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RIBAUCOUR TRANSFORMATION FOR FLAT m-SUBMANIFOLDS IN $\mathbb{H}^{m+n}$

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ABSTRACT. By using $O(m+n,1)/O(m)\times O(n,1)$-system, we give an analytic version of Ribaucour transformation for flat m-dimensional submanifolds in $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle and free of weak umbilics, where $n \geq m-1$.

1. Introduction

The study of immersions of space forms into space forms is one of the most important and interesting problems in classical differential geometry. E. Cartan showed that an n-dimensional hyperbolic space form can be locally immersed in $\mathbb{E}^{2n-1}$ and the dimension $2n-1$ cannot be lowered [2, 6]. It is a classical result due to Hilbert [5] that there are no complete isometric immersions $N^2(c') \to N^3(c)$ if $c' < c$ and $c' < 0$, but it is yet unknown (though conjectured) whether this result extends to complete isometric immersions $N^n(c') \to N^{2n-1}(c)$ for $c' < c$ and $c' < 0$. Notice that for the case $c' = 0$, one always has the clifford tori, and $c' > 0$ cannot occur due to the fact that such immersions induce global Chebyshev coordinates [7, 11]. In contrast, when $c' > c$, one always has the totally umbilical hypersurfaces. Especially if the immersion has no umbilic points, then the normal bundle is flat [7]. Later, in [13, 14], Tenenblet and Terng studied immersions of $\mathbb{H}^n$ into $\mathbb{E}^{2n-1}$ and $\mathbb{E}^n$ into $\mathbb{S}^{2n-1}(1)$, too.

Recently Terng et al in [12, 1] showed that to each symmetric space G/K one can associate to an m-dimensional first order systems of partial differential equations (PDEs in brief), the so-called G/K-system. By using $G^{p,q}_{m,n}$-systems they studied local non-degenerate isometric immersions $X : N^m_s(c') \to N^{m+k}_s(c)$ ($s = 0$ or 1) with flat normal bundle and

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$c' \leq c$, where $G^{p,q}_{m,n} = O(m+n, p+q)/O(m,p) \times O(n,q)$. On the other hand, the classical Ribaucour transformation for surfaces in Euclidean 3-space was extended for submanifolds of space forms with arbitrary dimension and codimension in [4], where it was applied to develop a transformation theory for submanifolds with constant sectional curvature. It was later shown that the dressing action of a simple element on a solution of the $G^{p,q}_{n,m}$-system corresponds to a Ribaucour transformation of the associated submanifold, see [12, 1, 8, 9, 10, 15] for details.

A natural question is whether Terng's method can be used to study local isometric immersions $X : N^m(c') \to N^{m+n}(c)$ with flat normal bundle and $c' > c$. In this paper we shall discuss $c' = 0$, $c = -1$ and $n \geq m-1$ and obtain an analogous results of [1] for flat m-dimensional submanifolds in $\mathbb{S}^{2n-1}$. We notice that the Gauss-Codazzi equations of X under certain conditions are gauge equivalent to the $G^1_{m,n} = O(m+n,1)/O(m) \times O(n,1)$-system and show that the dressing action of a simple element on the space of solutions of the $G^1_{m,n}$-system gives rise to a Ribaucour transformation of the flat m-dimensional submanifold X in $\mathbb{H}^{m+n}$.

2. The general cases of the $G^1_{m,n}$-system

G/K-systems were introduced for a symmetric space G/K by Terng et al in [12, 1]. To keep self-contained, below we give a short review of some known facts about the $G^1_{m,n}$-system, see [1, 15] for details.

DEFINITION 2.1. ([1, 12]) The $G^1_{m,n}$-system is the following PDEs for $\xi = (\xi_{ij}) : R^m \to \mathcal{M}_{m\times(n+1)}$ with $\xi_{ii} = 0$ for all $1 \leq i \leq m$ such that

$$\theta_\lambda = \sum_{i=1}^{m} \begin{pmatrix} D_i\xi^t - \xi D_i{}^t & -\lambda D_i I_{n,1} \\ \lambda D_i^t & D_i^t\xi - I_{n,1}\xi^t D_i I_{n,1} \end{pmatrix} dx_i \tag{2.1}$$

is a family of flat connections on R^m for all $\lambda \in \mathbf{C}$, i.e.,

$$d\theta_\lambda + \theta_\lambda \wedge \theta_\lambda = 0. \tag{2.2}$$

Here $D_i \in \mathcal{M}_{m\times(n+1)}$ is the matrix all whose entries are zero except that (i,i)-th entry is equal to 1.

It follows from (2.2) that for a given initial value $E(0,\lambda)$ there exists a smooth map $E : \mathbb{R}^m \times \mathbb{C} \to O(m+n,1)$, called the frame of θ_λ, such that $E^{-1}dE = \theta_\lambda$. The $G^1_{m,n}$-*reality condition* is

$$(2.3)\qquad \begin{cases} \overline{g(\bar\lambda)} = g(\lambda), \\ I_{m,n+1}g(\lambda)I_{m,n+1} = g(-\lambda), \\ g(\lambda)I_{m+n,1}g(\lambda)^t = I_{m+n,1}. \end{cases}$$

Let $g = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \in O(m) \times O(n,1)$ be a solution of $g^{-1}dg = \theta_0$. Write

$$\xi = (F, G), \quad D_i = (C_i, 0), \quad A = (A_1, A_2), \quad F, C_i \in gl(m),$$
$$G \in \mathcal{M}_{m\times(n+1-m)}, \ A_1 \in \mathcal{M}_{(n+1)\times m}, \quad A_2 \in \mathcal{M}_{(n+1)\times(n+1-m)},$$

and

$$h = \begin{pmatrix} I_m & 0 \\ 0 & A \end{pmatrix}.$$

To do the gauge transformation of θ_λ of by h, we have
(2.4)

$$\Omega_\lambda = h*\theta_\lambda = h\theta_\lambda h^{-1} - dhh^{-1} = \sum_{i=1}^m \begin{pmatrix} C_iF^t - FC_i^t & -C_iA_1{}^tI_{n,1}\lambda \\ A_1C_i^t\lambda & 0 \end{pmatrix} dx_i.$$

It is also a family of flat connections on $\mathbb{R}^m$ for all $\lambda \in \mathbb{C}$, i.e.,

$$(2.5)\qquad \begin{cases} (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_{k=1}^m f_{ki}f_{kj} = 0, & \text{if } i \neq j, \\ (f_{ij})_{x_k} = f_{ik}f_{kj}, & \text{if } i,j,k \text{ are distinct}, \\ (a_{ij})_{x_k} = a_{ik}f_{kj}, & \text{if } j \neq k, \end{cases}$$

where $A = (a_{ij}) \in O(n,1)$ and $F = (f_{ij}) \in gl_*(m) = \{(g_{ij}) \in gl(m) | g_{ii} = 0,\ 1 \le i \le m\}$. Note that Eh^{-1} is a frame of Ω_λ. The system (2.5) is called the **$G^1_{m,n}$-II system**.

In [15] we have obtained an explicit construction of the dressing action of a rational map with two simple poles of solutions of the $G^{p,q}_{m,n}$-system. Here we only state the result. Let $\mathbb{C}^{m+n+1}$ be equipped with the bilinear form

$$\langle u,\ v\rangle_1 = \sum_{i=1}^{m+n} \bar u_i v_i - \bar u_{m+n+1}v_{m+n+1}.$$

Let W and Z be unit space-like constant vectors in $\mathbb{R}^m$, $\mathbb{R}^{n,1}$ respectively, and π the orthogonal projection onto the space of $\mathbb{C}\begin{pmatrix} W \\ iZ \end{pmatrix}$ with respect to $\langle\ ,\ \rangle_1$. Define

$$(2.6)\quad g_{s,\pi} = \left(\pi + \frac{\lambda - is}{\lambda + is}(I - \pi)\right)\left(\bar\pi + \frac{\lambda + is}{\lambda - is}(I - \bar\pi)\right), \quad 0 \neq s \in R.$$

Lemma 2.2. ([15]) *Let* $\xi : \mathbb{R}^m \to \mathcal{M}_{m\times(n+1)}$ *be a solution of the* $G^1_{m,n}$-*system* (2.2), *and* $E(x,\lambda)$ *a frame of* ξ *such that* $E(x,\lambda)$ *is holomorphic for* $\lambda \in \mathbf{C}$ *and* $E(0,\lambda) = I$. *Let* $g_{s,\pi}$ as in (2.6) *and*

$$\left(\begin{array}{c} \tilde W \\ i\tilde Z \end{array}\right)(x) = E(x,-is)^{-1}\left(\begin{array}{c} W \\ iZ \end{array}\right), \tag{2.7}$$

and $\tilde\pi$ *the orthogonal projection onto the space of* $\mathbb{C}\left(\begin{array}{c} \tilde W \\ i\tilde Z \end{array}\right)$ *with respect to* $\langle\ ,\ \rangle_1$. *Then* $\tilde\xi = g_{s,\pi}\sharp\xi = \xi - 2s(\hat W \hat Z^t I_{n,1})_*$ *is a new solution of* (2.2) *and* $\tilde E = E g^{-1}_{s,\tilde\pi}$ *is a frame for* $\tilde\xi$, *where* $\hat W = \frac{\tilde W}{||\tilde W||_{R^m}}$ *and* $\hat Z = \frac{\tilde Z}{||\tilde Z||_{R^{n,1}}}$.

Note that both E and $\tilde E$ satisfy the $G^1_{m,n}$-reality conditions (2.3) which implies that $E(x,0)$ and $\tilde E(x,0)$ are in $O(m+n,1)$. Write

$$E(x,0) = \left(\begin{array}{cc} B(x) & 0 \\ 0 & A(x) \end{array}\right), \quad \tilde E(x,0) = \left(\begin{array}{cc} \tilde B(x) & 0 \\ 0 & \tilde A(x) \end{array}\right)$$

for some $A,\ B,\ \tilde A$ and $\tilde B$, and we have

$$\begin{cases} \tilde A = A(I - 2\hat Z \hat Z^t I_{n,1}), \\ \tilde B = B(I - 2\hat W \hat W^t). \end{cases}$$

Write

$$\xi = (F,G), \quad \tilde\xi = (\tilde F,\tilde G), \quad A = (A_1,A_2), \quad \tilde A = (\tilde A_1,\tilde A_2).$$

Here $A_1, \tilde A_1 \in \mathcal{M}_{(n+1)\times m}$, and $A_2, \tilde A_2 \in \mathcal{M}_{(n+1)\times(n+1-m)}$. Obviously (A_1,F) and $(\tilde A_1,\tilde F)$ are solutions of the $G^1_{m,n}$-II system (2.5), the corresponding frames are $E^{II}(x,\lambda)$ and $\tilde E^{II}(x,\lambda)$, where

$$E^{II}(x,\lambda) = E(x,\lambda)\left(\begin{array}{cc} I_m & 0 \\ 0 & A^{-1} \end{array}\right), \quad \tilde E^{II}(x,\lambda) = \tilde E(x,\lambda)\left(\begin{array}{cc} I_m & 0 \\ 0 & \tilde A^{-1} \end{array}\right).$$

It follows from lemma 2.2 that

$$\tilde E^{II}(x,\lambda) = E^{II}(x,\lambda)\left(I - \frac{2}{s^2+\lambda^2}\left(\begin{array}{c} s\tilde W \\ -\lambda A\tilde Z \end{array}\right)(s\tilde W^t, -\lambda \tilde Z^t A^t I_{n,1})\right). \tag{2.8}$$

In the following, we use the notation:

$$(\tilde A_1, \tilde F, \tilde E^{II}) = g_{s,\pi}\sharp(A_1, F, E^{II}).$$

3. Ribaucour transformations in $\mathbb{H}^{m+n}$

In this section we will show that the dressing action of $g_{s,\pi}$ on the spaces of solutions of the $G^1_{m,n}$-II system (2.2) gives rise to a Ribaucour transformation for flat m-dimensional submanifolds with flat and non-degenerate normal bundle in $\mathbb{H}^{m+n}$, where $n \geq m-1$.

Let X be a flat m-dimensional submanifold in $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle and free of weak umbilics. Recall from [7, 3] that a point $x \in X$ is said to be weak umbilics if there is a unit normal vector ζ at x such that $A_\zeta = Id$, where A_ζ denotes the shape operator in the direction ζ. Fix a local parallel normal frame $\{e_{m+1}, \ldots, e_{m+n}\}$, then there exist (see [3] for details) line of curvature coordinates $\{x_1, \ldots, x_m\}$ and a smooth map $A_1 \in \mathcal{M}_{(n+1)\times m}$ such that

$$A_1{}^t I_{n,1} A_1 = \begin{cases} I_m & \text{if } n \geq m \\ I_{m-1,1} & \text{if } n = m-1 \end{cases}$$

and the first and the second fundamental forms are

$$ds^2 = \sum_{k=1}^m a^2_{n+1,k} dx_k^2, \quad II = \sum_{k=1}^m \sum_{j=1}^n a_{n+1,k} a_{jk} dx_k^2 e_{m+j}. \tag{3.1}$$

A direct computation gives the following proposition.

Proposition 3.1. *If set $f_{ij} = \frac{(a_{n+1,j})_{x_i}}{a_{n+1,i}}$ for $i \neq j$, $f_{ii} = 0$ for all $1 \leq i \leq m$ and $F = (f_{ij})$, the Gauss-Codazzi-Ricci equations for the immersion X are the $G^1_{m,n}$-II system (2.5) for (A_1, F) which is called the generalized Laplace equation in [3].*

Suppose

$$X = e_{m+n+1}, \ e_k = \frac{X_{x_k}}{a_{n+1,k}}, \quad 1 \leq k \leq m$$

and

$$g = (e_1, \ldots, e_{m+n+1}) \in O(m+n, 1).$$

Then

$$g^{-1}dg = \Omega_\lambda|_{\lambda=1} = \sum_{i=1}^m \begin{pmatrix} C_iF^t - FC_i^t & -C_iA_1{}^t I_{n,1} \\ A_1 C_i^t & 0 \end{pmatrix} dx_i. \tag{3.2}$$

By the fundamental theorem of submanifolds, we have

Proposition 3.2. *Let (A_1, F) be a solution of the $G^1_{m,n}$-II system (2.5), then (3.2) is solvable. Let g be a solution of (3.2) and X the last column of g. If all entries of the last row of A_1 are non-zero, then X*

is a local isometric immersion of a flat m-dimensional submanifold in hyperbolic space $\mathbb{H}^{m+n}$ with flat, non-degenerate normal bundle such that the two fundamental forms are given by (3.1), where $A_1 = (a_{ij})$.

THEOREM 3.3. *Let E^{II} be a frame of the solution (A_1, F) of the $G^1_{m,n}$-II system (2.5), $g_{s,\pi}$ given by (2.6) and $(\tilde{A}_1, \tilde{F}, \tilde{E}^{II}) = g_{s,\pi}\sharp(A_1, F, E^{II})$. Write*

$$(3.3)\qquad \begin{aligned} E^{II}(x,1) &= (e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}, X), \\ \tilde{E}^{II}(x,1) &= (\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{e}_{m+1}, \ldots, \tilde{e}_{m+n}, \tilde{X}). \end{aligned}$$

Then:

(1) *both X and $\tilde{X}$ are flat m-dimensional submanifolds in $\mathbb{H}^{m+n}$ with flat and non-degenerate normal bundle, $\{x_1, \ldots, x_m\}$ line of curvature coordinates, $\{e_{m+k}\}_{k=1}^n$ and $\{\tilde{e}_{m+k}\}_{k=1}^n$ are parallel normal frames for X and $\tilde{X}$ respectively.*

(2) *The bundle morphism $P : \vartheta(X) \to \vartheta(\tilde{X})$ defined by $P(e_{m+k}(x)) = \tilde{e}_{m+k}(\mathcal{L}(x))$ for $1 \le k \le n$ is a Ribaucour transformation covering the map $\mathcal{L} : X(x) \to \tilde{X}(x)$.*

PROOF. (1) follows from lemma 2.2 and proposition 3.2.

(2) Firstly we show that if (A_1, F) is a solution of $G^1_{m,n}$-II system (2.5), then there exists an $\mathcal{M}_{m\times(n+1-m)}$-valued map G such that $\xi = (F, G)$ is a solution of the $G^1_{m,n}$-system (2.2). We consider two cases.

Case 1. $n = m - 1$. In this case $A_1 \in O(n,1)$ and we only need to prove that F is a solution of the $G^1_{m,n}$-system (2.2). Let $h_1 = \begin{pmatrix} I_m & 0 \\ 0 & A_1^{-1} \end{pmatrix}$ and the gauge transformation of h_1 on Ω_λ is

$$(3.4)\qquad \begin{aligned} h_1 * \Omega_\lambda &= h_1 \Omega_\lambda h_1^{-1} - dh_1 h_1^{-1} \\ &= \sum_{i=1}^m \begin{pmatrix} C_i F^t - F C_i^t & -C_i I_{n,1}\lambda \\ C_i^t \lambda & A_1^{-1} A_{1x_i} \end{pmatrix} dx_i \end{aligned}$$

By using (2.5), we have

$$(3.5)\qquad A_1^{-1} A_{1x_i} - (C_i^t F - I_{n,1} F^t C_i I_{n,1}) = Y C_i, \quad 1 \le i \le m,$$

where $Y : R^m \to gl(m)$. It follows from (3.5) and $A_1 \in O(n,1)$ that $YC_i I_{n,1} + (YC_i I_{n,1})^t = 0$ for all $1 \le i \le n$, which implies that $Y = 0$. Hence F is a solution of the $G^1_{m,n}$-system (2.2).

Case 2. $n > m - 1$. Choose $A_2 \in \mathcal{M}_{(n+1)\times(n+1-m)}$ such that $A = (A_1, A_2) \in O(n,1)$. Set $h = \begin{pmatrix} I_m & 0 \\ 0 & A^{-1} \end{pmatrix}$. The gauge transformation of h on Ω_λ is

$$\begin{aligned} h*\Omega_\lambda &= h\Omega_\lambda h^{-1} - dhh^{-1} \\ &= \sum_{i=1}^{m} \begin{pmatrix} C_iF^t - FC_i^t & -C_i\lambda & 0 \\ C_i^t\lambda & A_1^tI_{n,1}A_{1x_i} & A_1^tI_{n,1}A_{2x_i} \\ 0 & JA_2^tI_{n,1}A_{1x_i} & JA_2^tI_{n,1}A_{2x_i} \end{pmatrix} dx_i, \end{aligned} \tag{3.6}$$

where $J = I_{n-m,1}$. From (2.5), we have

$$dA_1 = A_1\sum_{i=1}^{m}(C_i^tF - F^tC_i)dx_i + \zeta\sum_{i=1}^{m} C_idx_i \tag{3.7}$$

for some $\zeta : R^m \to \mathcal{M}_{(n+1)\times m}$. Thus

$$A_2^tI_{n,1}dA_1 = A_2^tI_{n,1}\zeta\sum_{i=1}^{m} C_idx_i. \tag{3.8}$$

Notice that $A^{-1}dA$ is flat and

$$A^{-1}dA = \begin{pmatrix} A_1^tI_{n,1}dA_1 & A_1^tI_{n,1}dA_2 \\ JA_2^tI_{n,1}dA_1 & JA_2^tI_{n,1}dA_2 \end{pmatrix}, \tag{3.9}$$

we can get

$$JdA_2^t \wedge I_{n,1}dA_2 = JA_2^tI_{n,1}dA_1 \wedge A_1^tI_{n,1}dA_2 + JA_2^tI_{n,1}dA_2 \wedge JA_2^tI_{n,1}dA_2.$$

It follows from

$$A_1^tI_{n,1}dA_2 = (dA_2^tI_{n,1}A_1)^t = -(A_2^tI_{n,1}dA_1)^t = -\sum_{i=1}^{m} C_i\zeta^tJA_2dx_i,$$

that $JA_2^tI_{n,1}dA_2$ is flat. Therefore there exists $h_2 : \mathbb{R}^m :\to O(n-m,1)$ such that $h_2^{-1}dh_2 = JA_2^tI_{n,1}dA_2$. Take a gauge transformation by

$$\tilde{h}_2 = \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & h_2 \end{pmatrix}$$

on $h*\Omega_\lambda$, the resulting 1-form is

$$\begin{aligned} & \tilde{h}_2 * (h*\Omega_\lambda) \\ = {} & \tilde{h}_2(h*\Omega_\lambda)\tilde{h}_2^{-1} - d\tilde{h}_2\tilde{h}_2^{-1} \\ = {} & \sum_{i=1}^{m} \begin{pmatrix} C_iF^t - FC_i^t & -C_i\lambda & 0 \\ C_i^t\lambda & A_1^tI_{n,1}A_{1x_i} & -C_i\zeta^tI_{n,1}A_2h_2^{-1} \\ 0 & h_2JA_2^tI_{n,1}\zeta C_i & 0 \end{pmatrix} dx_i. \end{aligned}$$

Set $G = -\zeta^tI_{n,1}A_2h_2^{-1}$ and using (3.7), we have

$$A_1^tI_{n,1}A_{1x_i} - (C_i^tF - I_{n,1}F^tC_iI_{n,1}) = YC_i, \quad 1 \le i \le m, \tag{3.10}$$

where $Y = A_1^t I_{n,1}\zeta$. By using (3.10) and $A_1^t I_{n,1} A_1 = I_m$, we get $YC_i + (YC_i)^t = 0$ for all $1 \le i \le n$ which implies $Y = 0$. Hence $\xi = (F, G)$ is a solution of the $G^1_{m,n}$-system (2.2).

By using proposition 3.2, we know (A_1, F) and $(\tilde{A}_1, \tilde{F})$ are solutions of the $G^1_{m,n}$-II system (2.5) corresponding to X and $\tilde{X}$ respectively. Let A_2, G, $A = (A_1, A_2)$ be given as above and $\hat{W}, \hat{Z}$ as in lemma 2.2. Let

$$\gamma = (\gamma_1, \dots, \gamma_{m+n}, \gamma_{m+n+1}) = (\cos\rho \hat{W}^t, \sin\rho \hat{Z}^t A^t), \tag{3.11}$$

where $\cos\rho = \dfrac{s}{\sqrt{1+s^2}}$ and $\sin\rho = \dfrac{-1}{\sqrt{1+s^2}}$. Substituting $\lambda = 1$ into (2.8), we obtain

$$\tilde{E}^{II}(x, 1) = E^{II}(x, 1)(I_{m+n+1} - 2\gamma^t\gamma I_{m+n,1}). \tag{3.12}$$

Substituting (3.3) into (3.12), we have

$$\begin{aligned} \tilde{e}_k &= e_k - 2\gamma_k \sum_{j=1}^{m+n+1} \gamma_j e_j, \quad k = 1, \dots, m+n, \\ \tilde{X} &= X + 2\gamma_{m+n+1} \sum_{j=1}^{m+n+1} \gamma_j e_j. \end{aligned} \tag{3.13}$$

Notice that $X = e_{m+n+1}$ and $\tilde{X} = \tilde{e}_{m+n+1}$, we know

$$\gamma_k X - \gamma_{m+n+1} e_k = \gamma_k \tilde{X} - \gamma_{m+n+1}\tilde{e}_k, \quad k = 1, \dots, m+n. \tag{3.14}$$

Let $\Gamma_k = \operatorname{arctanh}\frac{\gamma_{m+n+1}}{\gamma_k}$, $1 \le k \le n+m$, (3.14) becomes

$$\cosh\Gamma_k X - \sinh\Gamma_k e_k = \cosh\Gamma_k \tilde{X} - \sinh\Gamma_k \tilde{e}_k. \tag{3.15}$$

Geometrically (3.19) means that the geodesic of $\mathbb{H}^{m+n}$ at $X(x)$ in the direction $e_k(x)$ intersects the geodesic of $\mathbb{H}^{m+n}$ at $\tilde{X}(x)$ in the direction $\tilde{e}_k(x)$ at a point equidistant to $X(x)$ and $\tilde{X}(x)$. Thus the bundle morphism $P : \vartheta(X) \to \vartheta(\tilde{X})$ is a Ribaucour transformation. □

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