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RIBAUCOUR TRANSFORMATIONS ON RIEMANNIAN SPACE FORMS IN LORENTZIAN SPACE FORM

JOONSANG PARK

ABSTRACT. We study Ribaucour transformations on nondegenerate local isometric immersions of Riemannian space forms into Lorentzian space forms with flat normal bundles. They can be explained by dressing actions on the solution space of Lorentzian Grassmannian systems.

1. Introduction

The study of isometric immersions of the space forms $N^n(c)$ with constant sectional curvature c into the space forms $N^{n+k}(c')$ has been a classical problem in differential geometry. Nonexistence of an isometric immersion of the hyperbolic space form $\mathbb{H}^2 = N^2(-1)$ into $\mathbb{R}^3 = N^3(0)$ by Hilbert [3], existence of local isometric immersions of $N^n(c)$ in $N^{2n-1}(c+1)$ and nonexistence of local immersions of $N^n(c)$ in $N^{2n-2}(c+1)$ by Cartan [2], and generalizations of Cartan's work by Tenenblat and Terng [7], [8], [9] are well-known, and many other results have been obtained in [10] and [1], too. On the other hand, the theory of transformations on the surfaces has been one of the main interests in the classical differential geometry, which has been extended to the immersions of space forms in space forms.

Recently the soliton theory in integrable systems has been developed extensively so that it can be applied to submanifold geometry. Notice that the sine-Gordon equation is a special kind of soliton equations, which is related to local immersions of $\mathbb{H}^2$ into $\mathbb{R}^3$. In this vein, the so-called n-dimensional system or G/K system on a symmetric space

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developed by Terng [10] has succeeded in explaining some geometry of submanifold $N^n(c)$ in $N^{n+k}(c')$ [10], [1]. The author also showed that some local isometric immersion of the Riemannian space form $N^n(c)$ in the Lorentzian space form $N^{n+k,1}(c)$ for $c = 1, 0$ or -1 is associated to a solution of the Lorentzian Grassmannian system [6].

In this paper, we study a Ribaucour transformation on a local isometric immersion of $N^n(c)$ which is nondegenerate and has flat normal bundle into $N^{n+k,1}(c)$ by using a dressing action on the solution of the Lorentzian Grassmannian system.

2. Preliminaries

In this section, we review basic knowledge and notations about Lorentzian submanifold geometry [4], [5] and introduce the G/K system [10]. Also, we briefly summarize the results in [6] to explain the relationship between the immersions of the Riemannian space form $N^n(c)$ into the Lorentzian space form $N^{n+k,1}(c)$ and the Lorentzian Grassmannian systems.

Denote by $\mathbb{R}^{m,r}$ the vector space $\mathbb{R}^{m+r}$ with the nondegenerate metric of index r, $\langle x, y\rangle_r = \sum_{i=1}^{m} x_i y_i - \sum_{i=m+1}^{m+r} x_i y_i$. It is well-known (cf. [4]) that the $(m+1)$-dimensional Lorentzian space form, that is, the simply-connected complete connected $(m+1)$-dimensional Lorentzian manifold $N^{m,1}(c)$ of the constant sectional curvature $c = 0, 1, -1$ is the Lorentzian space $\mathbb{R}^{m,1}$, the universal covering space of the Lorentzian sphere $\mathbb{S}^{m,1}$ or the Lorentzian hyperbolic space $\mathbb{H}^{m,1}$, respectively, where

$$\mathbb{S}^{m,1} = \{x \in \mathbb{R}^{m+1,1} \mid \langle x, x\rangle_1 = 1\},$$
$$\mathbb{H}^{m,1} = \{x \in \mathbb{R}^{m,2} \mid \langle x, x\rangle_2 = -1\}.$$

On the other hand, the Riemannian space forms $N^n(c)$ are the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$, and the hyperbolic space $\mathbb{H}^n$ for $c = 0, 1, -1$.

DEFINITION 2.1. Suppose $k + 1 \geq n$. A Riemannian submanifold M^n in $N^{n+k,1}(c)$ is called *nondegenerate* if the image of the second fundamental form $(\operatorname{Im} I\!I)_p = \{I\!I(X, Y) \mid X, Y \in T_pM\}$ has dimension n for any $p \in M^n$ and the inner product on $\operatorname{Im} I\!I$ induced by $\langle\ ,\ \rangle$ is nondegenerate. We say that the curvature normals of M are *spacelike* (or *Lorentzian*) if $\operatorname{Im} I\!I$ is a spacelike (or Lorentzian, respectively) subspace of the normal bundle $\nu(M)$.

We now explain the local geometry of submanifolds $N^n(c)$ in $N^{n+k,1}(c)$. We denote by I_p the $p\times p$ identity matrix and $J_p=\operatorname{diag}(1,\dots,1,-1)$ is a $p\times p$ diagonal matrix.

PROPOSITION 2.2. ([6]) *Let $X : N^n(c) \longrightarrow N^{n+k,1}(c)$ be a non-degenerate local isometric immersion with a flat normal bundle, and assume $k+1\geq n$. Then, for a local parallel normal frame e_α $(n+1\leq \alpha\leq n+k+1)$, there exist a curvature coordinate system $(x_1,\dots,x_n)$, a map $b=(b_1,\dots,b_n)^t$ and an $n\times(k+1)$ matrix-valued $B_1=(b_{ij})$ such that $B_1J_{k+1}B_1^t=I_n$ or $B_1J_{k+1}B_1^t=J_n$ and the first and second fundamental forms are given by*

$$I=\sum_{i=1}^{n} b_i^2 dx_i^2, \qquad II=\sum_{i=1}^{n}\sum_{j=1}^{k+1} b_{ij}b_i dx_i^2\otimes e_{n+j}.$$

The curvature normals are spacelike when $B_1J_{k+1}B_1^t = I_n$, and Lorentzian when $B_1J_{k+1}B_1^t=J_n$.

To describe Riemannian submanifolds in $N^{n+k,1}(c)$, we will use a special partial differential equation called G/K system, which is introduced by Terng in [10], and we mention some results from [10], which will be used in our case.

Let G/K be a rank n symmetric space with the involution $\sigma:\mathcal{G}\to\mathcal{G}$ on the Lie algebra $\mathcal{G}$ of G, $\mathcal{G}=\mathcal{K}+\mathcal{P}$ the Cartan decomposition, and $\mathcal{A}\subset\mathcal{P}$ a maximal abelian subalgebra with a basis $\{a_1,\dots,a_n\}$. Let $\mathcal{A}^\perp$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{G}$ with respect to the Killing form. *G/K system* for $v:R^n\to\mathcal{P}\cap\mathcal{A}^\perp$ is

$$[a_i,v_{x_j}]-[a_j,v_{x_i}]=\big[[a_i,v],[a_j,v]\big],\qquad 1\leq i\neq j\leq n, \tag{2.1}$$

where $v_{x_i}=\frac{\partial v}{\partial x_i}$.

It is known that v is a solution of (2.1) if and only if the $\mathcal{G}\otimes\mathbb{C}$-valued connection 1-form on the trivial principal bundle $\mathbb{R}^n\times\mathcal{G}$ on $\mathbb{R}^n$

$$\theta=\sum_{i=1}^{n}(a_i\lambda+[a_i,v])dx_i \tag{2.2}$$

is flat for any $\lambda\in\mathbb{C}$.

To apply the theory of G/K system to our case, we take the Lorentzian Grassmannian G_c/K system related to the isometry group G_c of $N^{n+k,1}(c)$.

Let $\mathcal{M}_{p\times q}$ be the set of $p\times q$ matrices, $\mathbb{R}^m=\mathcal{M}_{m\times 1}$, $J_{c,1}=\mathrm{diag}(c,1,\dots,1,-1)$. Denote $u=(u_0,u_1,\dots,u_m)^t\in\mathbb{R}^{m+1}$. For $c=0,1,-1$, the isometry groups G_c are given by

$$\mathrm{Isom}(\mathbb{R}^{n+k,1})=\left\{\begin{pmatrix}1&0\\ \xi&A\end{pmatrix}\;\middle|\;A\in O(n+k,1),\ \xi\in\mathbb{R}^{n+k,1}\right\},$$

$$O(n+k+1,1)=\left\{A\in GL(n+k+2,\mathbb{R})\;\middle|\;A^tJ_{1,1}A=J_{1,1}\right\},$$

$$O(n+k,2)=\left\{A\in GL(n+k+2,\mathbb{R})\;\middle|\;A^tJ_{-1,1}A=J_{-1,1}\right\},$$

respectively. Here the inner product on $\mathbb{R}^{n+k,2}$ is defined by

$$\langle u,v\rangle_2=u^tJ_{-1,1}\,v=-u_0v_0+\sum_{i=1}^{n+k}u_iv_i-u_{n+k+1}v_{n+k+1},$$

and we identify $\mathbb{R}^{n+k,1}$ with $\{1\}\times\mathbb{R}^{n+k,1}\subset\mathbb{R}^{n+k,2}$ by $X\leftrightarrow(1,X)$.

We can explain G_c in one way using $J_{c,1}$. That is,

$$G_c=\left\{A\in GL(n+k+2,\mathbb{R})\;\middle|\;AJ_{c,1}A^t=J_{c,1}\right\}.$$

The Lie algebra of G_c is

$$\mathcal{G}_c=\left\{\begin{pmatrix}0&-c\xi^tJ\\ \xi&Y\end{pmatrix}\;\middle|\;Y\in o(n+k,1),\ \xi\in\mathbb{R}^{n+k+1}\right\},$$

where $J=J_{1,1}$, which we will abuse the notation whatever the size is.

Now, we recall G_c/K systems related to the isometry group G_c of $N^{n+k,1}(c)$ defined in [6].

DEFINITION 2.3. Assume $k+1\ge n$. Put $\delta=\mathrm{diag}(dx_1,\dots,dx_n)$. For $b\in\mathbb{R}^n=\mathcal{M}_{n\times 1}$, $F=(f_{ij})\in\mathcal{M}_{n\times n}$ with $f_{ii}=0$ and $G\in\mathcal{M}_{(k+1-n)\times n}$,
(i) (F,G,b) is *a solution associated to* $\mathcal{G}_c$ *of spacelike type* if

$$\theta_s=\begin{pmatrix}0&-cb^t\delta&0&0\\ \delta b&\delta F-F^t\delta&\lambda\delta&0\\ 0&-\lambda\delta&\delta F^t-F\delta&\delta G^tJ\\ 0&0&-G\delta&0\end{pmatrix}\tag{2.3}$$

is a family of flat connection 1-form for any $\lambda\in\mathbb{C}$,

(ii) (F, G, b) is *a solution associated to* $\mathcal{G}_c$ *of Lorentzian type* if

$$(2.4)\qquad \theta_L = \begin{pmatrix} 0 & -cb^t\delta & 0 & 0 \\ \delta b & \delta F - F^t\delta & 0 & \lambda\delta \\ 0 & 0 & 0 & -G\delta \\ 0 & -\lambda J\delta & J\delta G^t & J\delta F^t J - F\delta \end{pmatrix}$$

is a family of flat connection 1-form for any $\lambda \in \mathbb{C}$.

Since θ_S in (2.3) is flat for any λ, we have

$$A^{-1}dA = \begin{pmatrix} 0 & -cb^t\delta \\ \delta b & \delta F - F^t\delta \end{pmatrix}, \quad BdB^{-1} = \begin{pmatrix} \delta F^t - F\delta & \delta G^t J \\ -G\delta & 0 \end{pmatrix}$$

for some $A \in \mathcal{M}_{(n+1)\times(n+1)}$ and $B = \binom{B_1}{B_2} \in O(k,1)$, where $B_1 \in \mathcal{M}_{n\times(k+1)}$, $B_2 \in \mathcal{M}_{(k+1-n)\times(k+1)}$. For θ_L in (2.4), there is $B = \binom{B_2}{B_1} \in O(k,1)$ such that

$$BdB^{-1} = \begin{pmatrix} 0 & -G\delta \\ J\delta G^t & J\delta F^t J - F\delta \end{pmatrix}.$$

The following fact is proved in [6];

PROPOSITION 2.4. *Let $k+1 \geq n$. Suppose X is a nondegenerate local isometric immersion of $N^n(c)$ into $N^{n+k,1}(c)$ with a flat normal bundle as in Proposition 2.2. Then there exists a solution (F, G, b) associated to $\mathcal{G}_c$ such that*

$$F = \left(\frac{(b_i)_{x_j}}{b_j} \right), \qquad B_1 dJB_1^t = \delta F^t - F\delta,$$

when M has spacelike curvature normals, and

$$F = \left(\frac{(b_i)_{x_j}}{b_j} \right), \qquad B_1 dJB_1^t J = J\delta F^t J - F\delta,$$

when M has Lorentzian curvature normals.

Conversely, if (F, G, b) is a solution associated to $\mathcal{G}_c$ of the spacelike type (or the Lorentzian type), then there exists a nondegenerate isometric immersion X of $N^n(c)$ with a flat normal bundle into $N^{n+k,1}(c)$, which has spacelike (or Lorentzian) curvature normals, a parallel normal frame $\{e_\alpha\}$, a coordinate system $(x_1, \dots, x_n)$, and an $\mathcal{M}_{n\times(k+1)}$-valued map B_1 with $B_1 JB_1^t = I$ (or $B_1 JB_1^t = J$, respectively) such that the first and second fundamental forms are given by

$$I = \sum_{i=1}^{n} b_i^2 dx_i^2, \qquad II = \sum_{i=1}^{n}\sum_{j=1}^{k+1} b_{ij} b_i dx_i^2 \otimes e_{n+j}.$$

3. Transformations

In this section, we construct a dressing action ([11]) on the space of solutions of the G_c/K system, which turns out to be a Ribaucour transformation of $N^n(c)$ in $N^{n+k,1}(c)$. This kind of action was used in [1] to get Ribaucour transformations of $N^n(c)$ in $N^{n+k}(c)$.

Put $J = \mathrm{diag}(1, \dots, 1, -1)$, $J_c = \mathrm{diag}(c, 1, \dots, 1)$ and $J_{c,1} = \mathrm{diag}(c, 1, \dots, -1)$. The G_c/K-reality conditions are

$$(3.1) \qquad \begin{cases} g(\lambda)\, J_{c,1}\, g(\lambda)^t = J_{c,1}, \\ \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{k+1} \end{pmatrix} g(\lambda) \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{k+1} \end{pmatrix} = g(-\lambda), \\ \overline{g(\bar{\lambda})} = g(\lambda). \end{cases}$$

Notice that trivializations E_S of θ_S and E_L of θ_L, which mean $E_S^{-1} dE_S = \theta_S$ and $E_L^{-1} dE_L = \theta_L$, satisfy the reality conditions (3.1). Also, E_S and E_L are holomorphic for any $\lambda \in \mathbb{C}$.

For $z = (z_0, z_1, \dots, z_n)^t \in \mathbb{R}^{n+1}$ and $w = (w_1, \dots, w_{k+1})^t \in \mathbb{R}^{k,1}$ such that

$\|z\|_{n,c}^2 = z^t J_c z = 1$ and $\|w\|_{k,1}^2 = w^t J w = 1$,

let $\pi = \frac{1}{2} \begin{pmatrix} z z^t J_c & -i\, z w^t J \\ i\, w z^t J_c & w w^t J \end{pmatrix}$.

Define $q_{s,\pi}(\lambda)$ for $s \in \mathbb{R}$ with $s \neq 0$ and $\lambda \in \mathbb{C}$ by

$$q_{s,\pi}(\lambda)^t = \left(\pi + \frac{\lambda + i\,s}{\lambda - i\,s}(1 - \pi) \right) \left(\bar{\pi} + \frac{\lambda - i\,s}{\lambda + i\,s}(1 - \bar{\pi}) \right).$$

Then it is easy to see that $q_{s,\pi}(\lambda)$ is invertible, satisfies the reality conditions (3.1), is holomorphic at $\lambda = \infty$, and

$$(3.2) \qquad q_{s,\pi}(\lambda) = q_{s,\pi}(-\lambda)^{-1} = I + \frac{2s}{\lambda^2 + s^2} \begin{pmatrix} -s J_c z z^t & -\lambda J_c z w^t \\ \lambda J w z^t & -s J w w^t \end{pmatrix}.$$

For convenience, we use $E = E_S$ or $E = E_L$. Let

$$\begin{pmatrix} \tilde{z} \\ i\,\tilde{w} \end{pmatrix} = E(x, -i\,s)^{-1} \begin{pmatrix} z \\ i\,w \end{pmatrix}.$$

Since $q_{s,\pi}(\lambda)$ satisfies the reality conditions (3.1), we can show that $\tilde{z} \in \mathbb{R}^{n+1}$ and $\tilde{w} \in \mathbb{R}^{k,1}$, and from

$$\begin{aligned} \|\tilde{z}\|_{n,c}^2 - \|\tilde{w}\|_{k,1}^2 &= \left(E(x, -i\,s)^{-1} \begin{pmatrix} z \\ i\,w \end{pmatrix} \right)^t J_{c,1} \left(E(x, -i\,s)^{-1} \begin{pmatrix} z \\ i\,w \end{pmatrix} \right) \\ &= \begin{pmatrix} z \\ i\,w \end{pmatrix}^t J_{c,1} \begin{pmatrix} z \\ i\,w \end{pmatrix} \\ &= \|z\|_{n,c}^2 - \|w\|_{k,1}^2 \\ &= 0, \end{aligned}$$

we have $\|\tilde{z}\|_{n,c} = \|\tilde{w}\|_{k,1}$.

Put $\hat{z} = \tilde{z}/\|\tilde{z}\|_{n,c}$, $\hat{w} = \tilde{w}/\|\tilde{w}\|_{k,1}$ and $\tilde{\pi} = \frac{1}{2}\begin{pmatrix} \hat{z}\hat{z}^t J_c & -i\,\hat{z}\hat{w}^t J \\ i\,\hat{w}\hat{z}^t J_c & \hat{w}\hat{w}^t J \end{pmatrix}$.

It is easy to show that $q_{s,\pi}(\lambda)Eq_{s,\tilde{\pi}}(\lambda)^{-1}$ is holomorphic in $\lambda \in \mathbb{C}$ by residue calculations at $\lambda = \pm i\, s$. Let $\tilde{E} = Eq_{s,\tilde{\pi}}(\lambda)^{-1}$, that is,

$$\tilde{E} = E\begin{pmatrix} I - \frac{2s^2}{\lambda^2+s^2} J_c\hat{z}\hat{z}^t & \frac{2s\lambda}{\lambda^2+s^2} J_c\hat{z}\hat{w}^t \\ -\frac{2s\lambda}{\lambda^2+s^2} J\hat{w}\hat{z}^t & I - \frac{2s^2}{\lambda^2+s^2} J\hat{w}\hat{w}^t \end{pmatrix}. \tag{3.3}$$

Now, from $E^{-1}dE = \sum_{i=1}^n (a_i\lambda + [a_i, v])dx_i$ and $q_{s,\tilde{\pi}}(\lambda)$ is holomorphic at $\lambda = \infty$, we can prove by direct calculation that $\tilde{\theta} = \tilde{E}^{-1}d\tilde{E}$ is of the form

$$\tilde{\theta} = \sum_{i=1}^{n} (a_i\lambda + [a_i, \tilde{v}])dx_i, \tag{3.4}$$

and when we write $v = \begin{pmatrix} 0 & -J_c\xi^t J \\ \xi & 0 \end{pmatrix}$ and $\tilde{v} = \begin{pmatrix} 0 & -J_c\tilde{\xi}^t J \\ \tilde{\xi} & 0 \end{pmatrix}$, we obtain

$$\tilde{\xi} = \xi - \frac{2}{s}\left(J\hat{w}\hat{z}^t\right)_*, \tag{3.5}$$

where $\begin{pmatrix} 0 & -(J_c\xi^t J)_* \\ \xi_* & 0 \end{pmatrix}$ is the orthogonal projection of $\begin{pmatrix} \eta & -J_c\xi^t J \\ \xi & \zeta \end{pmatrix}$ onto $\mathcal{P} \cap \mathcal{A}^\perp$.

By the above argument, we get

THEOREM 3.1. *Let (F, G, b) be a solution associated to $\mathcal{G}_c$ of the spacelike type (or the Lorentzian type) whose corresponding one parameter family of flat connections is θ as in (2.3) (or (2.4)). Then we have a new solution $(\tilde{F}, \tilde{G}, \tilde{b})$ corresponding to $\tilde{\theta}$ as in (3.4). In particular, when it is a spacelike type,*

$$\begin{pmatrix} \tilde{b} & \tilde{F} \\ 0 & \tilde{G} \end{pmatrix} = \begin{pmatrix} b & F \\ 0 & G \end{pmatrix} - \frac{2}{s}\left(J\hat{w}\hat{z}^t\right)_{*_S}, \tag{3.6}$$

and when it is a Lorentzian type,

$$\begin{pmatrix} 0 & \tilde{G} \\ \tilde{b} & \tilde{F} \end{pmatrix} = \begin{pmatrix} 0 & G \\ b & F \end{pmatrix} - \frac{2}{s}\left(J\hat{w}\hat{z}^t\right)_{*_L}, \tag{3.7}$$

where $(c_{ij})_{*_S}$ *means* $c_{i,i+1} = 0$ *and* $(c_{ij})_{*_L}$ *means* $c_{k+1-n+i,i+1} = 0$.

Hence we obtain a new solution $\tilde{X} : N^n(c) \to N^{n+k,1}(c)$ from a given immersion $X : N^n(c) \to N^{n+k,1}(c)$ by $(\tilde{F}, \tilde{G}, \tilde{b})$ from (F, G, b). We will investigate on how X and $\tilde{X}$ are related geometrically. Let E and $\tilde{E}$ be trivializations corresponding to θ and $\tilde{\theta}$ in (2.3) (or (2.4)) and (3.4), respectively. Write

$$E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x)^{-1} \end{pmatrix}, \quad \tilde{E}(x,0) = \begin{pmatrix} \tilde{A}(x) & 0 \\ 0 & \tilde{B}(x)^{-1} \end{pmatrix}.$$

From (3.3), we have

$$\tilde{A} = A(I - 2J_c\hat{z}\hat{z}^t), \quad \tilde{B}^{-1} = B^{-1}(I - 2J\hat{w}\hat{w}^t). \tag{3.8}$$

Put

$$E^I = E\begin{pmatrix} I_{n+1} & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \tilde{E}^I = \tilde{E}\begin{pmatrix} I_{n+1} & 0 \\ 0 & \tilde{B}^{-1} \end{pmatrix}.$$

In fact, if we write $E^I(x,1) = (X, e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k+1})$, then X is the immersion of $N^n(c)$ into $N^{n+k,1}(c)$, $e_i = \frac{1}{b_i}\frac{\partial}{\partial x_i}$ $(1 \le i \le n)$ are a tangent frame and e_α $(n+1 \le \alpha \le n+k+1)$ are a parallel normal frame to X. By a direct calculation, those trivializations are related by

$$\tilde{E}^I = E^I\Big(I - \frac{2}{\lambda^2 + s^2}\begin{pmatrix} sJ_c\hat{z} \\ \lambda B^t J\hat{w} \end{pmatrix}\begin{pmatrix} s\hat{z}^t & \lambda\hat{w}^t JBJ \end{pmatrix}\Big). \tag{3.9}$$

Now we conclude that

THEOREM 3.2. *Suppose $X : N^n(c) \to N^{n+k,1}(c)$ is a nondegenerate local isometric immersion with flat normal bundle. Then the $q_{s,\pi}(\lambda)$ action on X gives rise to a new immersion $\tilde{X} : N^n(c) \to N^{n+k,1}(c)$ of the same kind, and X and $\tilde{X}$ are in a Ribaucour transformation.*

PROOF. Suppose X has spacelike normals. Let E_S be a trivialization of θ_S in (2.4) and $\tilde{E}_S = E_S q_{s,\tilde{\pi}(x)}(\lambda)^{-1}$. Write

$$E^I_S(x,1) = (X(x), e_1(x), \ldots, e_n(x), e_{n+1}(x), \ldots, e_{n+k+1}(x)),$$
$$\tilde{E}^I_S(x,1) = (\tilde{X}(x), \tilde{e}_1(x), \ldots, \tilde{e}_n(x), \tilde{e}_{n+1}(x), \ldots, \tilde{e}_{n+k+1}(x)).$$

By (3.9), we have

(3.10)

$$\tilde{E}^I_S(x,1) = E^I_S(x,1)\Big(I - 2\begin{pmatrix} \cos\rho\, J_c\hat{z} \\ \sin\rho\, B^t J\hat{w} \end{pmatrix}\begin{pmatrix} \cos\rho\,\hat{z}^t & \sin\rho\,\hat{w}^t JBJ \end{pmatrix}\Big),$$

where $\rho = \arctan \frac{1}{s}$. Put

$$\beta = \cos\rho\, \hat{z}(x)^t, \quad \gamma = \sin\rho\, \hat{w}(x)^t JBJ, \quad \eta = 2E_s^I(x,1)\begin{pmatrix} \cos\rho\, J_c\hat{z}(x) \\ \sin\rho\, B^t J\hat{w}(x)\end{pmatrix}.$$

Then it follows from (3.10) that

$$\begin{aligned} \tilde{X} &= X - \beta_0\eta, & & \\ \tilde{e}_i &= e_i - \beta_i\eta, & & (1 \le i \le n), \\ \tilde{e}_\alpha &= e_\alpha - \gamma_\alpha\eta, & & (n+1 \le \alpha \le n+k+1). \end{aligned}$$

Hence $X + r_i e_i = \tilde{X} + r_i\tilde{e}_i$ and $X + r_\alpha e_\alpha = \tilde{X} + r_\alpha\tilde{e}_\alpha$ for $r_i = -\beta_0/\beta_i$ and $r_\alpha = -\beta_0/\gamma_\alpha$. Therefore, X and $\tilde{X}$ are in a Ribaucour transformation.

The case that X has Lorentzian normals is similar as above, so we will omit the proof. □

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Department of Mathematics
Dongguk University
Seoul 100-715, Korea
E-mail: jpark@dgu.edu