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A Theoretical Study on Free Gyroscopic Compass

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Abstract : *The authors aim to establish the theory necessary for developing the free gyroscopic compass and focus on mainly two points. One is to suggest north-finding principle by the angular velocity of the earth's rotation, and the other is to suggest orthogonal coordinate transformations of the motion rate of the spin axis, which transforms the components of motion rate in the free gyro frame into those in the platform frame and that this transformed rate is, in turn, transformed into the NED(north-east-down) navigation frame. Subsequently, ship's heading is obtained by using the fore-aft and athwartship components of the motion rate of the spin axis in the NED frame. In addition it was found how to solve the transformation matrix necessary for transforming each frame.*

Key words : *Free gyro, Angular velocity of the earth's rotation, Motion rate of spin axis, Gyro frame, Platform frame, NED navigation frame*

1. Introduction

North-seeking Compasses include traditional magnetic compass with bar magnets, flux gate compass with electric magnet, and conventional gyrocompass with its suppressed horizontal axis. Along with GPS compass by using GPS, nowadays ring laser and fiber optic gyrocompasses by using Sagnac effect are increasingly used.

This paper is to review the peculiar property of the motion of free gyro axis as part of free gyro positioning system, which was not dealt with in the previous studies by Park and Jeong(2004) and Jeong(2005).

Therefore this paper is to suggest the north-seeking principle by using the fore and aft and athwartship components of the earth's rotation rate and is also to suggest coordinate transformations of the motion rate of the spin axis of free gyro, which transforms the gyro frame into the north-east-down (hereafter NED) navigation frame via the platform frame

2. North-seeking principle by using the angular velocity of the earth's rotation

Considering the earth's rate ω_e, its north component is $\omega_e \cos L$, where L is the geodetic latitude of the point concerned.

Fig. 1 shows that the angular velocities of the fore and aft and the athwartship components are given by Eq.(1) (Titterson, et al., 1997), where ψ is ship's heading.

$$\begin{aligned} \omega_x &= \omega_e \cos L \cos \psi \\ \omega_y &= -\omega_e \cos L \sin \psi \end{aligned} \tag{1}$$

By taking the ratio of the two independent gyroscopic measurement, the latitude dependent terms cancel, allowing the heading, ψ, to be computed by Eq. (2).

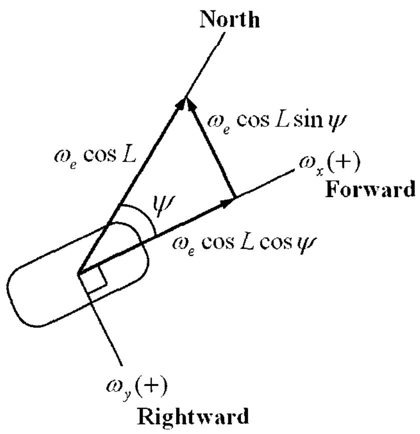

Fig. 1 North-seeking principle

$$\begin{aligned} \tan\psi &= \frac{\omega_y}{\omega_x} = -\frac{\omega_e \cos L \sin\psi}{\omega_e \cos L \cos\psi} \\ \psi &= \arctan\left(\omega_y/\omega_x\right) \end{aligned} \tag{2}$$

Ship's heading of Eq. (2) can be calculated in this way provided $\omega_x \neq 0$. In the event that ω_x is close to zero, the

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following equation may be used.

$$\psi = 90 - \arctan(\omega_x/\omega_y) \tag{3}$$

Therefore if the fore and aft and athwartship components of the earth's rate in the NED navigation plane are given, the ship's heading can be obtained.

Then how will the motion rate of the spin axis of free gyro be represented in the NED navigation plane? It will be explained in the next chapter.

3. Representation of the motion rate of the spin axis in the frames

3.1 Coordinate transformation from gyro frame to platform frame

In Fig. 2 the gyro frame refers to free gyro itself on the platform, whose axes are defined along the spin(x_g), horizontal(y_g), and downward(z_g) directions. The platform frame refers to the vehicle to be navigated, whose axes are defined along the forward(x_p), right(y_p), and through-the floor(z_p) directions.

The angle ξ is a rotation angle about the spin axis z_p and is positive in the counterclockwise sense as viewed along the axis toward the origin O, while the angle η is a rotation angle about the horizontal axis(y_g) and is positive in the same manner as above. Here the transformation matrix C_g^p from the gyro frame to platform frame is given by Eq.(4), using Euler angles and direction cosines.

$$\begin{aligned} C_g^p &= R_3(-\xi)R_2(-\eta) \\ &= \begin{pmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\eta & 0 & \sin\eta \\ 0 & 1 & 0 \\ -\sin\eta & 0 & \cos\eta \end{pmatrix} \\ &= \begin{pmatrix} \cos\xi\cos\eta & -\sin\xi & \cos\xi\sin\eta \\ \sin\xi\cos\eta & \cos\xi & \sin\xi\sin\eta \\ -\sin\eta & 0 & \cos\eta \end{pmatrix} \end{aligned} \tag{4}$$

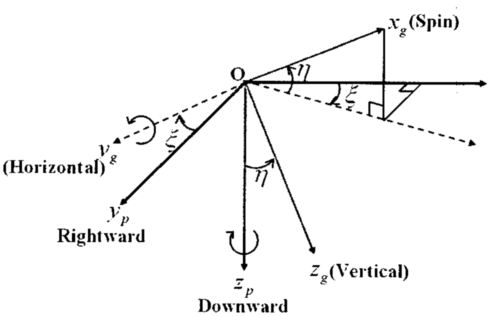

Fig. 2 Gyro & platform frames

Let the motion rate of spin axis in the gyro frame $\omega_{i/g}^g = [0\ \omega_{gy}\ \omega_{gz}]^T$, where we denote: $\omega_{i/g}^g$ = the motion rate of the gyro frame(g) relative to the inertial frame(i), with coordinates in the gyro frame(g), and hereafter the same notation of the angular velocity is applied. Then the motion rate of spin axis in the platform frame $\omega_{i/g}^p = [\omega_{px}\ \omega_{py}\ \omega_{pz}]^T$ is represented by the following.

$$\begin{aligned} \omega_{i/g}^p &= C_g^p \omega_{i/g}^g \\ &= \begin{pmatrix} -\omega_{gy}\sin\xi + \omega_{gz}\cos\xi\sin\eta \\ \omega_{gy}\cos\xi + \omega_{gz}\sin\xi\sin\eta \\ \omega_{gz}\cos\eta \end{pmatrix} \\ &= \begin{pmatrix} \omega_{px} \\ \omega_{py} \\ \omega_{pz} \end{pmatrix} \end{aligned} \tag{5}$$

This motion rate has to be transformed into the NED navigation frame.

3.2 Coordinate transformation into the NED navigation frame

With respect to the NED navigation frame whose axes are defined as the first axis points the north, the second axis points east and the third axis is aligned with the ellipsoidal normal at a point, in the downward direction. let's consider the platform frame axes point forward(x_p), to the right(y_p), and down(z_p). Euler angles define the transformation, that is, they are the roll(R), pitch(P), and yaw(Y) relative to the NED axes as shown in Fig. 3. Then the transformation matrix C_p^n is given by Eq. (6).

$$\begin{aligned} C_p^n &= R_3(-Y)R_2(-P)R_1(-R) \\ &= \begin{pmatrix} \cos P\cos Y & \sin R\sin P\cos Y - \cos R\sin Y & \cos R\sin P\cos Y + \sin R\sin Y \\ \cos P\sin Y & \sin R\sin P\sin Y + \cos R\cos Y & \cos R\sin P\sin Y - \sin R\cos Y \\ -\sin P & \sin R\cos P & \cos R\cos P \end{pmatrix} \end{aligned} \tag{6}$$

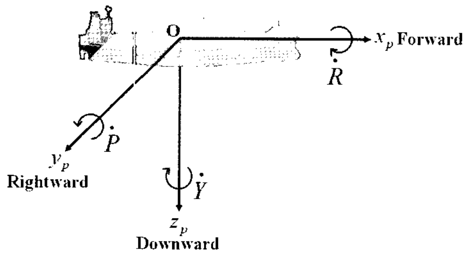

Fig. 3 Platform frame relative to NED frame

Eq. (6) is used only if the motion rate of spin axis is represented by north and east components.

Meantime because there is no need to force the axis x_p to be aligned with the axis N, the following equation is used.

$$C_p^n = R_2(-P)R_1(-R) = \begin{pmatrix} \cos P & \sin R\sin P & \cos R\sin P \\ 0 & \cos R & -\sin R \\ -\sin P & \sin R\cos P & \cos R\cos P \end{pmatrix} \quad (7)$$

By using Eq. (7) The motion rate of spin axis, $\omega_{i/g}^p = [\omega_{px}\ \omega_{py}\ \omega_{pz}]^T$, in the platform frame is transformed into the spin motion rate, $\omega_{i/g}^n$, in the navigation frame as follows.

$$\omega_{i/g}^n = C_p^n \omega_{i/g}^p = \begin{pmatrix} \omega_{px}\cos P + \omega_{py}\sin R\sin P + \omega_{pz}\cos R\sin P \\ \omega_{py}\cos R - \omega_{pz}\sin R \\ -\omega_{px}\sin P + \omega_{py}\sin R\cos P + \omega_{pz}\cos R\cos P \end{pmatrix} \quad (8)$$

And the spin motion rate of Eq. (8), $\omega_{i/g}^n$, is given by the following.

$$\begin{aligned} \omega_{i/g}^n &= \omega_{i/n}^n + \omega_{n/g}^n \\ &= \omega_{i/n}^n - \omega_{g/n}^n \\ \omega_{g/n}^n &= \omega_{i/n}^n - \omega_{i/g}^n \\ &= \begin{pmatrix} \omega_{nx} \\ \omega_{ny} \\ \omega_{nz} \end{pmatrix} \end{aligned} \quad (9)$$

Here $\omega_{i/n}^n$ is obtained from Eq. (20), and the spin motion rate in the NED frame is given by $\omega_{g/n}^n = [\omega_{nx}\ \omega_{ny}\ \omega_{nz}]^T$. The components of $\omega_{g/n}^n$ are forward, rightward, and downward respectively. However, when using Eq. (6), the components mean north, east, and down respectively.

3.3 Determination of ship's heading

Once the components of the spin motion rate are given from Eq. (9), ship's heading is defined as Eq. (10).

$$\tan\psi = \frac{\omega_{ny}}{\omega_{nx}}, \quad \psi = \arctan(\omega_{ny}/\omega_{nx}) \quad (10)$$

In case that ω_{nx} is close to zero, the following equation may be used.

$$\psi = 90 - \arctan(\omega_{nx}/\omega_{ny}) \quad (11)$$

4. Determination of transformation matrices

4.1 Transformation matrix from gyro frame to platform frame

For transformation we have to know the rotation angles of the gyro frame, ξ and η. They are obtained by integrating the respective components of the spin motion rate. In doing so consider the following differential equation.

$$\frac{dC_p^g}{dt} = -\Omega_{p/g}^g C_p^g \quad (12)$$

Where $\Omega_{p/g}^g$ is a skew-symmetric matrix and can be represented by $\omega_{p/g}^g$ in the vector form. As vectors the following are made of the angular velocity addition theorem.

$$\begin{aligned} \omega_{p/g}^g &= \omega_{p/i}^g + \omega_{i/g}^g \\ &= \omega_{i/g}^g - \omega_{i/p}^g \\ &= \omega_{i/g}^g - C_p^g \omega_{i/p}^p \end{aligned} \quad (13)$$

By the way, $\omega_{i/g}^g$ is actually measured from the angular velocity sensors installed in the horizontal and vertical axes of a free gyro respectively and $\omega_{i/p}^p$ is the sensed rates from the forward, rightward, and downward sensors of the platform. Here the transformation matrix C_p^g of Eq. (13) is obtained from the values of the one step earlier.

Here in order to solve the differential equation (12), the direction cosine differential equation is used. Generally the time increment or sampling interval for IMU (Inertial Measurement Unit) is very small, and it may be sufficient to approximate $\omega_{i/g}^g$ and $\omega_{i/p}^p$ as constant over this time interval. And we assume $\Omega_{p/g}^g$ is constant over the sampling interval.

$$\begin{aligned} C_p^g(t) &= \Psi(t,t_0)C_p^g(t_0) \\ \Psi(t,t_0) &= exp\left(\int_{t_0}^{t}(-\Omega_{p/g}^g)d\tau\right) \\ &= I + \frac{\sin(|a|)}{|a|}A + \frac{1-\cos(|a|)}{|a|^2}A^2 \end{aligned} \quad (14)$$

$$\begin{aligned} A &= \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} = \int_{t_0}^{t}(-\Omega_{p/g}^g)d\tau \\ &= -\Omega_{p/g}^g \Delta t \end{aligned} \quad (15)$$

$$\begin{aligned} a_i &= -\int_{t_0}^{t}\omega_{p/g}^g(i)d\tau \\ |a| &= \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned} \quad (16)$$

Where $\Delta t = t - t_0$, t_0 is the initial time and a_i is each component of rotation angle

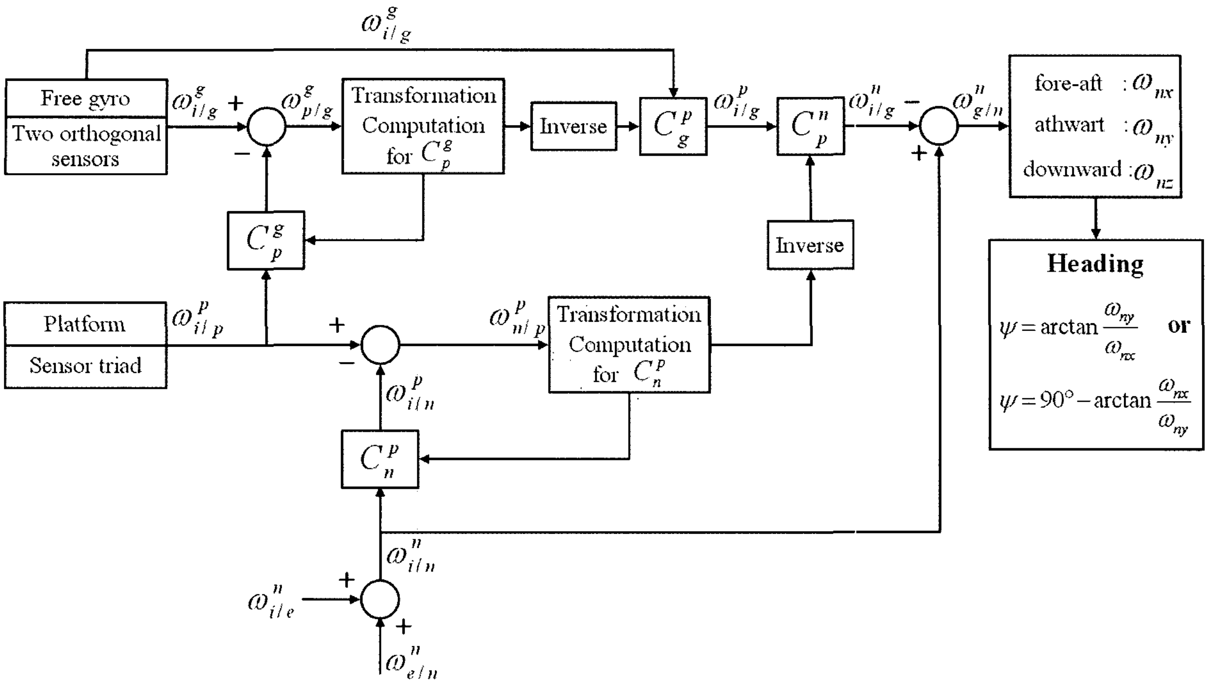

Fig. 4 Free gyroscopic compass mechanization

Once the transformation matrix C^g_p is obtained, the inverse matrix of it, C^p_g, is immediately calculated by doing the transpose of it since it is an orthogonal matrix. The relation between them is given by Eq. (17).

$$C^p_g = (C^g_p)^{-1} = (C^g_p)^T \tag{17}$$

4.2 Transformation matrix from platform frame to NED frame

For transformation we have to know the rotation angles of the platform frame Y, P, and R. They are obtained by integrating the respective components of the motion rate of the spin axis as shown in the previous section. In doing so consider the following differential equation as the previous section.

$$\frac{dC^p_n}{dt} = -\Omega^p_{n/p} C^p_n \tag{18}$$

Where $\Omega^p_{n/p}$ is also a skew-symmetric matrix and is represented by $\omega^p_{n/p}$ in the vector form. As vectors the following are made of the angular velocity addition theorem.

$$\begin{aligned} \omega^p_{n/p} &= \omega^p_{n/i} + \omega^p_{i/p} \\ &= \omega^p_{i/p} - \omega^p_{i/n} \\ &= \omega^p_{i/p} - C^p_n \omega^n_{i/n} \end{aligned} \tag{19}$$

$$\omega^n_{i/n} = \omega^n_{i/e} + \omega^n_{e/n} \tag{20}$$

$$\omega^n_{i/e} = \begin{pmatrix} \omega_e cosL \\ 0 \\ -\omega_e sinL \end{pmatrix} \tag{21}$$

$$\varpi^n_{e/n} = \begin{bmatrix} \dot{\lambda} \cos L \\ -\dot{L} \\ -\dot{\lambda} \sin L \end{bmatrix} \tag{22}$$

By the way, $\omega^p_{i/p}$ is the sensed rates from the forward, rightward, and downward sensors of the platform. $\omega^n_{i/n}$ of Eq. (20) is the sum of the rate of the earth rotation as Eq. (21) and the movement rate of a vehicle of Eq. (22). In Eq. (22) λ is the longitude and $\dot{\lambda}$ denotes the time rate of change of the longitude while L is the geodetic latitude and $\dot{L}$ is the time rate of change of the latitude. In this calculation the longitude and latitude are obtained from the values of the one step earlier.

Here in order to solve the differential equation (18), the direction cosine differential equation is also used. The same rule as the previous section is applied. Since the time increment or sampling interval is very small, it may be sufficient to approximate $\omega^p_{i/p}$ and $\omega^n_{i/n}$ as constant over this time interval. And also, we assume $\Omega^p_{n/n}$ is constant over the sampling interval.

$$C_n^p(t) = \Gamma(t,t_0)C_n^p(t_0)$$
$$\Gamma(t,t_0) = exp\left(\int_{t_0}^{t}(-\Omega_{n/p}^p)d\tau\right) = I + \frac{\sin(|b|)}{|b|}B + \frac{1-\cos(|b|)}{|b|^2}B^2 \quad (23)$$

$$B = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} = \int_{t_0}^{t}(-\Omega_{n/p}^p)d\tau = -\Omega_{n/p}^p \Delta t \quad (24)$$

$$b_i = -\int_{t_0}^{t}\omega_{n/p}^p(i)d\tau \qquad |b| = \sqrt{b_1^2+b_2^2+b_3^2} \quad (25)$$

Where $\Delta t = t - t_0$, t_0 is the initial time and b_i is each component of rotation angle. Once the transformation matrix C_n^p is obtained, the inverse matrix of it, C_p^n, is immediately calculated by doing the transpose of it since it is an orthogonal matrix.

In addition, the other methods to solve the differential equations (12) and (18) are also represented by the integration of four quaternions or three rotation vectors, the integration of three Euler angle equations, and etc.

Such equations suggested in chapter 3 are developed by referring to and using Farrell et al.(1999), Jekeli(2001) and Rogers(2003).

5. Algorithmic design of free gyroscopic compass

Fig. 4 shows the algorithmic design of free gyroscopic compass mechanization. In this mechanization two sensors for sensing the motion rate of the spin axis are mounted in the free gyro. Three sensors for sensing the motion rate of the platform are mounted in orthogonal triad. From the sensors in the gyro frame, the spin motion rate $\omega_{i/g}^g$ is obtained and from the ones in the platform frame $\omega_{i/p}^p$ is detected. By using the sum, $\omega_{p/g}^g$, of the rates from the free gyro and the ones detected from the platform sensors, the transformation matrix C_p^g is calculated and its inverse is determined. Therefore the spin motion rate, $\omega_{i/g}^g$, sensed from the free gyro is transformed into $\omega_{i/g}^p$ by using the inverse matrix C_g^p.

Meanwhile the rate of the earth's rotation $\omega_{i/e}^n$ and the rate of the vehicle movement $\omega_{e/n}^n$ are summed and transformed into $\omega_{i/n}^p$. It is added to the sensed rate from the platform $\omega_{i/p}^p$. As a result $\omega_{n/p}^p$ is generated. By using this, the transformation matrix C_n^p is calculated and the inverse of it, C_p^n, is obtained.

And the rate $\omega_{i/g}^p$ is transformed into $\omega_{i/g}^n$ by using the transformation matrix C_p^n. By using Eq. (9), the spin motion rate in the NED frame, $\omega_{g/n}^n$, is obtained from the rate, $\omega_{i/g}^n$. Finally, ship's heading is calculated by using the components of the spin motion rate according to Eq. (10) or Eq. (11).

6. Conclusion

This paper investigated and developed the algorithm regarding free gyroscopic compass theoretically and analytically. As a result conclusions are the following.

① Once the spin motion rate of free gyro is known, ship's heading is determined by using Eq. (10) or Eq. (11).

② In order to transform the spin motion rate of the gyro frame into the one of the NED navigation frame, the differential equations of Eq. (12) and Eq. (18) are solved by using Eq. (14) and Eq. (23) and the transformation matrices are obtained respectively.

③ In solving the above differential equations it is assumed the angular velocities are constant over the sampling time.

This paper ascertained the feasibility to set a stepping stone to the development of the free gyroscopic compass. However, several problems remain unsolved in the aspect of the following. Firstly a two-degree-of-freedom gyro is very expensive and is commercially disadvantageous in practice. Secondly the inherent errors caused by many elements complicated. Errors caused by free gyro itself, sensors of the platform, sensors of the free gyro, sampling time and etc. will be dealt with in the next study.

References

[1] Farrell, J. and Barth, M.(1999), "The Global Positioning System and Inertial Navigation", Mcgraw-hill, pp. 44-50.

[2] Jekeli, C.(2000), "Inertial Navigation Systems with Geodetic Applications", Walter de Gruyter, p.25.

[3] Jeong, T. G.(2005), "A Study on the Errors in the Free-Gyro Positioning System (I)", International Journal of Navigation and Port Research, Vol. 29, No. 7,

pp. 611-614.

[4] Park, S. C. and Jeong, T. G.(2004), "A Basic Study on Position Fixing by Free Gyros", Journal of Korean Navigation and Port Research, Vol.28, No.8, pp. 653-657.

[5] Rogers, R. M. (2003), "Applied Mathematics in Integrated Navigation Systems", 2nd ed., AIAA, pp.65-96.

[6] Titterton, D. H. and Weston, J. L.(1997), "Strapdown Inertial Navigation Technology", Peter Peregrinus Ltd., pp. 270.

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