Commun. Korean Math. Soc. **21** (2006), No. 4, pp. 631–640

ON BOUNDEDNESS FOR COMPLEX VALUED FUNCTIONS ON THE p-ADIC VECTOR SPACE

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ABSTRACT. In this paper, we prove sufficient conditions of boundedness of maximal operators on the p-adic vector space. We also consider weighted Hardy-Littlewood averages on the p-adic vector space.

1. Introduction and preliminaries

The p-adic numbers $\mathbb{Q}_p$, p a prime, is constructed by completing the rational numbers with respect to a non-Archimedean absolute value. These numbers and their finite algebraic extensions are locally compact, totally disconnected, and nondiscrete. For a somewhat more leisurely treatment of the construction of p-adic numbers, consult Gouvêa [2]. p-adic analysis is an area of mathematics that has gained a large progress in recent years because, apart from its great role from the mathematical point of view, it turns out to be a useful tool in unexpected fields such as theoretical physics. Fundamental results about the p-adic theory of physical problems can be found in [7] and [8] (see also the references therein).

In this paper, we consider the boundedness of maximal operators and weighted Hardy-Littlewood averages on the p-adic vector space. These are distinctly different from those on real numbers and complex numbers due to the nature of the underlying topology. For example, smooth functions are those which are locally constant; differentiability plays no role.

For a more complete introduction to the p-adic numbers, see [2] and [8].

Received February 20, 2006.
2000 Mathematics Subject Classification: 11E95.
Key words and phrases: Hardy-Littlewood averages, p-adic numbers.
This work was supported by Kyungnam University Research Fund, 2005.

Here we will outline what we need. Let $\mathbb{Q}_p^n$ be the vector space of all n-tuples of elements of $\mathbb{Q}_p$ consists of points $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$\|x\| = \max_{1 \leq i \leq n} |x_i|_p.$$

This is easily seen to be non-Archimedean property

$$\|x + y\| \leq \max(\|x\|, \|y\|)$$

for $x, y \in \mathbb{Q}_p^n$. Also, we say that $x, y \in \mathbb{Q}_p^n$ are congruent modulo p^γ, and write $\|x - y\| \leq p^\gamma$, if $|x_i - y_i|_p \leq p^\gamma$ for each $i = 1, \ldots, n$. For $\gamma \in \mathbb{Z}$, we denote by $B_\gamma^n(a)$ the ball of radius p^γ with the center at the point $a \in \mathbb{Q}_p^n$ and by $S_\gamma^n(a)$ its boundary (sphere), respectively:

$$B_\gamma^n(a) = \left\{x \in \mathbb{Q}_p^n \mid \|x - a\| \leq p^\gamma\right\}$$

and

$$S_\gamma^n(a) = \left\{x \in \mathbb{Q}_p^n \mid \|x - a\| = p^\gamma\right\} = B_\gamma^n(a) \setminus B_{\gamma-1}^n(a).$$

For $a = 0$ we set $B_\gamma^n(0) = B_\gamma^n$ and $S_\gamma^n(0) = S_\gamma^n$. The Haar measure dx_i $(i = 1, \ldots, n)$ on $\mathbb{Q}_p$ is extended to an invariant measure $dx = dx_1 \cdots dx_n$ on $\mathbb{Q}_p^n$ in the standard way. Its normalization is fixed by taking the Haar measure of B_0^n, the set of n-dimensional p-adic integers, as equal to 1:

$$\operatorname{vol}(B_0^n) = \int_{B_0^n} dx = 1.$$

It is now straightforward to calculate the measure of any n-ball and also of n-sphere from $\operatorname{vol}(B_0^n) = 1$ (see [1], [5] and [7]).

We say that $f \in L_{\mathrm{loc}}^1(\mathbb{Q}_p^n)$ is integrable on $\mathbb{Q}_p^n$ (improper integral) if there exists

$$\lim_{N \to \infty} \int_{B_N^n} f(x)dx = \lim_{N \to \infty} \sum_{-\infty < \gamma \leq N} \int_{S_\gamma^n} f(x)dx.$$

This limit is called an integral (improper) of the function f on $\mathbb{Q}_p^n$, and it is denoted by $\int_{\mathbb{Q}_p^n} f(x)dx$ so that $\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{-\infty < \gamma < \infty} \int_{S_\gamma^n} f(x)dx$.

Definition 1. (1) Let $f(x)$ be a complex-valued function on the p-adic space $\mathbb{Q}_p^n$. A function f is called locally-constant if for any point $x \in \mathbb{Q}_p^n$ there exists $l(x) \in \mathbb{Z}$ such that

$$f(x + x') = f(x), \quad \|x'\| \leq p^{l(x)}.$$

For the set of locally-constant functions on $\mathbb{Q}_p^n$ we denote by $\mathcal{E} = \mathcal{E}(\mathbb{Q}_p^n)$.

(2) A function $f \in \mathcal{E}(\mathbb{Q}_p^n)$ is called test function on $\mathbb{Q}_p^n$ if its support is compact on $\mathbb{Q}_p^n$. Let us denote by $\mathcal{D} = \mathcal{D}(\mathbb{Q}_p^n)$ the set of test functions on $\mathbb{Q}_p^n$.

A nonzero p-adic number $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p = p^{-\gamma}$ may uniquely be written in the form

$$\alpha = \sum_{k=\gamma}^{\infty} \alpha_k p^k,$$

where $0 \leq \alpha_k \leq p-1$ and $\alpha_\gamma \neq 0$. The standard additive character χ_p for α given by the above form is defined by

$$\chi_p(\alpha) = \begin{cases} \prod_{k=\gamma}^{-1} \exp(2\pi i \alpha_k p^k), & |\alpha|_p > 1, \\ 1, & |\alpha|_p \leq 1. \end{cases}$$

Let $\varphi \in \mathcal{D}$. Its p-adic Fourier-transform $\mathcal{F}[\varphi] = \widetilde{\varphi}$ is defined by the formula

$$\widetilde{\varphi}(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\langle \xi, x\rangle)\varphi(x)dx, \quad \xi \in \mathbb{Q}_p^n,$$

where $\chi_p(\langle \xi, x\rangle) = \chi_p(\xi_1 x_1)\cdots\chi_p(\xi_n x_n)$ and $\langle \xi, x\rangle$ is the scalar product of vectors. The p-adic Fourier-transform $\varphi \to \widetilde{\varphi}$ is the linear isomorphism from $\mathcal{D}$ onto $\mathcal{D}$, and also the inversion formula

$$\varphi(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\langle \xi, x\rangle)\widetilde{\varphi}(\xi)d\xi$$

for the p-adic Fourier-transform is valid. It gives the inverse mapping of the p-adic Fourier-transform $\widetilde{\varphi}$ of a test function φ on $\mathbb{Q}_p^n$ (see [1] and [7]). Let us introduce in $\mathcal{D}(\mathbb{Q}_p^n)$ a canonical δ-sequence $\delta_\gamma(x) = p^{n\gamma}\Omega(p^\gamma\|x\|)$ and a canonical 1-sequence $\Delta_\gamma(x) = \Omega(p^{-\gamma}\|x\|)$ for $\gamma \in \mathbb{Z}$ and $x \in \mathbb{Q}_p^n$, where $\Omega(t)$ the step function is 1 if $0 \leq t \leq 1$ and 0 if $t > 1$.

PROPOSITION 2. [5] *For all* $\gamma \in \mathbb{Z}$,

$$\widetilde{\Delta}_\gamma = \delta_\gamma.$$

DEFINITION 3. (1) Let $\varphi \in \mathcal{D}$. Then there exists $l \in \mathbb{Z}$, such that

$$\varphi(x + x') = \varphi(x),$$

where $x' \in B_l^n$ and $x \in \mathbb{Q}_p^n$. The largest one of such numbers $l = l(\varphi)$ is called the parameter of constancy of a function φ.

(2) Let us denote by $\mathcal{D}_\gamma^l = \mathcal{D}_\gamma^l(\mathbb{Q}_p^n)$ the set of test functions with support in the ball B_γ^n and with parameter of constancy $\geq l$.

LEMMA 4. [8] *Let* $\varphi \in \mathcal{D}_\gamma^l(\mathbb{Q}_p^n)$. *Then* $\widetilde{\varphi} \in \mathcal{D}_{-l}^{-\gamma}(\mathbb{Q}_p^n)$.

THEOREM 5. [8] *The Fourier-transform $\varphi \mapsto \widetilde{\varphi}$ is the linear isomorphism from $\mathcal{D}$ onto $\mathcal{D}$, and the Parseval-Steklov equalities are valid:*

$$\int_{\mathbb{Q}_p^n} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{Q}_p^n} \widetilde{\varphi}(x)\overline{\widetilde{\psi}(x)}dx$$

and

$$\int_{\mathbb{Q}_p^n} \varphi(x)\widetilde{\psi}(x)dx = \int_{\mathbb{Q}_p^n} \widetilde{\varphi}(x)\psi(x)dx,$$

where $\varphi, \psi \in \mathcal{D}$.

2. The p-adic bounded function on p-adic vector space

For every $\varphi \in \mathcal{D}_\gamma^l$, the canonical coverings of B_γ^n lead us to have the form

$$\begin{aligned}\varphi(x) &= \sum_{1\le v\le p^{n(\gamma-l)}} \varphi(a^v)\triangle_l(x_1 - a_1^v)\cdots\triangle_l(x_n - a_n^v)\\ &= \sum_{1\le v\le p^{n(\gamma-l)}} \varphi(a^v)\triangle_l(x - a^v),\end{aligned}$$

where $x \in \mathbb{Q}_p^n$ and $a^v = (a_1^v, \ldots, a_n^v) \in B_\gamma^n$, which does not depend on φ (see [4, p. 89, (5.2)]). Here $\triangle_l(x - a^v)$ is the characteristic function of the ball $B_l^n(a^v)$. By Lemma 4, the function $\widetilde{\varphi}$ can be expanded in the form

$$\begin{aligned}\widetilde{\varphi}(\xi) &= \sum_{1\le v\le p^{n(\gamma-l)}} \widetilde{\varphi}(a^v)\triangle_{-\gamma}(\xi_1 - a_1^v)\cdots\triangle_{-\gamma}(\xi_n - a_n^v)\\ &= \sum_{1\le v\le p^{n(\gamma-l)}} \widetilde{\varphi}(a^v)\triangle_{-\gamma}(\xi - a^v),\end{aligned}$$

where $\xi \in \mathbb{Q}_p^n$. Now, we easily see that

$$\mathcal{F}[\varphi(x - a^v)](\xi) = \int_{\mathbb{Q}_p^n} \varphi(x)\chi_p(\langle \xi, x + a^v\rangle)dx = \chi_p(\langle \xi, a^v\rangle)F[\varphi](\xi).$$

Applying the above equation and Proposition 2, we can readily show that every function $\varphi \in \mathcal{D}_\gamma^l$ is represented in the form

$$\varphi(x) = \sum_{1\le v\le p^{n(\gamma-l)}} \widetilde{\varphi}(a^v)\chi_p(\langle x, a^v\rangle)\delta_{-\gamma}(x)$$

for some $a^v \in B_{-l}^n$, which does not depend on φ.

Let K be a compact subset of $\mathbb{Q}_p^n$, and $C(K)$ the space of continuous complex valued functions defined on K with the norm

$$\|f\|_{C(K)} = \max_{x \in K} |f(x)|.$$

Note that for any compact $K \subset \mathbb{Q}_p^n$ the measure dx defines a positive linear continuous functional on $C(K)$ by the formula $\int_K f(x)dx$, $f \in C(K)$.

Therefore we obtain the following:

LEMMA 6. *Let M be any countable everywhere dense set in $\mathbb{Q}_p^n$ and let*

$$\mathcal{T} = \left\{ \sum_{v-\mathit{finite}} c_v \chi_p(\langle \xi^v, x \rangle), \ \ \xi^v \in M, x \in \mathbb{Q}_p^n \right\}$$

be the set of trigonometrical polynomials. Then

1. *$\mathcal{T}$ is dense in $C(K)$.*
2. *$\mathcal{T}$ is dense in $L^2(K)$.*

PROOF. For the proof, we refer to [8] and [4]. □

LEMMA 7. [5] *Let $1 \le r < \infty$. Then*

$$\int_{B_\gamma^n} \|x\|^r dx = p^{\gamma(r+n)}(1 - p^{-n})(1 - p^{-r-n})^{-1}.$$

DEFINITION 8. The maximal operator $\mathcal{M}_p(f)$ on $\mathbb{Q}_p^n$ is defined by

$$\mathcal{M}_p(f)(x,z) = \sup_{\substack{x \in B_\gamma^n(a) \\ \|z\|^{-1} \le p^{n\gamma}}} \frac{1}{\mathrm{vol}(B_\gamma^n(a))} \int_{B_\gamma^n(a)} |f(y)| dy,$$

where $x, z \in \mathbb{Q}_p^n$ and $z \neq 0$ (see [4]).

Note that

$$\mathrm{vol}(B_\gamma^n(a)) = \int_{B_\gamma^n} dy = p^{n\gamma}.$$

THEOREM 9. *Let B_γ^n and $B_{\gamma'}^n$ be balls of radius $p^\gamma, p^{\gamma'}$ respectively with center at $0 \in \mathbb{Q}_p^n$, and let $1 \le r < \infty$. Then*

(1) The maximal operator $\mathcal{M}_p$ is bounded from $C(B_\gamma^n)$ into $L^r(B_\gamma^n \times B_{\gamma'}^n)$. That is,

$$\|\mathcal{M}_p f\|_{L^r(B_\gamma^n \times B_{\gamma'}^n)} \le p^{n(\gamma(1+\frac{1}{r})+\gamma'(\frac{1}{n}+\frac{1}{r}))} \left(\frac{1-p^{-n}}{1-p^{-r-n}} \right)^{\frac{1}{r}} \cdot \|f\|_{C(B_\gamma^n)}.$$

(2) *The function* $\mathcal{M}_p$ *is bounded from* $L^2(B^n_\gamma)\cap\mathcal{T}$ *into* $L^r(B^n_\gamma\times B^n_{\gamma'})$, *where* $1\le r<\infty$. *That is, there exists a constant* $C>0$ *such that*

$$\|\mathcal{M}_p f\|_{L^r(B^n_\gamma\times B^n_{\gamma'})}\le Cp^{n(\gamma(1+\frac{1}{r})+\gamma'(\frac{1}{n}+\frac{1}{r}))}\left(\frac{1-p^{-n}}{1-p^{-r-n}}\right)^{\frac{1}{r}}\cdot\|f\|_{L^2(B^n_\gamma)}$$

for $f\in L^2(B^n_\gamma)\cap\mathcal{T}$.

PROOF. Part 1 follows immediately from Lemma 7 and Fubini's theorem, using essentially the same proof as for [4, Theorem 2.3].

For part 2, let $f\in L^2(B^n_\gamma)\cap\mathcal{T}$ and M be any countable everywhere dense set in $\mathbb{Q}_p^n$. Since by Lemma 6,

$$f(x)=\sum_{v-\text{finite}}c_v\chi_p(\langle\xi^v,x\rangle),\quad x\in\mathbb{Q}_p^n,\ \xi^v\in M,$$

Part 2 is essentially the same as [4, Theorem 2.4] using the fact that there exists $C>0$ such that $\sum|c_v|\le C(\sum|c_v|^2)^{1/2}$ and $\sum|c_v|\le C\|f\|_{L^2(B^n_\gamma)}$. □

COROLLARY 10. *Let* B^n_γ, *etc., be as in Theorem 9. Then the function* $\mathcal{M}_p$ *is bounded from* $L^2(B^n_\gamma)$ *into* $L^r(B^n_\gamma\times B^n_{\gamma'})$.

PROOF. See [4, Corollary 2.5]. □

3. The weighted Hardy-Littlewood averages on the p-adic vector space

A measurable complex-valued function f on $\mathbb{Q}_p^n$ is said to be in $L^r(\mathbb{Q}_p^n)$ $(1\le r<\infty)$ provided

$$\|f\|_{L^r(\mathbb{Q}_p^n)}=\left(\int_{\mathbb{Q}_p^n}|f(x)|^r dx\right)^{\frac{1}{r}}<\infty.$$

For $r=\infty$, $L^\infty(\mathbb{Q}_p^n)$ is given by the set of all measurable real-valued function f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)}=\operatorname{ess\,sup}|f(x)|<\infty,$$

where ess sup denotes the essential supremum.

Now let $\mathcal{O}$ be a compact open subset of $\mathbb{Q}_p$, such as $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$.

DEFINITION 11. Let $\psi : \mathcal{O} \to [0,\infty)$ be a function. We define the weighted Hardy-Littlewood average $U_\psi f$ on $\mathbb{Q}_p^n$ by

$$(U_\psi f)(x) = \int_{\mathcal{O}} f(tx)\psi(t)dt.$$

This is a p-adic version of the weighted Hardy-Littlewood average in [9]. On the real field $\mathbb{R}$, if $\psi \equiv 1$ then U_ψ is just reduced to the classical Hardy-Littlewood average Uf as

$$Uf(x) = \frac{1}{x}\int_0^x f(y)dy, \quad x \neq 0.$$

Let $\delta_\gamma(|t|_p - p^\gamma)$ $(t \neq 0)$ be the characteristic function of the circle S_γ and let $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p)$. Then one finds that

$$\int_{\mathbb{Q}_p} f(y)dy = \int_{\bigcup_{\gamma\in\mathbb{Z}} S_\gamma} f(y)dy = \sum_{\gamma\in\mathbb{Z}} p^\gamma (U_{\delta_0} f)(p^{-\gamma})$$

and

$$\int_{\mathbb{Q}_p^n} f(y)\|y\|^{-n}dy = \sum_{\gamma\in\mathbb{Z}}\int_{S_0^n} f(p^{-\gamma}y)dy.$$

EXAMPLE 1. Let $f_s(x) = \|x\|^{s-1}$ for $\operatorname{Re} s > 0$ and $\psi(t) = \log|1/t|_p$ $(t \neq 0)$. Then

$$(U_\psi f_s)(x) = \|x\|^{s-1}\int_{\mathbb{Z}_p\setminus\{0\}} |t|_p^{s-1}\log|1/t|_p dt = \|x/p\|^{s-1}\frac{(p-1)\log p}{(p^s-1)^2}.$$

Here we have used the equality

$$\sum_{\gamma=0}^{\infty}\gamma p^{-\gamma s} = \frac{p^s}{(p^s-1)^2}.$$

For $s \neq s_k = \frac{2k\pi i}{\log p}$ with $k \in \mathbb{Z}$, the $U_\psi f_s$ is defined by means of analytical continuation.

DEFINITION 12. For $f \in L^1_{\mathrm{loc}}(B_\gamma^n)$, we denote by $f_{B_\gamma^n}$ the average of f over B_γ^n,

$$f_{B_\gamma^n} = \frac{1}{\mathrm{vol}(B_\gamma^n)}\int_{B_\gamma^n} f(x)dx = \frac{1}{p^{n\gamma}}\int_{B_\gamma^n} f(x)dx.$$

EXAMPLE 2. Let $f_s(x) = \|x\|^{s-n}$, where $s \in \mathbb{C}$ and $\operatorname{Re} s > 0$. Then

$$f_{B_\gamma^n} = p^{\gamma(s-n)}(1-p^{-n})(1-p^{-s})^{-1}.$$

DEFINITION 13. A locally integrable function $f(x)$ on $\mathbb{Q}_p^n$ has p-adic bounded mean oscillation, $f \in \mathrm{BMO}$ (see [4]) if one has

$$\|f\|_* = \sup_{B_\gamma^n \subset \mathbb{Q}_p^n} \frac{1}{\mathrm{vol}(B_\gamma^n)} \int_{B_\gamma^n} |f(x) - f_{B_\gamma^n}| dx < \infty,$$

where the supremum is taken over a ball B_γ^n of $\mathbb{Q}_p^n$.

This notion was introduced by John and Nirenberg on the Euclidean space (cf. [9]).

Clearly, the bound $\|f\|_*$ in Definition 13 is the BMO norm of f. Because the constant functions have BMO norm zero, we identity $f \in \mathrm{BMO}$ with $f+$constant, and we view BMO as subset of $L^1_{\mathrm{loc}}(K)/\{\text{constant}\}$, where K is a compact subset of $\mathbb{Q}_p^n$.

THEOREM 14. *Let* $\psi : \mathbb{Z}_p^\times \to [0,\infty)$ *be a function. Suppose that* $\int_{\mathbb{Z}_p^\times} \psi(t)dt < \infty$. *Then*

1. $U_\psi f$ *is bounded from* $L^r(\mathbb{Q}_p^n)$ *into* $L^r(\mathbb{Q}_p^n)$.
2. $U_\psi f$ *is bounded from* BMO *into* BMO.

PROOF. By Minkowski's inequality, one sees that

$$\|U_\psi f\|_{L^r(\mathbb{Q}_p^n)} \le \int_{\mathbb{Z}_p^\times} \|f(tx)\psi(t)\|_{L^r(\mathbb{Q}_p^n)} dt = \|f\|_{L^r(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\times} \psi(t)dt.$$

Now put $\int_{\mathbb{Z}_p^\times} \psi(t)dt < \infty$. Then we obtain

$$\|U_\psi f\|_{L^r(\mathbb{Q}_p^n)} \le \|f\|_{L^r(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^\times} \psi(t)dt < \infty.$$

Thus the first part follows.

To see the second part, suppose that $\int_{\mathbb{Z}_p^\times} \psi(t)dt < \infty$. If $f \in \mathrm{BMO}$, then for any ball $B_\gamma^n \subset \mathbb{Q}_p^n$ we use Fubini's theorem to establish

$$(U_\psi f)_{B_\gamma^n} = \int_{\mathbb{Z}_p^\times} \left(\frac{1}{\mathrm{vol}(B_\gamma^n)} \int_{B_\gamma^n} f(tx)dx \right) \psi(t)dt = \int_{\mathbb{Z}_p^\times} f_{B_\gamma^n} \psi(t)dt$$

and

$$\begin{aligned} \int_{B_\gamma^n} |(U_\psi f)(x) - (U_\psi f)_{B_\gamma^n}| dx &\le \int_{B_\gamma^n} \left(\int_{\mathbb{Z}_p^\times} |f(tx) - f_{B_\gamma^n}| \psi(t)dt \right) dx \\ &= \int_{\mathbb{Z}_p^\times} \left(\int_{B_\gamma^n} |f(x) - f_{B_\gamma^n}| dx \right) \psi(t)dt \\ &\le \mathrm{vol}(B_\gamma^n) \|f\|_* \int_{\mathbb{Z}_p^\times} \psi(t)dt \end{aligned}$$

which establishes the result of Part 2. □

EXAMPLE 3. Set $f(x) = \chi_p(x)$ $(x \in \mathbb{Q}_p)$ and $\psi(t) = \Omega(|t|_p)$ in Definition 11. Then the following relation holds ([8, p. 42, (3.1)])

$$\Omega(x) = \int_{\mathbb{Q}_p} \chi_p(xt)\psi(t)dt = \int_{\mathbb{Z}_p} \chi_p(xt)dt = (U_\Omega \chi_p)(x).$$

So that $(U_\Omega \chi_p)(\varphi(x)) = \Omega(\varphi(x))$. From Fubini's theorem by changing the order of integration,

$$\int_{B_\gamma} (U_\Omega \chi_p)(\varphi(x))dx = \int_{\mathbb{Z}_p} \int_{B_\gamma} \chi_p(\varphi(x)t)dxdt.$$

In [7], for $\epsilon, \epsilon' \in \mathbb{R}$, the analogy of the Euler gamma and beta functions may be defined by means of the integrals

$$\gamma_p(\epsilon) = \int_{\mathbb{Z}_p} |x|_p^{\epsilon-1}\chi_p(x)dx, \quad b_p(\epsilon, \epsilon') = \int_{\mathbb{Z}_p} |x|_p^{\epsilon-1}|1-x|_p^{\epsilon'-1}dx,$$

respectively. Denote by $\zeta_p(s)$ the local zeta function on $\mathbb{Q}_p$, that is,

$$\zeta_p(s) = \frac{1}{1-p^{-1}} \int_{\mathbb{Q}_p} \Omega(x)|x|_p^{s-1}dx$$

for $\operatorname{Re} s > 0$.

EXAMPLE 4. [8] For $\operatorname{Re} s > 0$,

$$\int_{S_0} |1-x|_p^{s-1}dx = \frac{1-p^{-1}}{1-p^{-s}} - \frac{1}{p}.$$

EXAMPLE 5. Let $\epsilon, \epsilon' > 0$. Then using Example 4,

$$\begin{aligned} b_p(\epsilon, \epsilon') &= \sum_{\gamma=1}^{\infty} p^{-\gamma(\epsilon-1)} \int_{S_{-\gamma}} |1-x|_p^{\epsilon'-1}dx + \int_{S_0} |1-x|_p^{\epsilon'-1}dx \\ &= (1-p^{-1})(\zeta_p(\epsilon) + \zeta_p(\epsilon')) - 1. \end{aligned}$$

In particular, we see that $\zeta_p(\epsilon) = (1-p^{-\epsilon})^{-1}$ and $\gamma_p(\epsilon) = (1 - p^{-1})\zeta_p(\epsilon)$.

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