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ROUGHNESS IN SUBTRACTION ALGEBRAS

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ABSTRACT. As a generalization of ideals in subtraction algebras, the notion of rough ideals is discussed.

1. Introduction

B. M. Schein [10] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [11] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [5] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In 1982, Pawlak introduced the concept of a rough set (see [8]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [9]). Rough set theory is applied to semigroups and groups (see [6, 7]). In this paper, we apply the rough set theory to subtraction algebras, and we introduce the notion of upper/lower rough subalgebras/ideals which is an extended notion of subalgebras/ideals in a subtraction algebra.

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2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4, 5]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

A nonempty subset S of a subtraction algebra X is called a *subalgebra* of X if $x - y \in S$ whenever $x, y \in S$.

A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Note that every ideal of a subtraction algebra X is a subalgebra of X.

LEMMA 2.1. [5] *An ideal A of a subtraction algebra X has the following property:*

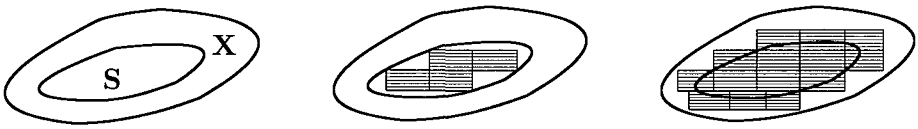
$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Rough sets in subtraction algebras

In what follows let X denote a subtraction algebra unless otherwise specified.

An equivalence relation ρ on X is called a *congruence relation* on X if whenever (x, y), $(u, v) \in \rho$ then $(x - u,\ y - v) \in \rho$. We denote by $[a]_\rho$ the ρ-congruence class containing the element $a \in X$. Let X/ρ denote the set of all ρ-congruence classes on X, i.e., $X/\rho := \{[a]_\rho \mid a \in X\}$. For any $[x]_\rho$, $[y]_\rho \in X/\rho$, if we define $[x]_\rho - [y]_\rho = [x - y]_\rho$, then $(X/\rho, -)$ is a subtraction algebra. Let ρ be an equivalence relation on X and let $\mathscr{P}(X)$ denote the power set of X and $\mathscr{P}^*(X) = \mathscr{P}(X) \setminus \{\emptyset\}$. For all $x \in X$, let $[x]_\rho$ denote the equivalence class of x with respect to ρ. Define the functions ρ_*, $\rho^* : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows: $\forall S \in \mathscr{P}(X)$,

$$\rho_*(S) = \{x \in X \mid [x]_\rho \subseteq S\} \text{ and } \rho^*(S) = \{x \in X \mid [x]_\rho \cap S \neq \emptyset\}.$$



$S \subset X$ $\qquad$ $\rho_*(S) \subseteq S$ $\qquad$ $S \subseteq \rho^*(S)$

$\rho_*(S)$ is called the *ρ-lower approximation* of S while $\rho^*(S)$ is called the *ρ-upper approximation* of S. For a nonempty subset S of X,

$$\rho(S) = (\rho_*(S),\ \rho^*(S))$$

is called a *rough set* with respect to ρ of $\mathscr{P}(X) \times \mathscr{P}(X)$ if $\rho_*(S) \neq \rho^*(S)$. A subset S of X is said to be *definable* if $\rho_*(S) = \rho^*(S)$. The pair (X, ρ) is called an *approximation space.*

The following property is useful for our research (cf. [8]).

PROPOSITION 3.1. *Let ρ and λ be congruence relations on X. Then the following assertions are true.*

(1) $(\forall F \in \mathscr{P}^*(X))\ (\rho_*(F) \subseteq F \subseteq \rho^*(F))$,
(2) $(\forall F, G \in \mathscr{P}^*(X))\ (\rho^*(F \cup G) = \rho^*(F) \cup \rho^*(G))$,
(3) $(\forall F, G \in \mathscr{P}^*(X))\ (\rho_*(F \cap G) = \rho_*(F) \cap \rho_*(G))$,
(4) $(\forall F, G \in \mathscr{P}^*(X))\ (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,

(5) $(\forall F, G \in \mathscr{P}^*(X))\ (F \subseteq G \Rightarrow \rho^*(F) \subseteq \rho^*(G))$,
(6) $(\forall F, G \in \mathscr{P}^*(X))\ (\rho_*(F) \cup \rho_*(G) \subseteq \rho_*(F \cup G))$,
(7) $(\forall F, G \in \mathscr{P}^*(X))\ (\rho^*(F \cap G) \subseteq \rho^*(F) \cap \rho^*(G))$,
(8) $(\forall F \in \mathscr{P}^*(X))\ (\rho \subseteq \lambda \Rightarrow \lambda_*(F) \subseteq \rho_*(F),\ \ \rho^*(F) \subseteq \lambda^*(F))$.

PROOF. Straightforward. □

COROLLARY 3.2. *If ρ and λ are congruence relations on X, then*

(i) $(\forall F \in \mathscr{P}^*(X))\ ((\rho \cap \lambda)^*(F) \subseteq \rho^*(F) \cap \lambda^*(F))$.
(ii) $(\forall F \in \mathscr{P}^*(X))\ (\rho_*(F) \cap \lambda_*(F) \subseteq (\rho \cap \lambda)_*(F))$.

PROOF. It follows immediately from Proposition 3.1. □

For any $F, G \in \mathscr{P}^*(X)$, we define $F - G := \{a - b \mid a \in F,\ b \in G\}$.

THEOREM 3.3. *If ρ is a congruence relation on X, then*

$$(\forall F, G \in \mathscr{P}^*(X))\ (\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)).$$

PROOF. Let $c \in \rho^*(F) - \rho^*(G)$. Then there exist $a \in \rho^*(F)$ and $b \in \rho^*(G)$ such that $c = a - b$. It follows that $[a]_\rho \cap F \neq \emptyset$ and $[b]_\rho \cap G \neq \emptyset$ so that $x \in [a]_\rho \cap F$ and $y \in [b]_\rho \cap G$ for some $x, y \in X$. Hence $x - y \in [a]_\rho - [b]_\rho = [a - b]_\rho$ and $x - y \in F - G$, that is, $x - y \in [a - b]_\rho \cap (F - G)$. Thus $c = a - b \in \rho^*(F - G)$, and so $\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)$. □

THEOREM 3.4. *If ρ is a congruence relation on X, then*

$$(\forall F, G \in \mathscr{P}^*(X))\ (\rho_*(F - G) \neq \emptyset \Rightarrow \rho_*(F) - \rho_*(G) \subseteq \rho_*(F - G)).$$

PROOF. Let $c \in \rho_*(F) - \rho_*(G)$. Then $c = a - b$ for some $a \in \rho_*(F)$ and $b \in \rho_*(G)$. Thus we get $[a]_\rho \subseteq F$ and $[b]_\rho \subseteq G$. It follows that

$$[a - b]_\rho = [a]_\rho - [b]_\rho \subseteq F - G$$

so that $c = a - b \in \rho_*(F - G)$. Therefore the result is valid. □

The following example shows the condition that $\rho_*(F - G) \neq \emptyset$ in Theorem 3.4 is necessary.

EXAMPLE 3.5. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the following Cayley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let ρ be a congruence relation on X such that $\{0, a\}$, $\{b\}$, and $\{c\}$ are all ρ-congruences of X. Taking $F = \{b, c\}$ and $G = \{c\}$, we have $F - G = \{0\}$, $\rho_*(F - G) = \emptyset$, $\rho_*(F) = \{b, c\}$, $\rho_*(G) = \{c\}$, and $\rho_*(F) - \rho_*(G) = \{0\}$.

For any congruence relation ρ on X, we note that

- $(\forall F \in \mathscr{P}^*(X))\ (\rho_*(F) \subseteq F)$,
- $(\forall F, G \in \mathscr{P}^*(X))\ (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,
- $(\forall F \in \mathscr{P}^*(X))\ (\rho_*(\rho_*(F)) = \rho_*(F))$,

which means that ρ_* is an interior operator on X. This operation induces a topology $\mathscr{T}$ on X such that

$$F \in \mathscr{T} \iff \rho_*(F) = F.$$

LEMMA 3.6. *For any congruence relation ρ on X, ρ^* is a closure operator on the topological space $(X, \mathscr{T})$.*

PROOF. For any $F \in \mathscr{P}^*(X)$ we have

$$x \in \rho^*(F) \Leftrightarrow [x]_\rho \cap F \neq \emptyset \Leftrightarrow [x]_\rho \nsubseteq F^c \Leftrightarrow x \notin \rho_*(F^c) \Leftrightarrow x \in (\rho_*(F^c))^c,$$

that is, $\rho^*(F) = (\rho_*(F^c))^c$, which completes the proof. □

LEMMA 3.7. *For any congruence relation ρ on X, we have*

(i) $(\forall F \in \mathscr{P}(X))\ (\rho_*(F) = F \iff \rho^*(F^c) = F^c)$,
(ii) $(\forall F \in \mathscr{P}(X))\ (\rho_*(F) = F \iff \rho^*(F) = F)$.

PROOF. Straightforward. □

Based on the above two lemmas we have the following result.

THEOREM 3.8. *For any $F \subseteq X$ and a congruence relation ρ on X, the following assertions are equivalent.*

(i) *F is definable with respect to ρ.*
(ii) *F is open in the topological space $(X, \mathscr{T})$.*
(iii) *F is closed in the topological space $(X, \mathscr{T})$.*

According to [7], we say that an open set F of X is said to be *free* in an approximation space (X, ρ) if $x \notin \rho^*(F \setminus \{x\})$ for all $x \in X$. Since $\rho^*(F \setminus \{x\}) = (\rho_*((F \setminus \{x\})^c))^c$, a nonempty subset F of X is free if and only if $x \in \rho_*(F^c \cup \{x\})$, i.e., if and only if $[x]_\rho \subseteq F^c \cup \{x\}$ for every $x \in F$. Thus for a free subset F and any $(x, y) \in \rho \cap (F \times F)$ we have $y \in F$, which together with $y \in [x]_\rho \subseteq F^c \cup \{x\}$ implies that $y = x$. Therefore $\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$. Conversely, let

$$\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$$

and let y be an arbitrary element of $[x]_\rho$. If $y \in F$, then $y = x$, i.e., $y \in \{x\} \subseteq F^c \cup \{x\}$. If $y \notin F$, then $y \in F^c \subseteq F^c \cup \{x\}$. Thus, in each case $[x]_\rho \subseteq F^c \cup \{x\}$, which means that F is free. Consequently, we obtain the following characterization of free subsets.

THEOREM 3.9. *$F \subseteq X$ is free if and only if $\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$.*

COROLLARY 3.10. *If X is free, then any subset of X is free.*

4. Roughness of ideals

Let A be an ideal of X. Define a relation $\mathscr{R}$ on X by

$$(\forall x, y \in X)\,((x, y) \in \mathscr{R} \Leftrightarrow x - y \in A,\ y - x \in A).$$

Then $\mathscr{R}$ is an equivalence relation on X related to an ideal A of X. Moreover $\mathscr{R}$ satisfies

$$(\forall x, y, u, v \in X)\,((x, y) \in \mathscr{R},\ (u, v) \in \mathscr{R} \Rightarrow (x - u, y - v) \in \mathscr{R}).$$

Hence $\mathscr{R}$ is a congruence relation on X. Let A_x denote the equivalence class of x with respect to the equivalence relation $\mathscr{R}$ related to the ideal A of X, and X/A denote the collection of all equivalence classes, that is, $X/A = \{A_x \mid x \in X\}$. Then $A_0 = A$. If $A_x \ominus A_y$ is defined as the class containing $x - y$, that is, $A_x \ominus A_y = A_{x-y}$, then it is easy to verify that $(X/A, -, A_0)$ is a subtraction algebra. Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. For any nonempty subset S of X, the lower and upper approximations of S are denoted by $\underline{\mathscr{R}}(A; S)$ and $\overline{\mathscr{R}}(A; S)$ respectively, that is,

$$\underline{\mathscr{R}}(A; S) = \{x \in X \mid A_x \subseteq S\} \text{ and } \overline{\mathscr{R}}(A; S) = \{x \in X \mid A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\mathscr{R}}(A; S)$ and $\overline{\mathscr{R}}(A; S)$ are denoted by $\underline{\mathscr{R}}(A)$ and $\overline{\mathscr{R}}(A)$, respectively.

EXAMPLE 4.1. (1) Let $X = \{0, a, b, c\}$ be a set with the Cayley table as follows:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $(X, -, 0)$ is a subtraction algebra. Consider an ideal $A = \{0, a\}$ of X and let $\mathscr{R}$ be an equivalence relation on X related to A. Then $A_0 = A_a = A$, $A_b = \{b\}$, and $A_c = \{c\}$. Hence

- $\underline{\mathscr{R}}(A; \{0, b\}) = \{b\} = \underline{\mathscr{R}}(A; \{b\})$,
- $\underline{\mathscr{R}}(A; \{0, c\}) = \{c\} = \underline{\mathscr{R}}(A; \{c\})$,
- $\underline{\mathscr{R}}(A; \{0, a, b\}) = \{0, a, b\} \lhd X$,
- $\underline{\mathscr{R}}(A; \{0, a\}) = \{0, a\} \lhd X$,
- $\overline{\mathscr{R}}(A; \{0, c\}) = \{0, a, c\} \lhd X$,
- $\overline{\mathscr{R}}(A; \{a\}) = A \lhd X$,
- $\underline{\mathscr{R}}(A; \{0\}) = \emptyset = \underline{\mathscr{R}}(A; \{a\})$,
- $\underline{\mathscr{R}}(A; \{0, a, c\}) = \{0, a, c\} \lhd X$,
- $\underline{\mathscr{R}}(A; \{0, b, c\}) = \{b, c\}$,
- $\overline{\mathscr{R}}(A; \{0, b\}) = \{0, a, b\} \lhd X$,
- $\overline{\mathscr{R}}(A; \{0, a\}) = A \lhd X$,
- $\overline{\mathscr{R}}(A; \{b\}) = \{b\}$.

(2) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Consider $A = \{0, b, d\} \lhd X$ and let $\mathscr{R}$ be an equivalence relation on X related to A. Then the equivalence classes are as follows: $A_0 = A_b = A_d = A$, $A_a = \{a, c, d\}$, and $A_c = \{a, c\}$. Thus

- $\underline{\mathscr{R}}(A; \{0, a\}) = \emptyset$,
- $\underline{\mathscr{R}}(A; \{0, a, d\}) = \emptyset$,
- $\underline{\mathscr{R}}(A; \{0, b, d\}) = A \lhd X$,
- $\underline{\mathscr{R}}(A; \{0, b, c, d\}) = A \lhd X$,
- $\overline{\mathscr{R}}(A; \{0, b\}) = A \lhd X$,
- $\overline{\mathscr{R}}(A; \{0, d\}) = \{0, a, b, d\}$,
- $\overline{\mathscr{R}}(A; \{b\}) = A \lhd X$,
- $\overline{\mathscr{R}}(A; \{d\}) = A \lhd X$.
- $\underline{\mathscr{R}}(A; \{0, b, c\}) = \emptyset$,
- $\underline{\mathscr{R}}(A; \{0, a, c\}) = \{c\}$,
- $\underline{\mathscr{R}}(A; \{0, a, b, c\}) = \{c\}$,
- $\overline{\mathscr{R}}(A; \{0, a\}) = X$,
- $\overline{\mathscr{R}}(A; \{0, c\}) = X$,
- $\overline{\mathscr{R}}(A; \{0, a, d\}) = X$,
- $\overline{\mathscr{R}}(A; \{c\}) = \{a, c\}$,

In Example 4.1, we know that there exists a non-ideal U of X such that $\underline{\mathscr{R}}(A; U) \lhd X$; and there exists a non-ideal V of X such that $\overline{\mathscr{R}}(A; V) \lhd X$, where $\mathscr{R}$ is an equivalence relation on X related to $A \lhd X$.

PROPOSITION 4.2. *Let $\mathscr{R}$ and $\mathscr{Q}$ be equivalence relations on X related to ideals A and B of X, respectively. If $A \subseteq B$, then $\mathscr{R} \subseteq \mathscr{Q}$.*

PROOF. If $(x, y) \in \mathscr{R}$, then $x - y \in A \subseteq B$ and $y - x \in A \subseteq B$. Hence $(x, y) \in \mathscr{Q}$, and so $\mathscr{R} \subseteq \mathscr{Q}$. □

PROPOSITION 4.3. *Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. Then*

(1) $(\forall S \in \mathscr{P}(X))\ (\underline{\mathscr{R}}(A;S) \subseteq S \subseteq \overline{\mathscr{R}}(A;S))$,
(2) $(\forall S,T \in \mathscr{P}(X))\ (\overline{\mathscr{R}}(A;S\cup T) = \overline{\mathscr{R}}(A;S) \cup \overline{\mathscr{R}}(A;T))$,
(3) $(\forall S,T \in \mathscr{P}(X))\ (\underline{\mathscr{R}}(A;S\cap T) = \underline{\mathscr{R}}(A;S) \cap \underline{\mathscr{R}}(A;T))$,
(4) $(\forall S,T \in \mathscr{P}(X))\ (S \subseteq T \Rightarrow \underline{\mathscr{R}}(A;S) \subseteq \underline{\mathscr{R}}(A;T), \overline{\mathscr{R}}(A;S) \subseteq \overline{\mathscr{R}}(A;T))$,
(5) $(\forall S,T \in \mathscr{P}(X))\ (\underline{\mathscr{R}}(A;S\cup T) \supseteq \underline{\mathscr{R}}(A;S) \cup \underline{\mathscr{R}}(A;T))$,
(6) $(\forall S,T \in \mathscr{P}(X))\ (\overline{\mathscr{R}}(A;S\cap T) \subseteq \overline{\mathscr{R}}(A;S) \cap \overline{\mathscr{R}}(A;T))$,
(7) *If $\mathscr{Q}$ is an equivalence relation on X related to an ideal B of X and if $A \subseteq B$, then $\overline{\mathscr{R}}(A;S) \subseteq \overline{\mathscr{Q}}(B;S)$ for all $S \in \mathscr{P}(X)$.*

PROOF. (1) is straightforward.

(2) For any subsets S and T of X, we have

$$\begin{aligned} x \in \overline{\mathscr{R}}(A;S\cup T) &\Leftrightarrow A_x \cap (S\cup T) \neq \emptyset \\ &\Leftrightarrow (A_x \cap S) \cup (A_x \cap T) \neq \emptyset \\ &\Leftrightarrow A_x \cap S \neq \emptyset \ \text{ or } \ A_x \cap T \neq \emptyset \\ &\Leftrightarrow x \in \overline{\mathscr{R}}(A;S) \ \text{ or } \ x \in \overline{\mathscr{R}}(A;T) \\ &\Leftrightarrow x \in \overline{\mathscr{R}}(A;S) \cup \overline{\mathscr{R}}(A;T), \end{aligned}$$

and hence $\overline{\mathscr{R}}(A;S\cup T) = \overline{\mathscr{R}}(A;S) \cup \overline{\mathscr{R}}(A;T)$.

(3) For any subsets S and T of X we have

$$\begin{aligned} x \in \underline{\mathscr{R}}(A;S\cap T) &\Leftrightarrow A_x \subseteq S \cap T \\ &\Leftrightarrow A_x \subseteq S \ \text{ and } A_x \subseteq T \\ &\Leftrightarrow x \in \underline{\mathscr{R}}(A;S) \ \text{ and } \ x \in \underline{\mathscr{R}}(A;T) \\ &\Leftrightarrow x \in \underline{\mathscr{R}}(A;S) \cap \underline{\mathscr{R}}(A;T). \end{aligned}$$

Hence $\underline{\mathscr{R}}(A;S\cap T) = \underline{\mathscr{R}}(A;S) \cap \underline{\mathscr{R}}(A;T)$.

(4) Let $S,T \in \mathscr{P}(X)$ be such that $S \subseteq T$. Then $S \cap T = S$ and $S \cup T = T$. It follows from (3) and (2) that

$$\underline{\mathscr{R}}(A;S) = \underline{\mathscr{R}}(A;S\cap T) = \underline{\mathscr{R}}(A;S) \cap \underline{\mathscr{R}}(A;T)$$

and

$$\overline{\mathscr{R}}(A;T) = \overline{\mathscr{R}}(A;S\cup T) = \overline{\mathscr{R}}(A;S) \cup \overline{\mathscr{R}}(A;T),$$

which yield $\underline{\mathscr{R}}(A;S) \subseteq \underline{\mathscr{R}}(A;T)$ and $\overline{\mathscr{R}}(A;S) \subseteq \overline{\mathscr{R}}(A;T)$, respectively.

(5) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, it follows from (4) that

$$\underline{\mathscr{R}}(A;S) \subseteq \underline{\mathscr{R}}(A;S\cup T) \ \text{ and } \ \underline{\mathscr{R}}(A;T) \subseteq \underline{\mathscr{R}}(A;S\cup T).$$

Thus $\underline{\mathscr{R}}(A;S) \cup \underline{\mathscr{R}}(A;T) \subseteq \underline{\mathscr{R}}(A;S\cup T)$.

(6) Since $S \cap T \subseteq S, T$, it follows from (4) that

$$\overline{\mathscr{R}}(A;S\cap T) \subseteq \overline{\mathscr{R}}(A;S) \ \text{ and } \ \overline{\mathscr{R}}(A;S\cap T) \subseteq \overline{\mathscr{R}}(A;T)$$

so that $\overline{\mathscr{R}}(A;S\cap T) \subseteq \overline{\mathscr{R}}(A;S) \cap \overline{\mathscr{R}}(A;T)$.

(7) If $x \in \overline{\mathscr{R}}(A;S)$, then $A_x \cap S \neq \emptyset$, and so there exists $a \in S$ such that $a \in A_x$. Hence $(a,x) \in \mathscr{R}$, that is, $a - x \in A$ and $x - a \in A$. Since $A \subseteq B$, it follows that $a - x \in B$ and $x - a \in B$ so that $(a,x) \in \mathscr{Q}$, that is, $a \in B_x$. Therefore $a \in B_x \cap S$, which means $x \in \overline{\mathscr{Q}}(B;S)$. This completes the proof. □

PROPOSITION 4.4. *Let $\mathscr{R}$ be an equivalence relation on X related to any ideal A of X. Then $\underline{\mathscr{R}}(A;X) = X = \overline{\mathscr{R}}(A;X)$, that is, X is definable.*

PROOF. It is straightforward. □

PROPOSITION 4.5. *Let $\mathscr{R}$ be an equivalence relation on X related to the trivial ideal $\{0\}$ of X. Then $\underline{\mathscr{R}}(\{0\};S) = S = \overline{\mathscr{R}}(\{0\};S)$ for every nonempty subset S of X, that is, every nonempty subset of X is definable.*

PROOF. Note that $\{0\}_x = \{x\}$ for all $x \in X$, since if $a \in \{0\}_x$ then $(a,x) \in \mathscr{R}$ and hence $a - x = 0$ and $x - a = 0$. It follows that $a = x$. Hence

$$\underline{\mathscr{R}}(\{0\};S) = \{x \in X \mid \{0\}_x \subseteq S\} = S$$

and

$$\overline{\mathscr{R}}(\{0\};S) = \{x \in X \mid \{0\}_x \cap S \neq \emptyset\} = S.$$

This completes the proof. □

REMARK 4.6. Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. If B is an ideal of X such that $A \neq B$, then $\underline{\mathscr{R}}(A;B)$ is not an ideal of X in general. For, consider a subtraction algebra X in Example 4.1(2) and an equivalence relation $\mathscr{R}$ on X related to the ideal $A = \{0,1,2\}$. If we take an ideal $B = \{0,1,3\}$ of X, then $A \neq B$ and $\underline{\mathscr{R}}(A;B) = \{3\}$ which is not an ideal of X.

DEFINITION 4.7. Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. A nonempty subset S of X is called an *upper* (resp. a *lower*) *rough subalgebra/ideal* of X if the upper (resp. nonempty lower) approximation of S is a subalgebra/ideal of X. If S is both an upper and a lower rough subalgebra/ideal of X, we say that S is a *rough subalgebra/ideal* of X.

THEOREM 4.8. *Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. Then every subalgebra S of X is a rough subalgebra of X.*

PROOF. Let $x, y \in \underline{\mathscr{R}}(A;S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a subalgebra of X, it follows that $A_{x-y} = A_x \ominus A_y \subseteq S$ so that $x - y \in$

$\underline{\mathscr{R}}(A;S)$. Hence $\underline{\mathscr{R}}(A;S)$ is a subalgebra of X. Now if $x,y \in \overline{\mathscr{R}}(A;S)$, then $A_x \cap S \neq \emptyset$ and $A_y \cap S \neq \emptyset$, and so there exist $a,b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a,x) \in \mathscr{R}$ and $(b,y) \in \mathscr{R}$. Since $\mathscr{R}$ is a congruence relation on X, we have $(a-b, x-y) \in \mathscr{R}$. Hence $a-b \in A_{x-y}$. Since S is a subalgebra of X, we get $a-b \in S$, and therefore $a-b \in A_{x-y} \cap S$, that is, $A_{x-y} \cap S \neq \emptyset$. This shows that $x-y \in \overline{\mathscr{R}}(A;S)$, and consequently $\overline{\mathscr{R}}(A;S)$ is a subalgebra of X. This completes the proof. □

COROLLARY 4.9. *Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. Then $\underline{\mathscr{R}}(A)$ $(\neq \emptyset)$ and $\overline{\mathscr{R}}(A)$ are subalgebras of X, that is, A is a rough subalgebra of X.*

PROOF. It is straightforward. □

THEOREM 4.10. *Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. If U is an ideal of X containing A, then*

(1) *$\underline{\mathscr{R}}(A;U)$ $(\neq \emptyset)$ is an ideal of X, that is, U is a lower rough ideal of X.*
(2) *$\overline{\mathscr{R}}(A;U)$ is an ideal of X, that is, U is an upper rough ideal of X.*

PROOF. Let U be an ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq U$, and so $A_0 \subseteq U$. Hence $0 \in \underline{\mathscr{R}}(A;U)$. Let $x,y \in X$ be such that $y \in \underline{\mathscr{R}}(A;U)$ and $x-y \in \underline{\mathscr{R}}(A:U)$. Then $A_y \subseteq U$ and $A_x \ominus A_y = A_{x-y} \subseteq U$. Let $a \in A_x$ and $b \in A_y$. Then $(a,x) \in \mathscr{R}$ and $(b,y) \in \mathscr{R}$, which implies $(a-b, x-y) \in \mathscr{R}$. Hence $a-b \in A_{x-y} \subseteq U$. Since $b \in A_y \subseteq U$ and U is an ideal, it follows that $a \in U$, so that $A_x \subseteq U$. Thus $x \in \underline{\mathscr{R}}(A;U)$. This shows that $\underline{\mathscr{R}}(A;U)$ is an ideal of X, that is, U is a lower rough ideal of X. Now, obviously $0 \in \overline{\mathscr{R}}(A;U)$. Let $x,y \in X$ be such that $y \in \overline{\mathscr{R}}(A;U)$ and $x-y \in \overline{\mathscr{R}}(A;U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x-y} \cap U \neq \emptyset$, and so there exist $a,b \in U$ such that $a \in A_y$ and $b \in A_{x-y}$. Hence $(a,y) \in \mathscr{R}$ and $(b, x-y) \in \mathscr{R}$, which implies $y-a \in A \subseteq U$ and $(x-y)-b \in A \subseteq U$. Since $a,b \in U$ and U is an ideal, we get $y \in U$ and $x-y \in U$; hence $x \in U$. Note that $x \in A_x$, thus $x \in A_x \cap U$, that is, $A_x \cap U \neq \emptyset$. Therefore $x \in \overline{\mathscr{R}}(A;U)$, and consequently U is an upper rough ideal of X. □

COROLLARY 4.11. *Let $\mathscr{R}$ be an equivalence relation on X related to an ideal A of X. Then $\underline{\mathscr{R}}(A)$ $(\neq \emptyset)$ and $\overline{\mathscr{R}}(A)$ are ideals of X, that is, A is a rough ideal of X.*

Theorem 4.10 shows that the notion of an upper (resp. a lower) rough ideal is an extended notion of an ideal in a subtraction algebra.

The following example shows that if A and U are ideals of X such that $A \not\subseteq U$, then $\underline{\mathscr{R}}(A;U)$ may not be an ideal of X.

EXAMPLE 4.12. (1) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra described in Example 4.1(2). Consider two ideals $A = \{0, b\}$ and $U = \{0, d\}$ of X. Then $\underline{\mathscr{R}}(A;U) = \{d\}$ which is not an ideal of X.

(2) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Consider $A = \{0, a, b\} \lhd X$ and let $\mathscr{R}$ be an equivalence relation on X related to A. Then the equivalence classes are as follows: $A_0 = A_a = A_b = A$, $A_c = \{c\}$, and $A_d = \{d\}$. Then $U = \{0, a, c\}$ is an ideal of X which does not contain A, and $\underline{\mathscr{R}}(A;U) = \{c\}$ which is not an ideal of X.

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