obtaining new mode functions there arises a certain pattern in the spectrum of eigenvalues. Since a flat spectrum of eigenvalues, together with the completeness of eigenfunctions, implies the complete incoherence, any pattern in the spectrum of new eigenvalues indicates a certain amount of coherence generated upon propagation. However, in general, the change from one shape to another does not always imply the development of coherence. Except two extreme cases, fully coherent and completely incoherent cases, we do not have any quantitative measure of coherence within the coherent mode representation. Even whether coherence is enhanced or reduced is not well understood in the framework of coherent mode representation. The only statement that we can make is whether there arises something out of nothing, i.e., the change from a flat one to a structured one certainly indicates the generation of coherence. The van Cittert-Zernike theorem exactly applies to this situation.

VI. SUMMARY

We have examined the variation of global coherence on propagation in the framework of coherent mode representation. The entropic measure is used to describe global coherence quantitatively. In the angular spectrum representation, the wave propagation from the input plane to the output plane leads to a mapping between two sets of functions defined in the two planes. The mapping does not in general yield a coherent mode representation in the output plane from the coherent mode representation in the input plane.

It has been shown that the global coherence may be enhanced, may be reduced, or may not change upon propagation. Some explicit examples have been provided to explain the three cases within the class of input fields which are degenerate. The entropic measure of the degenerate field with N mode functions is log N. When all the mode functions are band limited, the entropic measure in the output plane is also given by the same value, however, when they are not all band limited, the entropic measure becomes less than log N. A specific example for reduction of the entropic measure has also been provided.

When the mode functions in the input plane form a complete set, the degenerate field becomes completely incoherent and its cross spectral density is proportional to a delta function in its two spatial variables. The number of mode functions is then necessarily infinite and the band limitedness condition is no longer satisfied by all of the mode functions. The global coherence should then be generated upon propagation and this is the essence of van Cittert-Zernike theorem in the coherent mode representation. It is interesting to recognize that the non-unitary propagation kernel is responsible for van Cittert-Zernike theorem.

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