Commun. Korean Math. Soc. **21** (2006), No. 4, pp. 675–678

CERTAIN REPRESENTATION OF A REAL POLYNOMIAL

YOUNG JOON AHN AND SEON-HONG KIM

ABSTRACT. We show that every real polynomial of degree n can be represented by the sum of two real polynomials of degree n, each having only real zeros. As an example of this, we consider a real polynomial with even degree whose all zeros lie on the imaginary axes except the origin.

1. Introduction and statement of result

To what extent can a sum and its factorization both be known? More precisely, if A and B belong to a ring, to what extent can we simultaneously know the factorizations of A, B and C where $A+B=C$? There is an inverse problem: Given C, find A and B with factorizations of a specific type such that $A+B=C$. Much of the original motivation comes from results and questions about integers and power series such as Goldbach's conjecture, Fermat's last theorem and Jacobi's theta-function identity (see p. 35 of [1]).

It easily follows from the result of Fell [3] about a line that if all zeros of monic polynomials A and B with the same degree are distinct, real and form good pairs (see Definition 4.2 of [4]) then $A+B$ has all its zeros real. An inverse problem of this, given polynomial C having only real zeros, whether polynomials A and B having only real zeros such that $A+B=C$ exist, is easily solved by just taking both A and B as $C/2$. The result of this short paper is an extension of this. More precisely, we answer the question: Can every real polynomial with degree n be represented as the sum of two real polynomials with degree n, each having only real zeros and having the same sign of leading coefficients? Without the restriction "the same sign of leading coefficients" the question can be

Received July 27, 2006.
2000 Mathematics Subject Classification: Primary 30C15; Secondary 26C10.
Key words and phrases: polynomials, zeros.

easily answered by just taking $p = T + (p - T)$, where T is some large multiple of a first kind of Chebyshev polynomial of same degree with p. However it does not seem to be obvious to find an example about the question above.

It is of interest whether a polynomial can be represented by the sum of several polynomials having specific properties. For representing a nonnegative valued real polynomial by sums of squares of polynomials (see pp. 85-86 of [2]). For representing a positive valued real polynomial on a closed interval by sums of polynomials with real zeros (see pp. 346-348 of [2]). Also Videnskii [7] showed that a real polynomial $q(x)$ with degree n and $q(x) > 0$ on $[a, b]$ could be represented by $q(x) = q_1(x) + q_2(x)$, where each q_i is a real polynomial with deg $q_1(x) =$ deg $q_2(x) = n$, is nonnegative on $[a, b]$, and has only real zeros (in $[a, b]$). Answering to the question above in this direction, we show that, in Section 2,

THEOREM 1. *Every real polynomial $p(x)$ with degree n has the form*

$$p(x) = p_1(x) + p_2(x),$$

where $p_1(x)$ and $p_2(x)$ are different real polynomials with deg $p_1(x) =$ deg $p_2(x) = n$ *and have only real zeros, and $p(x)$, $p_1(x)$, $p_2(x)$ have the same sign of leading coefficients.*

Given a complex polynomial $f(x)$, Stong [6] has shown in replying to a question of Rubinstein that there always exist three polynomials $f_i(x) \in \mathbb{C}[x]$, $1 \leq i \leq 3$, each having all zeros on the unit circle, such that $f(x) = f_1(x) + f_2(x) + f_3(x)$. Others have shown that the three is best possible. We will use a modification of Stong's method to show Theorem 1 above.

As an example of Theorem 1, for a real polynomial p with even degree whose all zeros lie on the imaginary axes except the origin, we will find polynomials p_1 and p_2 satisfying the conclusion of Theorem 1.

2. Proof and example

PROOF OF THEOREM 1. We may assume that $p(x)$ is monic. Let

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Choose a constant C such that $|p(x)| \leq C$ for $x \in [-1, n-1]$. Let

$$p_1(x) = \frac{1}{2}x^n + Kx(x-1)\cdots(x-n+2)$$

for some large K. Take K large enough so that

$$|p_1(x)| > C$$

for $x = -\frac{1}{2}, \frac{1}{2}, 1 + \frac{1}{2}, \dots, n-1-\frac{1}{2}$ and $p_1(x)$ at these points alternate in sign. So $p_1(x)$ has a zero in each $[k-\frac{1}{2}, k+\frac{1}{2}]$, $0 \le k \le n-2$. This guarantees that $p_1(x)$ is a real polynomial of degree n with $n-1$ real zeros. Hence $p_1(x)$ in fact has n real zeros. Since $|p(x)| \le C$ at these n real zeros,

$$p_1(x) \ne p_2(x),$$

where $p_2(x) = p(x) - p_1(x)$. By the largeness of K and using same method above we can easily show that the polynomial of degree n

$$p_2(x) = \frac{1}{2}x^n + a_{n-1}x^{n-1} + \cdots + a_0 - Kx(x-1)\cdots(x-n+2)$$

has all its zeros real. □

Next, we give an example of Theorem 1 for a real polynomial with even degree whose all zeros lie on the imaginary axes except the origin. For this, we need the following lemma (see p. 182 of [5] for the proof).

LEMMA 2. *The condition that all zeros of the polynomial* $R(x)+iS(x)$ *are contained in* $\operatorname{Im} x > 0$ (*or* $\operatorname{Im} x < 0$) *is equivalent to the condition that all zeros of the equations* $R(x) = 0$ *and* $S(x) = 0$ *are real and separating each other.*

EXAMPLE 3. Let $a_k > 0$ for $k = 0, 1, 2, \dots$. For $n = 0, 1, 2, \dots$, we define polynomials

$$A_n = A_n(x) \quad \text{and} \quad B_n = B_n(x)$$

by $A_0(x) = x - a_0$, $B_0(x) = x + a_0$ and

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} x & -a_{n+1} \\ a_{n+1} & x \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \quad n \ge 0.$$

Then we can easily check that, by using induction on n,

(1) $$B_n(x) = (-1)^{n+1} A_n(-x) \quad (n \ge 0)$$

and, by (1) and using induction again,

$$A_n(x)^2 + B_n(x)^2 = 2\,(x^2 + a_0^2)(x^2 + a_1^2)\cdots(x^2 + a_n^2). \tag{2}$$

A real monic polynomial $p(x)$ with even degree whose all zeros lie on the imaginary axes except the origin is represented by the right of (2) with deleting 2. Moreover, we have $A_n(x) + i\,B_n(x) = \sqrt{2}\,(x + i\,a_0)(x + i\,a_1)\cdots(x + i\,a_n)$. So all the zeros of $A_n(x) + i\,B_n(x)$ are contained in $\operatorname{Im} x < 0$. By Lemma 2, the zeros of $A_n(x)$, and of $B_n(x)$ are all real, and the zeros of $A_n(x)$ interlace the zeros of $B_n(x)$. Choosing $p_1(x) = \frac{1}{2}A_n(x)^2$ and $p_2(x) = \frac{1}{2}B_n(x)^2$ gives conclusion of Theorem 1.

References

[1] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, Inc., New York, 1998.

[2] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.

[3] H. J. Fell, *On the zeros of convex combinations of polynomials*, Pacific J. Math. **89** (1980), 43–50.

[4] S.-H. Kim, *Factorization of sums of polynomials*, Acta Appl. Math. **73** (2002), no. 3, 275–284.

[5] G. V. Milovanovic, D. S. Mitrinovic, and T. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[6] R. Stong, *Sums of C-polynomials*, Amer. Math. Monthly **102** (1995), no. 3, 279–280.

[7] V. S. Videnskii, *On estimates of the derivative of a polynomial*, Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 401–420.

Young Joon Ahn
Department of Mathematics Education
College of Education
Chosun University
GwangJu 501-759, Korea
E-mail: ahn@chosun.ac.kr

Seon-Hong Kim
Mathematics Major
Division of Mathematics & Statistics, College of Science
Sookmyung Women's University
Seoul 140-742, Korea
E-mail: shkim17@sookmyung.ac.kr