

GENERALIZED NONLINEAR MULTIVALUED MIXED QUASI-VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. In this paper, we introduce a new class of generalized nonlinear multivalued mixed quasi-variational-like inequalities and prove the existence and uniqueness of solutions for the class of generalized nonlinear multivalued mixed quasi-variational-like inequalities in reflexive Banach spaces using Fan-KKM Theorem.

1. Introduction and preliminaries

Variational inequality theory, which constituted a significant extension of the variational principle, has described a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics, regional and engineering sciences. The ideas and techniques are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. Hence variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, and unilateral, obstacle, optimization, etc. [1, 2, 3, 7, 8, 11, 12, 17, 18]. Among many kinds of variational inequalities, variational-like inequalities were firstly posed by Parida and Sen [13] and quasi-variational inequalities were firstly introduced by Aubin and Ekeland [1]. In fact, the quasi-variational-like inequality, which is an extension and a generalization of the variational inequalities mentioned above, is desirable to be considered more deeply.

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On the other hand, recently a new class of monotone nonlinear variational inequalities was introduced and the existence of solutions to them was considered in [16].

In this paper, a new class of generalized nonlinear multivalued mixed quasi-variational-like inequalities are introduced and studied. Also the solvability of this class of generalized nonlinear multivalued mixed quasi-variational-like inequalities in reflexive Banach spaces is given. Our results improve and extend the corresponding results of [6, 10, 14, 15, 16].

Throughout this paper, X is a real Banach space with dual space X^* and K is a nonempty convex closed subset of X . Denote $\langle \ell, x \rangle = \ell(x)$, for all $\ell \in X^*$ and $x \in X$. Let $S, T : K \rightarrow 2^{X^*}$ be two multivalued mappings, $N : X^* \times X^* \rightarrow X^*$ and $g : K \rightarrow X^*$ be mappings. Let a mapping $\eta : K \times K \rightarrow K$ be affine with respect to the first argument satisfying $\eta(u, v) = -\eta(v, u)$ for all $u, v \in K$. We consider the following generalized nonlinear multivalued mixed quasi-variational-like inequality problem (in short, GNMMQVLIP):

For any $\ell \in X^*$, find $u \in K$ such that

$$(1) \quad \sup_{x \in S(u), y \in T(u)} \langle (g(u) + N(x, y)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K,$$

where $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex functional.

We note that, if g is a zero mapping then (1) is reduced to the following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(2) \quad \sup_{x \in S(u), y \in T(u)} \langle N(x, y) - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K,$$

which is called a generalized nonlinear multivalued quasi-variational-like inequality problem.

We remark that if $\eta(v, u) = v - u$, then (2) is collapsed to the following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(3) \quad \sup_{x \in S(u), y \in T(u)} \langle N(x, y) - \ell, v - u \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K,$$

which was studied by Cho et al. [4].

We note that, if $N(x, y) = x - y$, then (2) is equivalent to following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(4) \quad \sup_{x \in S(u), y \in T(u)} \langle x - y - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K,$$

which was considered by Fang et al. [6].

If S and T are single-valued mappings, then (2) is equivalent to following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(5) \quad \langle N(S(u), T(u)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K,$$

which was studied by Huang et al. [10].

We remark that if $\eta(v, u) = G(v) - G(u)$, where $G : K \rightarrow K$ is a mapping, then (1) is collapsed to the following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(6) \quad \sup_{x \in S(u), y \in T(u)} \langle (g(u) + N(x, y)) - \ell, G(v) - G(u) \rangle + f(v) - f(u) \geq 0$$

for all $v \in K$.

Again, we note that if G is an identity mapping, then $G(v) - G(u) = v - u$, therefore (6) is collapsed to the following problem:

For any $\ell \in X^*$, find $u \in K$ such that

$$(7) \quad \sup_{x \in S(u), y \in T(u)} \langle (g(u) + N(x, y)) - \ell, v - u \rangle + f(v) - f(u) \geq 0 \text{ for all } v \in K.$$

REMARK 1.1. For a suitable choice of g, N, S, T, η and f , the problem (1) includes many known variational inequalities as special cases (see [4, 6, 10, 15, 16] and the references therein).

In the sequel, we recall some definitions needed.

DEFINITION 1.1. A mapping $S : K(\subset X) \rightarrow 2^{X^*}$ is said to be η - φ - p -monotone with respect to the first argument of a mapping $N : X^* \times X^* \rightarrow X^*$ if there exist a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$, a mapping $\eta : K \times K \rightarrow K$ and a constant $p > 1$ such that

$$(8) \quad \langle N(x, \cdot) - N(y, \cdot), \eta(u, v) \rangle \geq \varphi(\|\eta(u, v)\|) \|\eta(u, v)\|^p$$

for all $u, v \in K, x \in S(u)$ and $y \in S(v)$.

DEFINITION 1.2. A mapping $T : K(\subset X) \rightarrow 2^{X^*}$ is said to be η - ψ - p -monotone with respect to the second argument of a mapping $N : X^* \times X^* \rightarrow X^*$ if there exist a function $\psi : [0, +\infty) \rightarrow [0, +\infty)$, a mapping $\eta : K \times K \rightarrow K$ and a constant $p > 1$ such that

$$(9) \quad \langle N(\cdot, x) - N(\cdot, y), \eta(u, v) \rangle \geq -\psi(\|\eta(u, v)\|) \|\eta(u, v)\|^p$$

for all $u, v \in K, x \in T(u)$ and $y \in T(v)$.

DEFINITION 1.3. A mapping $g : K(\subset X) \rightarrow X^*$ is said to be η - ϕ - p -relaxed Lipschitzian if there exist a function $\phi : [0, +\infty) \rightarrow [0, +\infty)$, a mapping $\eta : K \times K \rightarrow K$ and a constant $p > 1$ such that

$$(10) \quad \langle g(v) - g(u), \eta(v, u) \rangle \geq \phi(\|\eta(v, u)\|) \|\eta(v, u)\|^p \text{ for all } u, v \in K.$$

REMARK 1.2. For a suitable choice of N , S , T , g and η we will provide many concepts which are special cases of our concepts (see [4, 6, 9, 10, 14, 16] and the references therein).

DEFINITION 1.4. Let X and Y be topological spaces. A mapping $F : X \rightarrow 2^Y$ is said to be lower semi-continuous at $x \in X$ if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in X converging to x , there exists a subset $\{x_\beta\} \subset \{x_\alpha\}$ and $y_\beta \in F(x_\beta)$ for each β such that $\{y_\beta\}$ converging to y .

DEFINITION 1.5 [14]. A mapping $g : K \rightarrow X^*$ is said to be hemi-continuous if for all $u, v, z \in K$, the mapping $t \rightarrow \langle g(u + tv), z \rangle$ is continuous on $[0, 1]$. A mapping $T : K \rightarrow 2^{X^*}$ is said to be lower hemi-continuous if for all $u, v, z \in K$, the multivalued mapping

$$t \rightarrow \langle T(u + tv), z \rangle$$

is lower semi-continuous on $[0, 1]$.

2. Main results

Now, we give our main results in this paper.

THEOREM 2.1. *Let X be a reflexive Banach space, X^* be its dual and K be a nonempty convex closed subset of X , let $g : K \rightarrow X^*$ be a hemi-continuous mapping satisfying (10) and also let S and $T : K \rightarrow 2^{X^*}$ be lower semi-continuous multivalued mappings satisfying (8) and (9), respectively, where for functions $\varphi, \psi, \phi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $\varphi(t) + \phi(t) > \psi(t)$ for all $t > 0$, $\varphi + \phi - \psi$ is bounded in $[0, \delta]$ for some $\delta > 0$. In addition, suppose that $\eta(u, v) = -\eta(v, u)$ for all $u, v \in K$, $\eta : K \times K \rightarrow K$ is affine with respect to the first argument, $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex functional and $N : X^* \times X^* \rightarrow X^*$ is continuous with respect to the weak* topology of X . Let a multivalued mapping $v \mapsto \{N(z, w) \in X^* : z \in S(v), w \in T(v)\}$ be lower hemi-continuous. Then for any $\ell \in X^*$, $u \in K$ is a solution of problem (1) if and only if $u \in K$ is a solution of the following problem:*

Find $u \in K$ such that

$$(11) \quad \begin{aligned} & \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ & \geq (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|))\|\eta(v, u)\|^p \end{aligned}$$

for all $v \in K, z \in S(v)$ and $w \in T(v)$.

PROOF. Suppose that the problem (1) holds. Since mappings S, T and g satisfy (8), (9) and (10), respectively, then for all $u, v \in K, x \in S(u), z \in S(v), y \in T(u)$ and $w \in T(v)$, we have

$$\begin{aligned} & \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ & = \langle -\ell, \eta(v, u) \rangle + \langle g(v), \eta(v, u) \rangle + \langle N(z, w), \eta(v, u) \rangle + f(v) - f(u) \\ & = \langle -\ell, \eta(v, u) \rangle - \langle N(x, w) - N(z, w), \eta(v, u) \rangle \\ & \quad - \langle N(x, y) - N(x, w), \eta(v, u) \rangle \\ & \quad + \langle N(x, y), \eta(v, u) \rangle + \langle g(v) - g(u), \eta(v, u) \rangle \\ & \quad + \langle g(u), \eta(v, u) \rangle + f(v) - f(u) \\ & = \langle -\ell, \eta(v, u) \rangle + \langle N(x, w) - N(z, w), \eta(v, u) \rangle \\ & \quad + \langle N(x, y) - N(x, w), \eta(v, u) \rangle + \langle N(x, y), \eta(v, u) \rangle \\ & \quad + \langle g(v) - g(u), \eta(v, u) \rangle + \langle g(u), \eta(v, u) \rangle + f(v) - f(u) \\ & \geq \langle (g(u) + N(x, y)) - \ell, \eta(v, u) \rangle + f(v) - f(u) + (\varphi(\|\eta(v, u)\|) \\ & \quad - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|))\|\eta(v, u)\|^p. \end{aligned}$$

$$\text{Put } A = \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u)$$

$$B = \langle (g(u) + N(x, y)) - \ell, \eta(v, u) \rangle + f(v) - f(u)$$

$$C = (\varphi - \psi + \phi)(\|\eta(v, u)\|)\|\eta(v, u)\|^p.$$

Taking suprema on both sides of the following inequality;

$$A \geq B + C,$$

we have

$$A = \sup_{x \in S(u), y \in T(u)} A \geq \sup_{x \in S(u), y \in T(u)} (B + C) = \sup_{x \in S(u), y \in T(u)} B + C.$$

Since $\sup_{x \in S(u), y \in T(u)} B \geq 0$ for all $v \in K$ from (1), $A \geq C$. Hence

$$\begin{aligned} & \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ & \geq (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|))\|\eta(v, u)\|^p \end{aligned}$$

for all $v \in K, z \in S(v)$ and $w \in T(v)$, i.e., (11) is true.

Conversely, suppose that (11) holds, without loss of generality, choose a point $v \in K$ such that $f(v) < +\infty$ and so $f(u) < +\infty$. Letting $v_n =$

$(1 - \frac{1}{n})u + \frac{1}{n}v$ for $n \in \mathbb{N}$, we have $v_n \in K$. Since $\eta : K \times K \rightarrow K$ is affine with respect to the first argument and $\eta(u, u) = 0$, $\eta(v_n, u) = \frac{1}{n}\eta(v, u)$ for $n \in \mathbb{N}$.

For any $x \in S(u)$ and $y \in T(u)$, since the mapping $v \mapsto \{N(z, w) \in X^* : z \in S(v), w \in T(v)\}$ is lower hemi-continuous, the mapping $v \mapsto g(v)$ is hemi-continuous and $v_n \rightarrow u$ as $n \rightarrow \infty$, there exists a subsequence $\{v_{n_j}\} \subset \{v_n\}$ and there are $z_{n_j} \in S(v_{n_j})$, $w_{n_j} \in T(v_{n_j})$ such that for any $\tau \in X$

$$(12) \quad z_{n_j} \rightarrow x, w_{n_j} \rightarrow y, \langle g(v_{n_j}) + N(z_{n_j}, w_{n_j}), \tau \rangle \rightarrow \langle g(u) + N(x, y), \tau \rangle$$

as $j \rightarrow \infty$. It follows from (11) that

$$(13) \quad \begin{aligned} & \langle g(v_{n_j}) + N(z_{n_j}, w_{n_j}) - \ell, \eta(v_{n_j}, u) \rangle + f(v_{n_j}) - f(u) \\ & \geq (\varphi(\|\eta(v_{n_j}, u)\|) - \psi(\|\eta(v_{n_j}, u)\|) + \phi(\|\eta(v_{n_j}, u)\|)) \|\eta(v_{n_j}, u)\|^p \\ & = \left(\frac{1}{n_j}\right)^p \left(\varphi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) - \psi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) \right. \\ & \quad \left. + \phi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) \right) \|\eta(v, u)\|^p, \end{aligned}$$

since $\eta(v_{n_j}, u) = \frac{1}{n_j}\eta(v, u)$.

Since f is convex and $v_{n_j} = (1 - \frac{1}{n_j})u + \frac{1}{n_j}v$,

$$\begin{aligned} f(v) - f(u) &= n_j \left(\left(1 - \frac{1}{n_j}\right) f(u) + \frac{1}{n_j} f(v) - f(u) \right) \\ &\geq n_j f\left(\left(1 - \frac{1}{n_j}\right)u + \frac{1}{n_j}v\right) - n_j f(u) \\ &= n_j f(v_{n_j}) - n_j f(u) \\ &= n_j (f(v_{n_j}) - f(u)), \end{aligned}$$

from (13) it follows that

$$(14) \quad \begin{aligned} & \langle g(v_{n_j}) + N(z_{n_j}, w_{n_j}) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ & \geq \left(\frac{1}{n_j}\right)^{p-1} \left(\varphi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) - \psi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) \right. \\ & \quad \left. + \phi\left(\frac{1}{n_j}\|\eta(v, u)\|\right) \right) \|\eta(v, u)\|^p. \end{aligned}$$

It follows from (12) and (14) that

$$\langle (g(u) + N(x, y) - \ell, \eta(v, u)) \rangle + f(v) - f(u) \geq 0$$

for all $v \in K$, $x \in S(u)$, $y \in T(u)$. This completes the proof. \square

REMARK 2.1. Theorem 2.1 improves and extends Theorem 2.1 of [4, 5, 6, 10, 16].

From Theorem 2.1, we have the following theorem.

COROLLARY 2.2. Let $G : K \rightarrow K$ be an affine mapping and put $\eta(v, u) = G(v) - G(u)$ for $u, v \in K$ in Definition 2.1, Definition 2.2, Definition 2.3 and Theorem 2.1. Then for any $\ell \in X^*$, $u \in K$ is a solution of the problem (6) if and only if $u \in K$ is a solution of the following problem.

For any $\ell \in X^*$, find $u \in K$ such that

$$(15) \quad \begin{aligned} & \langle (g(v) + N(z, w)) - \ell, G(v) - G(u) \rangle + f(v) - f(u) \\ & \geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) \\ & \quad + \phi(\|G(v) - G(u)\|)) \|G(v) - G(u)\|^p \end{aligned}$$

for all $v \in K, z \in S(v)$ and $w \in T(v)$.

From Corollary 2.2, we obtain the main result Theorem 2.1 of Cho et al. [4] as a corollary.

COROLLARY 2.3 [4]. Let G be an identity mapping, $g \equiv 0, \phi \equiv 0$ and $N(x, y) = x - y$ for $x, y \in X^*$ in Corollary 2.2. Then for any $\ell \in X^*, u \in K$ is a solution of

$$\sup_{x \in S(v), y \in T(u)} \langle N(x, y) - \ell, v - u \rangle + f(v) - f(u) \geq 0 \quad \text{for all } v \in K$$

if and only if $u \in K$ is a solution of

$$\langle N(z, w) - \ell, v - u \rangle + f(v) - f(u) \geq (\varphi\|v - u\| - \psi(\|v - u\|))\|v - u\|^p$$

for all $v \in K, z \in S(v)$ and $w \in T(v)$.

The following definition and result are essential for our further result.

DEFINITION 2.1 [17]. Let X be a topological vector space. A mapping $F : X \rightarrow 2^X$ is called a KKM mapping if for any $\{x_1, x_2, \dots, x_n\} \subset X$,

$$co\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n F(x_i).$$

FAN-KKM THEOREM 2.4 [5]. Let K be a nonempty subset of a topological vector space X and $F : K \rightarrow 2^X$ be a KKM-mapping. If $F(x)$ is closed in X for every x in K and there exists at least a point $x_0 \in K$ such that $F(x_0)$ is compact, then

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

THEOREM 2.5. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty bounded closed convex subset of X . Let $S, T, g, N, \varphi, \psi, \phi$ and η be the same as those in Theorem 2.1. Suppose that the mapping η and a function $\varphi - \psi + \phi$ are continuous and $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous. Then the problem (1) has a solution. Moreover if $\eta(u, v) = 0$ implies $u = v$, then the problem (1) has a unique solution.*

PROOF. We first prove the existence of a solution of the problem (1). Define two multivalued mappings $F, H : K \rightarrow 2^K$ by, for any $\ell \in X^*$,

$$F(v) = \{u \in K : \langle (g(u) + N(x, y)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0 \\ \text{for some } x \in S(u), y \in T(u)\}$$

for all $v \in K$, and

$$H(v) = \{u \in K : \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ \geq (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|)) \|\eta(v, u)\|^p \\ \text{for all } z \in S(v), w \in T(v)\}$$

for all $v \in K$.

We show that F is a KKM-mapping. Assume that F is not a KKM-mapping. Then there exists $\{v_1, v_2, \dots, v_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \dots, n$, such that

$$\sum_{i=1}^n t_i = 1, v = \sum_{i=1}^n t_i v_i \notin \bigcup_{i=1}^n F(v_i).$$

For any $z \in S(u)$ and $w \in T(u)$, by the definition of F , we have

$$\langle (g(v) + N(z, w)) - \ell, \eta(v_i, v) \rangle + f(v_i) - f(v) < 0$$

for $i = 1, 2, \dots, n$. It follows that

$$\begin{aligned} 0 &= \langle (g(v) + N(z, w)) - \ell, \eta(v, v) \rangle \\ &= \langle (g(v) + N(z, w)) - \ell, \eta(\sum_{i=1}^n t_i v_i, v) \rangle \\ &= \sum_{i=1}^n t_i \langle (g(v) + N(z, w)) - \ell, \eta(v_i, v) \rangle < \sum_{i=1}^n t_i (f(v) - f(v_i)) \\ &= f(v) - \sum_{i=1}^n t_i f(v_i) \leq f(v) - f(v) = 0, \end{aligned}$$

which is a contradiction. This implies that F is a KKM-mapping. Now we prove that $F(v) \subset H(v)$ for all $v \in K$. Let $u \in F(v)$. Then there exist $x \in S(u)$, $y \in T(u)$ such that

$$\langle (g(u) + N(x, y)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \geq 0.$$

Since the mappings g, S and T satisfy (10), (8) and (9), respectively, we have

$$\begin{aligned} & \langle (g(v) + N(z, w)) - \ell, \eta(v, u) \rangle + f(v) - f(u) \\ &= \langle -\ell, \eta(v, u) \rangle + \langle N(x, w) - N(z, w), \eta(u, v) \rangle \\ & \quad + \langle N(x, y) - N(x, w), \eta(u, v) \rangle + \langle N(x, y), \eta(v, u) \rangle \\ & \quad + \langle g(v) - g(u), \eta(v, u) \rangle + \langle g(u), \eta(v, u) \rangle + f(v) - f(u) \\ & \geq (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|)) \|\eta(v, u)\|^p \\ & \quad + \langle (g(u) + N(x, y) - \ell, \eta(v, u)) + f(v) - f(u) \rangle \\ & \geq (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|) + \phi(\|\eta(v, u)\|)) \|\eta(v, u)\|^p \end{aligned}$$

for all $v \in K, z \in S(v)$ and $w \in T(v)$. This implies that $u \in H(v)$ and so H is also a KKM-mapping.

From the assumption, it follows that $H(v)$ is weakly closed for all $v \in K$. Since K is bounded closed convex, we know that K is weakly compact and so $H(v)$ is weakly compact in K for all $v \in K$. It follows from Fan-KKM Theorem that

$$\bigcap_{v \in K} H(v) \neq \emptyset.$$

Hence for any $\ell \in X^*$ there exists a point $u_0 \in K$ such that

$$\begin{aligned} & \langle (g(v) + N(z, w)) - \ell, \eta(v, u_0) \rangle + f(v) - f(u_0) \\ & \geq (\varphi(\|\eta(v, u_0)\|) - \psi(\|\eta(v, u_0)\|) + \phi(\|\eta(v, u_0)\|)) \|\eta(v, u_0)\|^p, \end{aligned}$$

for all $z \in S(v), w \in T(v)$ for all $v \in K$. Thus

$$\langle (g(u_0) - N(x, y)) - \ell, \eta(v, u_0) \rangle + f(v) - f(u_0) \geq 0,$$

for all $v \in K$, for some $x \in S(v)$ and for some $y \in T(v)$, which shows that u_0 is a solution of (1).

Let u_1 and $u_2 \in K$ be solutions of the problem (1). Since

$$k_1(v) := \sup_{\substack{x \in S(u_1) \\ y \in T(u_1)}} \langle (g(u_1) + N(x, y)) - \ell, \eta(v, u_1) \rangle + f(v) - f(u_1) \geq 0$$

for all $v \in K$ and

$$k_2(v) := \sup_{\substack{x \in S(u_2) \\ y \in T(u_2)}} \langle (g(u_2) + N(x, y)) - \ell, \eta(v, u_2) \rangle + f(v) - f(u_2) \geq 0$$

for all $v \in K$,

by the definition of supremum, for any positive number ε , there exist $x_1 \in S(u_1)$ and $y_1 \in T(u_1)$ such that

$$(16) \quad k_1(v) - \varepsilon < \langle (g(u_1) + N(x_1, y_1)) - \ell, \eta(v, u_1) \rangle + f(v) - f(u_1) \leq k_1(v)$$

and, there exist $x_2 \in S(u_2)$ and $y_2 \in T(u_2)$ such that

$$(17) \quad k_2(v) - \varepsilon < \langle (g(u_2) + N(x_2, y_2)) - \ell, \eta(v, u_2) \rangle + f(v) - f(u_2) \leq k_2(v)$$

Setting $v = u_2$ in (16) and $v = u_1$ in (17) and adding, we have

$$\begin{aligned} & k_1(u_2) + k_2(u_1) - 2\varepsilon \\ & < \langle g(u_1) - g(u_2) + N(x_1, y_1) - N(x_2, y_2), \eta(u_2, u_1) \rangle \\ & \leq k_1(u_2) + k_2(u_1). \end{aligned}$$

Since ε is arbitrary,

$$(18) \quad \begin{aligned} & \langle g(u_1) - g(u_2) + N(x_1, y_1) - N(x_2, y_2), \eta(u_2, u_1) \rangle \\ & = k_1(u_2) + k_2(u_1) \geq 0. \end{aligned}$$

By (8), (9) and (10), we obtain

$$(19) \quad \begin{aligned} & \langle g(u_1) - g(u_2) + N(x_1, y_1) - N(x_2, y_2), \eta(u_2, u_1) \rangle \\ & = \langle g(u_1) - g(u_2), \eta(u_2, u_1) \rangle + \langle N(x_1, y_1) - N(x_2, y_1), \eta(u_2, u_1) \rangle \\ & \quad + \langle N(x_2, y_1) - N(x_2, y_2), \eta(u_2, u_1) \rangle \\ & = -\langle g(u_1) - g(u_2), \eta(u_1, u_2) \rangle - \langle N(x_1, y_1) - N(x_2, y_1), \eta(u_1, u_2) \rangle \\ & \quad - \langle N(x_2, y_1) - N(x_2, y_2), \eta(u_1, u_2) \rangle \\ & \leq (-\phi(\|\eta(u_1, u_2)\|) - \varphi(\|\eta(u_1, u_2)\|) + \psi(\|\eta(u_1, u_2)\|)) \|\eta(u_1, u_2)\|^p \\ & = -(\phi(\|\eta(u_1, u_2)\|) + \varphi(\|\eta(u_1, u_2)\|) - \psi(\|\eta(u_1, u_2)\|)) \|\eta(u_1, u_2)\|^p. \end{aligned}$$

Due to the inequality $\phi(t) + \varphi(t) > \psi(t)$ for all $t > 0$, it follows from (8) and (9) that $\|\eta(u_1, u_2)\|^p = 0$.

By the assumption that $\eta(u, v) = 0$ implies $u = v$, we have $u_1 = u_2$. This completes the proof. \square

REMARK 2.2. Theorem 2.4 also improves and extends Theorem 2.4 of [4, 6, 10, 16].

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