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MODULI SPACES OF 3-DIMENSIONAL FLAT MANIFOLDS

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ABSTRACT. For 3-dimensional Bieberbach groups, we study the deformation spaces in the group of isometries of $\mathbb{R}^3$. First we calculate the discrete representation spaces and the automorphism groups. Then for each of these Bieberbach groups, we give complete descriptions of Teichmüller spaces, Chabauty spaces, and moduli spaces.

1. Introduction

Let M be a 3-dimensional manifold with an effective circle action with $M = \pi \setminus \widetilde{M}$, where π is a discrete subgroup of the group $\mathrm{Isom}(\widetilde{M})$ of isometries of $\widetilde{M}$. It is known that the evaluation map of the circle action at the base point, $i : S^1 \longrightarrow M$, induces an injective homomorphism

$$i_\sharp : \mathbb{Z} \longrightarrow \pi = \pi_1(M)$$

unless $\widetilde{M}$ is the three-sphere S^3. Furthermore, the Seifert structure lifts to an $\mathbb{R}$-action on the universal covering $\widetilde{M}$ of M.

Let π be a cocompact discrete subgroup of $\mathrm{Isom}(\widetilde{M})$ which acts on $\widetilde{M}$ properly discontinuously as above. The quotient space $M = \pi \setminus \widetilde{M}$ is said to have a *geometric structure* modelled on $(\widetilde{M}, \mathrm{Isom}(\widetilde{M}))$. We use the notation $\mathcal{I}$ for the group of isometries; that is,

$$\mathcal{I} = \mathrm{Isom}(\widetilde{M}).$$

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The space of discrete representations, *the Weil space*, is defined as follows:

$\mathcal{R}(\pi;\mathcal{I})$ = the space of all injective discrete homomorphisms θ of π into $\mathcal{I}$ such that $\theta(\pi)$ is discrete in $\mathcal{I}$ and $\mathcal{I}/\theta(\pi)$ is compact.

Every element of $\mathcal{R}(\pi;\mathcal{I})$ gives rise to an orbifold modelled on $(\widetilde{M}, \mathrm{Isom}(\widetilde{M}))$.

The group of automorphisms of π, $\mathrm{Aut}(\pi)$, acts on $\mathcal{R}(\pi;\mathcal{I})$ on the right; for $\theta \in \mathcal{R}(\pi;\mathcal{I})$ and $\varphi \in \mathrm{Aut}(\pi)$,

$$\begin{array}{ccc} \mathcal{R}(\pi;\mathcal{I}) \times \mathrm{Aut}(\pi) & \to & \mathcal{R}(\pi;\mathcal{I}). \\ (\theta,\varphi) & \longmapsto & \theta\circ\varphi \end{array}$$

On the other hand, the group $\mathrm{Inn}(\mathcal{I})$ of inner automorphisms of $\mathcal{I}$ acts on the space $\mathcal{R}(\pi;\mathcal{I})$ from the left by

$$\begin{array}{ccc} \mathrm{Inn}(\mathcal{I}) \times \mathcal{R}(\pi;\mathcal{I}) & \to & \mathcal{R}(\pi;\mathcal{I}), \\ (\mu(g),\theta) & \longmapsto & \mu(g)\circ\theta \end{array}$$

where $\mu(g)$ is the conjugation by $g \in \mathcal{I}$.

Definition 1.1. The deformation spaces of π are the orbit spaces defined as follows:

$$\begin{aligned} \mathcal{T}(\pi;\mathcal{I}) &= \mathrm{Inn}(\mathcal{I}) \setminus \mathcal{R}(\pi;\mathcal{I}) \\ \mathcal{S}(\pi;\mathcal{I}) &= \mathcal{R}(\pi;\mathcal{I}) / \mathrm{Aut}(\pi) \\ \mathcal{M}(\pi;\mathcal{I}) &= \mathrm{Inn}(\mathcal{I}) \setminus \mathcal{R}(\pi;\mathcal{I}) / \mathrm{Aut}(\pi). \end{aligned}$$

These are the *Teichmüller space*, *Chabauty space* (or *space of discrete subgroups*), and the *moduli space* of π, respectively.

The Chabauty space is the space of all distinct discrete subgroups of $\mathcal{I}$ isomorphic to π. If $\theta, \theta' \in \mathcal{R}(\pi;\mathcal{I})$ represent the same point in $\mathcal{T}(\pi;\mathcal{I})$, then $\theta' = \mu(g)\circ\theta$ for some $g \in \mathcal{I}$. This implies

$$g\circ\theta(\alpha) = \theta'(\alpha)\circ g$$

for all $\alpha \in \pi$. Then, g induces a map $\bar{g}$

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{g} & \widetilde{M} \\ \downarrow & & \downarrow \\ \theta(\pi)\setminus\widetilde{M} & \xrightarrow[\bar{g}]{} & \theta'(\pi)\setminus\widetilde{M} \end{array}$$

which is an isometry.

Two embeddings θ and θ' represent the same point in $\mathcal{M}(\pi;\mathcal{I})$ if and only if $\theta(\pi)\backslash\widetilde{M}$ and $\theta'(\pi)\backslash\widetilde{M}$ are isometric. Therefore, the moduli space $\mathcal{M}(\pi;\mathcal{I})$ of π is the space of isometry classes of the orbifolds $\{\ \theta(\pi)\backslash\widetilde{M} : \theta \in \mathcal{R}(\pi;\mathcal{I})\ \}$.

Let M be a closed oriented 3-manifold. Thurston's classification yields 8 geometries: $\mathbb{R}\times S^2$, $\mathbb{R}^3$, $\mathbb{R}\times\mathbb{H}^2$, S^3, Nil, $\widetilde{\mathrm{PSL}_2\mathbb{R}}$, Sol and $\mathbb{H}^3$. It is known that if a closed 3-dimension manifold M admits a geometric structure modelled on one of the eight geometries, then the geometry involved is unique. The classical closed 3-dimensional Seifert manifolds encompass the first 6 of the 8 geometries. In cases of

$$\widetilde{M} \quad = \quad \mathbb{R}\times S^2, \quad \mathrm{Nil}, \quad \mathbb{R}\times\mathbb{H}^2, \quad \text{and} \quad \widetilde{\mathrm{PSL}_2\mathbb{R}},$$

R. Kulkarni, K. B. Lee and F. Raymond computed the deformation spaces $\mathcal{R}(\pi;\mathcal{I})$, $\mathcal{T}(\pi;\mathcal{I})$, $\mathcal{S}(\pi;\mathcal{I})$ and $\mathcal{M}(\pi;\mathcal{I})$ in their paper [5]. And the Weil space$\mathcal{R}(\pi;\mathcal{I})$ and the Teichmüller space $\mathcal{T}(\pi;\mathcal{I})$ of the case of $\mathbb{R}^3$ are calculated in [3] and [4].

The aim of this work is to calculate the Chabauty spaces $\mathcal{S}(\pi;\mathcal{I})$ and the moduli spaces $\mathcal{M}(\pi;\mathcal{I})$ for 3-dimensional Bieberbach groups π with $(\mathbb{R}^3, \mathrm{Isom}(\mathbb{R}^3))$-geometry.

2. Preliminaries

A *rigid motion* is an ordered pair $(\mathbf{a}, A)$ with $\mathbf{a}\in\mathbb{R}^n$ and $A\in \mathrm{O}(n)$, which acts on $\mathbb{R}^n$ by

$$(\mathbf{a}, A)\cdot\mathbf{x} = A\mathbf{x} + \mathbf{a} \quad \text{ for } \mathbf{x}\in\mathbb{R}^n,$$

and these are the isometries of $\mathbb{R}^n$. For $n = 3$,

$$\mathcal{I} = \mathrm{Isom}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{O}(3).$$

The group $\mathrm{Isom}(\mathbb{R}^3)$ is a subgroup of the affine group

$$\mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R}).$$

A subgroup π of $\mathrm{Isom}(\mathbb{R}^3)$ is said to be a *Bieberbach* group if π is cocompact, discrete and torsion free. If π is a Bieberbach subgroup of $\mathrm{Isom}(\mathbb{R}^3)$, then the quotient space $\pi\setminus\mathbb{R}^3$ is a Riemannian manifold of sectional curvature $\kappa = 0$. Conversely, a flat closed Riemannian manifold of dimension 3 can be expressed a quotient space of $\mathbb{R}^3$ by a Bieberbach subgroup of $\mathrm{Isom}(\mathbb{R}^3)$. See e.g., [7; Chapter 3].

A Bieberbach group π contains a unique maximal normal abelian subgroup $\mathbb{Z}^3$, fitting the following commutative diagram of groups with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^3 \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "\theta"'] & \Phi \arrow[r] \arrow[d] & 1 \\ 0 \arrow[r] & \mathbb{R}^3 \arrow[r] & \mathbb{R}^3 \rtimes O(3) \arrow[r] & O(3) \arrow[r] & 1 \end{tikzcd} \tag{2.1}$$

where Φ is called the *holonomy group* of π. It is finite and $\Phi \to O(3)$ is injective. The Bieberbach's second theorem says that any isomorphism between Bieberbach groups on $\mathbb{R}^n$ is conjugation by an element of the affine group $\mathrm{Aff}(n)$. See [2] or [7].

There are only 10 Bieberbach groups in dimension 3 up to affine change of coordinates. Out of them six are orientable and the others are non-orientable. See [6] or [7]. Let I be the 3×3 identity matrix, $\{\mathbf{e}_i\}$ the standard basis in $\mathbb{R}^3$ and R_k the rotation matrix of rotation of $\mathbb{R}^3$ about the x-axis through $\frac{2\pi}{k}$; namely,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$R_k = R\Big(\frac{2\pi}{k}\Big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{2\pi}{k}) & -\sin(\frac{2\pi}{k}) \\ 0 & \sin(\frac{2\pi}{k}) & \cos(\frac{2\pi}{k}) \end{bmatrix}$$

and let $\mathbf{t}_i = (\mathbf{e}_i, I)$, for $i = 1, 2, 3$.

LEMMA 2.1. *We list all the 3-dimensional Bieberbach groups embedded in $\mathbb{R}^3 \rtimes O(3)$ and their holonomy groups* ([3] *and* [4]). *Let*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

1) $\mathfrak{G}_1$. $\Phi = \{1\}$ *and* π *is generated by* $\mathbf{t}_1, \mathbf{t}_2$ *and* $\mathbf{t}_3$,
2) $\mathfrak{G}_2$. $\Phi = \mathbb{Z}_2$ *and* π *is generated by* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ *and* $\alpha = (\frac{1}{2}\mathbf{e}_1, R_2)$,
3) $\mathfrak{G}_3$.$\Psi = \mathbb{Z}_3$ *and* π *is generated by* $\mathbf{t}_1, \mathbf{s}_1 = (R_3\mathbf{e}_2, I), \mathbf{s}_2 = ((R_3)^2\mathbf{e}_2, \mathbf{I})$ *and* $\beta = (\frac{1}{3}\mathbf{e}_1, R_3)$,
4) $\mathfrak{G}_4$. $\Phi = \mathbb{Z}_4$ *and* π *is generated by* $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ *and* $\alpha = (\frac{1}{4}\mathbf{e}_1, R_4)$,
5) $\mathfrak{G}_5$. $\Phi = \mathbb{Z}_6$ *and* π *is generated by* $\mathbf{t}_1, \mathbf{s}_1 = (R_6\mathbf{e}_2, I), \mathbf{s}_2 = ((R_6)^2\mathbf{e}_2, \mathbf{I})$ *and* $\beta = (\frac{1}{6}\mathbf{e}_1, R_6)$,
6) $\mathfrak{G}_6$. $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* π *is generated by* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\frac{1}{2}\mathbf{e}_1, R_2)$, $\beta = (\frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3), -ER_2)$,

7) $\mathfrak{B}_1$. *$\Phi = \mathbb{Z}_2$ and π is generated by* $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ *and* $\varepsilon = (\frac{1}{2}\mathbf{e}_1, E)$,
8) $\mathfrak{B}_2$. *$\Phi = \mathbb{Z}_2$ and π is generated by* $\mathbf{t}_1, \mathbf{t}_2, \mathbf{s} = (\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) + \mathbf{e}_3, I)$ *and* $\varepsilon = (\frac{1}{2}\mathbf{e}_1, E)$,
9) $\mathfrak{B}_3$. *$\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ and π is generated by* $\mathbf{t}_1, \mathbf{t}_2$, $\mathbf{t}_3$, $\alpha = (\frac{1}{2}\mathbf{e}_1, R_2)$ *and* $\varepsilon = (\frac{1}{2}\mathbf{e}_2, E)$,
10) $\mathfrak{B}_4$. *$\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ and π is generated by* $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, $\alpha = (\frac{1}{2}\mathbf{e}_1, R_2)$ *and* $\varepsilon = (\frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3), E)$.

In the above list, the matrices which span the holonomy groups are integral except two cases $\mathfrak{G}_3$, $\mathfrak{G}_5$. We want to conjugate π's of type $\mathfrak{G}_3$ and $\mathfrak{G}_5$ into $\mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$.

LEMMA 2.2. *The Bieberbach group π of type $\mathfrak{G}_3$ or $\mathfrak{G}_5$ is conjugates to a subgroup of the form $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha\rangle$ of $\mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$, where if π is of type $\mathfrak{G}_3$ then $\alpha = (\frac{1}{3}\mathbf{e}_1, A_3)$, and if $\mathfrak{G}_5$ then $\alpha = (\frac{1}{6}\mathbf{e}_1, A_5)$,*

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \textit{and} \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Proof. For $(\mathbf{0}, P) \in \mathrm{Aff}(3)$ let $\mu_{(\mathbf{0},P)}$ be the conjugation by $(\mathbf{0}, P)$; so,

$$\mu_{(\mathbf{0},P)}(\mathbf{a}, A) = (\mathbf{0}, P)(\mathbf{a}, A)(\mathbf{0}, P)^{-1}$$

for $(\mathbf{a}, A) \in \mathrm{Aff}(3)$. If we take the following P_3 and P_5

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & \frac{1}{\sqrt{3}} \\ 0 & -1 & -\frac{1}{\sqrt{3}} \end{bmatrix} \qquad P_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{3}} \\ 0 & -1 & \frac{1}{\sqrt{3}} \end{bmatrix},$$

it is easy to check that

$$\mu_{(\mathbf{0},P_3)} : \mathfrak{G}_3 \longrightarrow \langle \mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{t}_3,\ \alpha = (\frac{1}{3}\mathbf{e}_1, A_3)\rangle$$
$$\mu_{(\mathbf{0},P_5)} : \mathfrak{G}_5 \longrightarrow \langle \mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{t}_3,\ \alpha = (\frac{1}{6}\mathbf{e}_1, A_5)\rangle.$$

The proof is complete. □

Let $\mathcal{N}(\pi)$ denote the normalizer of π in $\mathrm{Aff}(3)$. If $\xi \in \mathrm{Aff}(3)$, then

$$\xi \mathcal{N}(\pi)\xi^{-1} = \mathcal{N}(\xi\pi\xi^{-1}).$$

And so, in order to compute the normalizer $\mathcal{N}(\pi)$ of a Bieberbach group π with type $\mathfrak{G}_3$ or $\mathfrak{G}_5$, we shall use the integral representation as in Lemma 2.2.

3. Automorphisms of a 3-dimensional Bieberbach group

Let $\mathcal{N}(\pi) = \mathcal{N}_{\mathrm{Aff}(3)}(\pi)$ and $\mathcal{C}(\pi) = \mathcal{C}_{\mathrm{Aff}(3)}(\pi)$ be the normalizer and the centralizer of a Bieberbach group π in the affine group $\mathrm{Aff}(3)$. The following is a commutative diagram in which all rows and columns are exact [2] :

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{Z}(\pi) \arrow[r] \arrow[d] & C(\pi) \arrow[r] \arrow[d] & \mathrm{Aff}_0(M) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi \arrow[r] \arrow[d] & N(\pi) \arrow[r] \arrow[d] & \mathrm{Aff}(M) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\pi) \arrow[r] \arrow[d] & \mathrm{Aut}(\pi) \arrow[r] \arrow[d] & \mathrm{Out}(\pi) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{3.1}$$

where $\mathcal{Z}(\pi)$ is the center of π and $C(\pi)$ is the centralizer of π in $\mathrm{Aff}(3)$. The top row is always of the form

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{R}^k \longrightarrow T^k \longrightarrow 1,$$

where k is the rank of the center of π. (So, $k = 3$ for $\mathfrak{G}_1$; $k = 2$ for $\mathfrak{B}_1$, $\mathfrak{B}_2$; $k = 1$ for $\mathfrak{G}_2$, $\mathfrak{G}_3$, $\mathfrak{G}_4$, $\mathfrak{G}_5$, $\mathfrak{B}_3$, $\mathfrak{B}_4$; and $k = 0$ for $\mathfrak{G}_6$).

In order to calculate $\mathrm{Aut}(\pi)$, it is enough to compute $\mathcal{N}(\pi)$ and $\mathcal{C}(\pi)$ because

$$\mathrm{Aut}(\pi) = \mathcal{N}(\pi)/\mathcal{C}(\pi).$$

We have the exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \pi \xrightarrow{\ p\ } \Phi \longrightarrow 1.$$

Let $j : \Phi \longrightarrow \mathrm{Aut}(\mathbb{Z}^3)$ be the map induced by the action of Φ on $\mathbb{Z}^3$ which assigns a conjugation by an element of π;

$$[j(A)](\mathbf{x}) = \sigma(\mathbf{x},\ I)\sigma^{-1} \quad \text{for } \mathbf{x} \in \mathbb{Z}^3 \text{ and } \sigma \in p^{-1}(A).$$

For our rigid motions, Φ-action on $\mathbb{Z}^3$ is given by multiplications of matrices on 3-vectors of $\mathbb{Z}^3$ on the left. And it can be lifted to an action on $\mathbb{R}^3$. The centralizer $\mathcal{C}(\pi)$ of π in $\mathrm{Aff}(3)$ is a subgroup of pure translations, $(\mathbb{R}^3)^\Phi$, the fixed point set of the Φ-action on $\mathbb{R}^3$. Let $\mathcal{N}(\Phi)$

be the normalizer of $j(\Phi)$ in $\mathrm{Aut}(\mathbb{Z}^3) = \mathrm{GL}(3,\mathbb{Z})$. We need to figure out the normalizer $\mathcal{N}(\pi)$ of π in $\mathrm{Aff}(3)$. In the following theorem,

$$\mathcal{N}^+(\Phi) = \{X \in \mathcal{N}(\Phi) \mid \det X > 0\}.$$

THEOREM 3.1. *Let M be a 3-dimensional orientable flat manifold with $\pi = \pi_1(M)$. Then the normalizer $\mathcal{N}(\pi)$ in $\mathrm{Aff}(3)$ of π is a semi-direct product*

$$T \rtimes \mathcal{M} \subset \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{Z}),$$

where the pure translations T and the matrix group $\mathcal{M}$ are as follows:

1. *If $\pi = \mathfrak{G}_1$, then $T = \mathbb{R}^3$ and $\mathcal{M} = \mathcal{N}(\Phi) = \mathrm{GL}(3,\mathbb{Z})$.*
2. *If $\pi = \mathfrak{G}_2$, then $T = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3)$, $\mathcal{M} = \mathcal{N}(\Phi) = \mathbb{Z}_2 \times \mathrm{GL}(2,\mathbb{Z})$, where $\mathcal{M} = \mathcal{N}(\Phi)$ is the matrices of the form*

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

3. *If $\pi = \mathfrak{G}_3$, then $T = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\frac{1}{3}\mathbf{e}_2 + \frac{2}{3}\mathbf{e}_3) \oplus \mathbb{Z}(\frac{2}{3}\mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3)$, $\mathcal{M} = \mathcal{N}^+(\Phi) = \mathbb{Z}_6 \rtimes \mathbb{Z}_2 = \boldsymbol{D}_6$, where the dihedral group $\boldsymbol{D}_6$ is generated by the matrices*

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

4. *If $\pi = \mathfrak{G}_4$, then $T = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2 + \frac{1}{2}\mathbf{e}_3)$, $\mathcal{M} = \mathcal{N}^+(\Phi) = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \boldsymbol{D}_4$, where the dihedral group $\boldsymbol{D}_4$ is generated by the matrices*

$$D = R_4 \text{ and } J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

5. *If $\pi = \mathfrak{G}_5$, then $T = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\mathbf{e}_2) \oplus \mathbb{Z}(\mathbf{e}_3)$, $\mathcal{M} = \mathcal{N}^+(\Phi) = \mathbb{Z}_6 \rtimes \mathbb{Z}_2 = \boldsymbol{D}_6$, where the dihedral group $\boldsymbol{D}_6$ is generated by the matrices*

$$D = A_5 \text{ and } J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

6. *If* $\pi = \mathfrak{G}_6$, *then* $T = \mathbb{Z}(\frac{1}{2}\mathbf{e}_1) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3)$, $\mathcal{M} = \mathcal{N}(\Phi) = (\mathbb{Z}_2)^3 \rtimes \mathcal{S}_3$, *where*
$$(\mathbb{Z}_2)^3 = \begin{bmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}$$
and $\mathcal{S}_3$ *is the permutation group of 3 letters.*

Proof. In order for an element $(\mathbf{x}, X) \in \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R}) = \mathrm{Aff}(3)$ to lie in the normalizer $\mathcal{N}(\pi)$, it should conjugate the pure translational subgroup $\mathbb{Z}^3$ to itself. This implies $X \in \mathrm{GL}(3,\mathbb{Z})$. So, we first find all such X which normalizes the holonomy group $\Phi \subset \mathrm{GL}(3,\mathbb{Z})$. For each of such matrices X, it is not hard to see $(\mathbf{0}, X)$ normalizes the whole group (with $\mathbf{x} = \mathbf{0}$). This means that the group $\mathcal{N}(\pi)$ splits as a semi-direct product $T \rtimes \mathcal{M}$. To find the pure translations T, one just solves the equation
$$(\mathbf{x}, I)(\mathbf{a}, A)(-\mathbf{x}, I) \in \pi$$
for $\mathbf{x}$ and $(\mathbf{a}, A) \in \pi$, for every non-trivial generator A of the holonomy group.

If π is of type $\mathfrak{G}_n$ $(n = 1, 2, 6)$, then the matrix part $\mathcal{M}$ is equal to the normalizer $\mathcal{N}(\Phi)$ of the holonomy group Φ in $\mathrm{GL}(3,\mathbb{Z})$. For the cases of $\mathfrak{G}_n$ $(n = 3, 4, 5)$, $X \in \mathcal{N}(\Phi)$ if and only if, for a generator A of Φ, $XAX^{-1} = A^r$ for r coprime to the order of A. Thus r can only be 1 or -1. The centralizer
$$\mathcal{C}(\Phi) = \{X \in \mathrm{GL}(3,\mathbb{R}) : \; XAX^{-1} = A\}.$$
of Φ in $\mathrm{GL}(3,\mathbb{Z})$ is isomorphic to $\langle D \rangle \times \mathbb{Z}_2$, where $-I \in \mathbb{Z}_2$ and $\langle D \rangle$ is a finite cyclic group of order 6 ($\mathfrak{G}_3$, $\mathfrak{G}_5$) or 4 ($\mathfrak{G}_4$), and $\mathcal{C}(\Phi)$ has index 2 in the normalizer $\mathcal{N}(\Phi)$. More precisely, there is $J \in \mathrm{GL}(3,\mathbb{R})$ such that $JAJ^{-1} = A^{-1}$ with $\det(J) = 1$. Consequently,
$$\mathcal{N}(\Phi) = (\langle \pm I \rangle \times \mathbb{Z}_m) \rtimes \mathbb{Z}_2 = \langle \pm I \rangle \times (\mathbb{Z}_m \rtimes \mathbb{Z}_2) = \langle \pm I \rangle \times \mathcal{N}^+(\Phi),$$
where the $\mathbb{Z}_2$ is generated by J.

Now consider $(\mathbf{x}, -I)$. For the cases of $\mathfrak{G}_3$, $\mathfrak{G}_4$ and $\mathfrak{G}_5$,
$$(\mathbf{x}, -I)(\mathbf{a}, A)(\mathbf{x}, -I)^{-1}(\mathbf{a}, A)^{-1} \in \mathbb{Z}^3$$
does not have a solution for $\mathbf{x} \in \mathbb{R}^3$. Therefore, we only need to look at the subgroup $\mathcal{N}^+(\Phi) = \mathbb{Z}_m \rtimes \mathbb{Z}_2 = \mathbf{D}_m$. $\square$

For non-orientable Bieberbach groups, we need a special subgroup of $\mathrm{GL}(2,\mathbb{Z})$. Consider the natural homomorphism
$$\rho : \; \mathrm{GL}(2,\mathbb{Z}) \longrightarrow \mathrm{GL}(2,\mathbb{Z}_2)$$

induced from the natural homomorphism $\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. It is easy to see that

$$\operatorname{kernel}(\rho) = \widehat{\mathrm{GL}}(2,\mathbb{Z}) = \left\{ \begin{bmatrix} 2a+1 & 2b \\ 2c & 2d+1 \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid a,b,c,d \in \mathbb{Z} \right\}$$

and

$$\operatorname{image}(\rho) = \mathrm{GL}(2,\mathbb{Z}_2) \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2,$$

which is generated by

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore,

LEMMA 3.2. *We have an exact sequence*

$$1 \longrightarrow \widehat{\mathrm{GL}}(2,\mathbb{Z}) \longrightarrow \mathrm{GL}(2,\mathbb{Z}) \xrightarrow{\rho} \mathbf{D}_3 \longrightarrow 1,$$

where $\mathbf{D}_3$ is the dihedral group.

Then the normalizer $\mathcal{N}(\pi)$ in $\mathrm{Aff}(3)$ of π is as follows.

LEMMA 3.3. *Let M be a non-orientable flat manifold with $\pi_1(M) = \pi$. Let*

$$\overline{\mathrm{GL}}(2,\mathbb{Z}) = \left\{ \begin{bmatrix} A & 0 \\ 0 & \pm 1 \end{bmatrix} : A \in \widehat{\mathrm{GL}}(2,\mathbb{Z}) \right\}.$$

1. *If $\pi = \mathfrak{B}_1$, then $\mathcal{N}(\pi) = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3) \rtimes \overline{\mathrm{GL}}(2,\mathbb{Z})$.*
2. *If $\pi = \mathfrak{B}_2$, then $\mathcal{N}(\pi) = \Big((\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{Z}\mathbf{e}_3) \rtimes \overline{\mathrm{GL}}(2,\mathbb{Z})\Big) \rtimes \mathbb{Z}_2$,*

   where $\mathbb{Z}_2$ is generated by $\xi = \left(\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right)$.
3. *If $\pi = \mathfrak{B}_3$ or $\mathfrak{B}_4$, then $\mathcal{N}(\pi) = (\mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3)) \rtimes (\mathbb{Z}_2)^3$,*

   where $(\mathbb{Z}_2)^3 = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$.

Proof. When π is of type $\mathfrak{B}_1$, similarly to the proof of Theorem 3.1, if $(\mathbf{x}, X) \in \mathcal{N}(\pi)$ for some $\mathbf{x} \in \mathbb{R}^3$, then $(\mathbf{0}, X)$ normalizes the whole group π. That is,

$$(\mathbf{0}, X)(\tfrac{1}{2}\mathbf{e}_1, E)(\mathbf{0}, X)^{-1} \in \pi.$$

Hence we obtain an exact sequence

$$0 \longrightarrow \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{Z}(\tfrac{1}{2}\mathbf{e}_3) \longrightarrow \mathcal{N}(\pi) \longrightarrow \overline{\mathrm{GL}}(2,\mathbb{Z}) \longrightarrow 1$$

which splits.

Now if π is of type $\mathfrak{B}_2$, each element of $(\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{Z}\mathbf{e}_3) \rtimes \overline{\mathrm{GL}}(2,\mathbb{Z})$ conjugates the whole group π and furthermore, $\xi \in \mathrm{Isom}(\mathbb{R}^3)$ does conjugate π into itself. So, the result (2) holds. Notice that ξ has a non-trivial translation part.

For π of type $\mathfrak{B}_3$ or $\mathfrak{B}_4$, let $(\mathbf{x}, X) \in \mathcal{N}(\pi)$. By the similar computation to Theorem 3.1, we see $X \in \mathcal{C}(\pi)$, and so X has to be diagonal. We get $\mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3)$ as the translation part of $\mathcal{N}(\pi)$. □

It is worth emphasizing that the translation part of the normalizer $\mathcal{N}(\pi)$ can be expressed by the direct sum of the centralizer $\mathcal{C}(\pi) = \mathbb{R}^k$ and $\mathbb{Z}^{3-k}$ for the rank k of the center of π.

If the holonomy group Φ of π has order greater than two, then the normalizer of Φ in $\mathrm{GL}(3,\mathbb{R})$ is finite and $\mathrm{Aut}(\pi)$ itself becomes a crystallographic group. There are only 17 crystallographic groups in dimension 2 and 219 in dimensional 3.

THEOREM 3.4. *Let π be a 3-dimensional Bieberbach group. If the order of the holonomy group Φ of π is greater than two, then the automorphism group* $\mathrm{Aut}(\pi)$ *of π is a crystallographic group of dimension 2 or 3:*

1. *If Φ is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_6$, then* $\mathrm{Aut}(\pi) \cong \mathbb{Z}^2 \rtimes \mathbf{D}_6$.
2. *If Φ is isomorphic to $\mathbb{Z}_4$, then* $\mathrm{Aut}(\pi) \cong \mathbb{Z}^2 \rtimes \boldsymbol{D}_4$.
3. *If $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{SO}(3)$, then* $\mathrm{Aut}(\pi) \cong \mathbb{Z}^3 \rtimes ((\mathbb{Z}_2)^3 \rtimes \mathcal{S}_3)$.
4. *If $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \not\subset \mathrm{SO}(3)$, then* $\mathrm{Aut}(\pi) \cong \mathbb{Z}^2 \rtimes (\mathbb{Z}_2)^3$.

Proof. Recall that

$$\mathrm{Aut}(\pi) = \mathcal{N}(\pi)/\mathcal{C}(\pi),$$

where a list of $\mathcal{C}(\pi) = (\mathbb{R}^3)^\Phi$'s was made.

(1) From Theorem 3.1, the exact sequence

$$0 \longrightarrow \mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\tfrac{1}{3}\mathbf{e}_2 + \tfrac{2}{3}\mathbf{e}_3) \oplus \mathbb{Z}(\tfrac{2}{3}\mathbf{e}_2 + \tfrac{1}{3}\mathbf{e}_3) \longrightarrow \mathcal{N}(\mathfrak{G}_3) \longrightarrow \mathbf{D}_6 \longrightarrow 1$$

splits. The centralizer $\mathcal{C}(\pi) = \mathbb{R}\mathbf{e}_1$ consists of the first axis (See 4.2). Therefore we have the exact sequence

$$0 \longrightarrow (\mathbb{Z})^2 \longrightarrow \mathrm{Aut}(\mathfrak{G}_3) \longrightarrow \mathbf{D}_6 \longrightarrow 1,$$

where $\mathbb{Z}^2$ which is generated by $\{\frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix}, \quad \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}\}$ and the dihedral group $\mathbf{D}_6$ of degree 12 is generated by D and J. Here if Φ is isomorphic to $\mathbb{Z}_3$ (or $\mathbb{Z}_6$), then

$$D = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix} \quad (\text{or } D = \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}) \quad \text{and } J = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Note that D has order 6, $JDJ^{-1} = D^{-1}$ and $\mathrm{Aut}(\pi)$ is a 2-dimensional crystallographic group.

(2) The automorphism group $\mathrm{Aut}(\mathfrak{G}_4)$ is a 2-dimensional crystallographic group

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathrm{Aut}(\mathfrak{G}_4) \longrightarrow \mathbf{D}_4 \longrightarrow 1,$$

where $\mathbb{Z}^2$ is generated by $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right\}$, and the quotient $\mathbf{D}_4$ is the dihedral group of order 8 which is generated by $D = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \quad J = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$. Note that D has order 4, and $JDJ^{-1} = D^{-1}$.

(3) Since the centralizer $\mathcal{C}(\pi)$ is trivial, $\mathcal{N}(\pi) \cong \mathrm{Aut}(\pi)$.

(4) The centralizer $\mathcal{C}(\pi) = \mathbb{R}\mathbf{e}_1$ consists of the first axis and the translation part of the normalizer $\mathcal{N}(\pi)$ is equal to $\mathbb{R}\mathbf{e}_1 \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_2) \oplus \mathbb{Z}(\frac{1}{2}\mathbf{e}_3)$. □

If the holonomy group Φ of π is isomorphic to a cyclic group $\mathbb{Z}_2$ with order two, then $\mathrm{Aut}(\pi)$ is as follows:

THEOREM 3.5. *Let π be a 3-dimensional Bieberbach group. The automorphism groups* $\mathrm{Aut}(\pi)$*'s are as follow;*

1. *If π is of type $\mathfrak{G}_1$,* $\mathrm{Aut}(\pi) \cong \mathrm{GL}(3,\mathbb{Z})$.
2. *If π is of type $\mathfrak{G}_2$,* $\mathrm{Aut}(\pi) \cong \mathbb{Z}^2 \rtimes (\mathrm{GL}(2,\mathbb{Z}) \times \mathbb{Z}_2)$.
3. *If π is of type $\mathfrak{B}_1$,* $\mathrm{Aut}(\pi) \cong \mathbb{Z} \rtimes \overline{\mathrm{GL}}(2,\mathbb{Z})$.
4. *If π is of type $\mathfrak{B}_2$,* $\mathrm{Aut}(\pi) \cong (\mathbb{Z} \rtimes \overline{\mathrm{GL}}(2,\mathbb{Z})) \times \mathbb{Z}_2$.

Proof. (1) If Φ is trivial, then $\mathcal{C}(\pi) = (\mathbb{R}^3)^{\Phi} = \mathbb{R}^3$.

(2) Since $\Phi \cong \mathbb{Z}_2 \subset \mathrm{SO}(3)$ and $\mathcal{C}(\pi) = \mathbb{R}\mathbf{e}_1$ consists of the first axis,

$$\begin{aligned}\mathrm{Aut}(\pi) =& \left\langle \tfrac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}0\\1\end{bmatrix} \right\rangle \rtimes (\mathbb{Z}_2 \times \mathrm{GL}(2,\mathbb{Z}))\\ \cong& \mathbb{Z}^2 \rtimes (\mathrm{GL}(2,\mathbb{Z}) \times \mathbb{Z}_2).\end{aligned}$$

(3), (4) If $\Phi \cong \mathbb{Z}_2 \not\subseteq \mathrm{SO}(3)$, then the centralizer is $\mathcal{C}(\pi) = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2$. If $\pi = \mathfrak{B}_1$,

$$\begin{aligned}\mathrm{Aut}(\pi) =& \left\langle \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right\rangle \rtimes \overline{\mathrm{GL}}(2, \mathbb{Z}) \\ \cong& \mathbb{Z} \rtimes \overline{\mathrm{GL}}(2, \mathbb{Z}).\end{aligned}$$

If $\pi = \mathfrak{B}_2$,

$$\begin{aligned}\mathrm{Aut}(\pi) =& (\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \rtimes \overline{\mathrm{GL}}(2, \mathbb{Z})) \rtimes \mathbb{Z}_2 \\ \cong& (\mathbb{Z} \rtimes \overline{\mathrm{GL}}(2, \mathbb{Z})) \times \mathbb{Z}_2,\end{aligned}$$

where $\mathbb{Z}_2$ is generated by $\left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \in \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})$. $\square$

REMARK 3.6. Let M be a 3-dimensional orientable flat manifold with $\pi = \pi_1(M)$. From 3.1 we are able to compute the group $\mathrm{Aff}(M)$ of affinities of a 3-dimensional flat manifold M;

$$1 \longrightarrow T^k \longrightarrow \mathrm{Aff}(M) \longrightarrow \mathcal{M}/\Phi \longrightarrow 1,$$

where k is the rank of the center of π.

4. Chabauty spaces and moduli spaces

For a 3-dimensional Bieberbach group π with holonomy group Φ, let θ_0 be the embedding of π into $\mathbb{R}^3 \rtimes \mathrm{O}(3)$ as in Lemma 2.1. In fact, we think of $\theta_0(\pi)$ as π. Let

$$\mathcal{X}(\Phi) = \{X \in \mathrm{GL}(3, \mathbb{R}) |\ XAX^{-1} \in \mathrm{O}(3) \text{ for all } A \in \Phi\}.$$

It is not a subgroup in general, but is a nice algebraic sub-variety of $\mathrm{GL}(3, \mathbb{R})$.

NOTATION 4.1. Consider the set ([3] and [4])

$$\mathbb{R}^3 \rtimes \mathcal{X}(\Phi) = \{(\mathbf{r}, R) |\ \mathbf{r} \in \mathbb{R}^3 \text{ and } R \in \mathcal{X}(\Phi)\} \subset \mathbb{R}^3 \rtimes \mathrm{O}(3).$$

This is not a subgroup, but a topological subspace.

It is shown that the Weil space is

$$\begin{aligned}\mathcal{R}(\pi; \mathcal{I}) &= \{\mu_\xi \circ \theta_0 |\ \xi \in \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)\} \\ &\approx (\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathcal{C}(\pi),\end{aligned}$$

where μ_ξ is the conjugation by ξ and $\mathcal{C}(\pi)$ is the centralizer of π in $\mathrm{Aff}(3)$. The action of the automorphism group $\mathrm{Aut}(\pi) = \mathcal{N}(\pi)/\mathcal{C}(\pi)$

on $\mathcal{R}(\pi;\mathcal{I})$ is inherited from the action of the normalizer $\mathcal{N}(\pi)$, which is given by the rule:

$$\begin{array}{ccccc} \pi & \xrightarrow{\varphi} & \pi & & \\ \big\downarrow{\scriptstyle \theta_0} & & \big\downarrow{\scriptstyle \theta_0} & & \\ \theta_0(\pi) & \xrightarrow[\mu_\eta]{} & \theta_0(\pi) & \xrightarrow[\mu_\xi]{} & \theta(\pi) \end{array}$$

for each $[\xi = (\mathbf{r}, R)] \in \mathcal{R}(\pi;\mathcal{I})$ and $\eta = (\mathbf{x}, X) \in \mathcal{N}(\pi)$. Recall that μ denotes conjugation.

For the readers' conveniences, we write $\mathcal{X}(\Phi)$ for each of the Bieberbach groups. See [3; Proposition 2.3] and [4; Theorem 3.1].

PROPOSITION 4.2. *For each of the 3-dimensional Bieberbach groups* π,

1. *For* $\pi = \mathfrak{G}_1$, $\mathcal{X}(\Phi) = \mathrm{GL}(3,\mathbb{R})$ *and* $(\mathbb{R}^3)^\Phi = \mathbb{R}^3$.
2. *For* $\pi = \mathfrak{G}_2$, $\mathcal{X}(\Phi) = \mathrm{SO}(3) \cdot \Big(\mathrm{GL}(1,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})\Big)$ *and* $(\mathbb{R}^3)^\Phi = \mathbb{R}\mathbf{e}_1$.
3. *For* $\pi = \mathfrak{G}_3$, $\mathfrak{G}_4$, *and* $\mathfrak{G}_5$,
$$\mathcal{X}(\Phi) = \mathrm{SO}(3) \cdot \Big(\mathbb{R}^* \cdot \mathrm{SO}(1) \times \mathbb{R}^* \cdot \mathrm{SO}(2)\Big) \quad \textit{and} \quad (\mathbb{R}^3)^\Phi = \mathbb{R}\mathbf{e}_1.$$
4. *For* $\pi = \mathfrak{G}_6$, $\mathcal{X}(\Phi) = \mathrm{SO}(3) \cdot (\mathbb{R}^*)^3$ *and* $(\mathbb{R}^3)^\Phi = \{0\}$.
5. *For* $\pi = \mathfrak{B}_1$ *and* $\mathfrak{B}_2$,
$$\mathcal{X}(\Phi) = \mathrm{SO}(3) \cdot \Big(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})\Big) \quad \textit{and} \quad (\mathbb{R}^3)^\Phi = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2.$$
6. *For* $\pi = \mathfrak{B}_3$ *and* $\mathfrak{B}_4$, $\mathcal{X}(\Phi) = \mathrm{SO}(3) \cdot (\mathbb{R}^*)^3$ *and* $(\mathbb{R}^3)^\Phi = \mathbb{R}\mathbf{e}_1$.

Here $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})$ *are the blocked diagonal matrices,* $\mathbb{R}^*$ *is the set of all non-zero real numbers and* $(\mathbb{R}^+)^3$ *are the group of 3-dimensional diagonal matrices with positive entries.*

Note that in (3) of the above theorem, $\mathcal{X}(\Phi)$ is computed using the orthogonal representation as in Lemma 2.1.

Now we can get the Chabauty space as follows:

$$\begin{aligned} \mathcal{S}(\pi;\mathcal{I}) &= \mathcal{R}(\pi;\mathcal{I}) / \operatorname{Aut}(\pi) \\ &= (\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)) / \mathcal{N}(\pi). \end{aligned}$$

THEOREM 4.3. *Let* M *be a 3-dimensional flat manifold with* $\pi_1(M) = \pi$. *The Chabauty spaces are as follow:*

1. *If π is of type $\mathfrak{G}_1$, $\mathcal{S}(\pi;\mathcal{I}) \approx \mathrm{GL}(3,\mathbb{R})/\mathrm{GL}(3,\mathbb{Z})$. This is 9-dimensional.*
2. *If π is of type $\mathfrak{G}_2$, $\mathcal{S}(\pi;\mathcal{I}) \approx \mathbf{T}^2 \times \mathrm{SO}(3)\cdot\Big(\mathbb{R}^+ \times \mathrm{GL}(2,\mathbb{R})/\mathrm{GL}(2,\mathbb{Z})\Big)$.*
   Since $\mathrm{SO}(3)\cap\mathrm{GL}(2,\mathbb{R}) \cong \mathrm{SO}(2)$, $\mathcal{S}(\pi;\mathcal{I})$ is 9-dimensional.
3. *If π is of type $\mathfrak{G}_3$ or $\mathfrak{G}_5$,*
   $$\mathcal{S}(\pi;\mathcal{I}) \approx \mathbf{T}^2 \times \mathrm{SO}(3)\cdot\Big(\mathbb{R}^+ \times \mathbb{R}^+\cdot\mathrm{SO}(2)/\mathbb{Z}_6\Big).$$
   Since $\mathrm{SO}(3)\cap\mathrm{SO}(2) \cong \mathrm{SO}(2)$, $\mathcal{S}(\pi;\mathcal{I})$ is 7-dimensional.
4. *If π is of type $\mathfrak{G}_4$,*
   $$\mathcal{S}(\pi;\mathcal{I}) \approx \mathbf{T}^2 \times \mathrm{SO}(3)\cdot\Big(\mathbb{R}^+ \times \mathbb{R}^+\cdot\mathrm{SO}(2)/\mathbb{Z}_4\Big).$$
   Since $\mathrm{SO}(3)\cap\mathrm{SO}(2) \cong \mathrm{SO}(2)$, $\mathcal{S}(\pi;\mathcal{I})$ is 7-dimensional.
5. *If π is of type $\mathfrak{G}_6$, $\mathcal{S}(\pi;\mathcal{I}) \approx \mathbf{T}^3 \times \Big(\mathrm{SO}(3)/\mathbb{Z}_3\Big)\cdot(\mathbb{R}^+)^3$.*
   This is 9-dimensional.
6. *If π is of type $\mathfrak{B}_1$,*
   $$\begin{aligned}\mathcal{S}(\pi;\mathcal{I}) &\approx \mathbf{T}^1 \times \mathrm{SO}(3)\cdot\Big(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})/\overline{\mathrm{GL}}(2,\mathbb{Z})\Big)\\ &\approx \mathbf{T}^1 \times \mathrm{SO}(3)\cdot\Big(\mathrm{GL}(2,\mathbb{R})/\widehat{\mathrm{GL}}(2,\mathbb{Z}) \times \mathbb{R}^+\Big).\end{aligned}$$
   Since $\mathrm{SO}(3)\cap\mathrm{GL}(2,\mathbb{R}) \cong \mathrm{SO}(2)$, $\mathcal{S}(\pi;\mathcal{I})$ is 8-dimensional.
7. *If π is of type $\mathfrak{B}_2$,*
   $$\begin{aligned}\mathcal{S}(\pi;\mathcal{I}) &\approx \Big(\mathbf{T}^1 \times \mathrm{SO}(3)\cdot\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})/\overline{\mathrm{GL}}(2,\mathbb{Z})\Big)/\mathbb{Z}_2\\ &\approx \widehat{\mathbf{T}}^1 \times \mathrm{SO}(3)\cdot\Big(\mathrm{GL}(2,\mathbb{R})/\widehat{\mathrm{GL}}(2,\mathbb{Z}) \times \mathbb{R}^+\Big),\end{aligned}$$
   where $\widehat{\mathbf{T}}^1$ is a circle doubly covered by $\mathbf{T}^1$. $\mathcal{S}(\pi;\mathcal{I})$ is 8-dimensional.
8. *If π is of type $\mathfrak{B}_3$ or $\mathfrak{B}_4$,*
   $$\mathcal{S}(\pi;\mathcal{I}) \approx \mathbf{T}^2 \times \mathrm{SO}(3)\cdot(\mathbb{R}^+)^3.$$
   This is 8-dimensional.

Proof. Except for the case when $\pi = \mathfrak{B}_2$. the Chabauty space is obtained by

$$\begin{aligned}\mathcal{S}(\pi;\mathcal{I}) &= (\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathcal{N}(\pi)\\ &= (\mathbb{R}/\mathbb{Z})^{3-k} \times [\mathcal{X}(\Phi)/\mathcal{M}],\end{aligned}$$

where k is the rank of the center $\mathcal{Z}(\pi)$ of π and $\mathcal{M}$ is the matrix part of $\mathcal{N}(\pi)$. The proof is obvious from Proposition 4.2 and Theorem 3.1.

For the cases of $\mathfrak{G}_3$ and $\mathfrak{G}_5$, since the spaces $\mathcal{X}(\Phi)$ are computed using by not integral representations of Lemma 2.2 but orthogonal those of Lemma 2.1 and $\mathcal{N}(\xi \cdot \pi \cdot \xi^{-1}) = \xi \cdot \mathcal{N}(\pi) \cdot \xi^{-1}$, the matrix parts of $\mathcal{N}(\pi)$ become $\mathbf{D}_6$ generated by $R(\frac{\pi}{3})$ and a half turn H. (If π is of $\mathfrak{G}_3$ then H is the matrix of rotation of $\mathbb{R}^3$ about y-axis through 180°, and If π is of $\mathfrak{G}_5$ then H is the matrix of rotation of $\mathbb{R}^3$ about z-axis through 180°.)

For the case of $\mathfrak{B}_2$, the finite group $\mathbb{Z}_2$ acts on the circle $\mathbf{T}^1$ yielding a doubly covered circle $\widehat{\mathbf{T}}^1$. $\square$

For completeness, we state the Teichmüller spaces of our groups. They appeared in [3] and [4].

THEOREM 4.4. *Let M be a 3-dimensional flat manifold with $\pi_1(M) = \pi$. Then the Teichmüller spaces are as follow:*

1. *For $\pi = \mathfrak{G}_1$, $\mathcal{T}(\pi;\mathcal{I}) = \mathrm{O}(3)\backslash\mathrm{GL}(3,\mathbb{R}) \approx \mathbb{R}^6$.*
2. *For $\pi = \mathfrak{G}_2$, $\mathcal{T}(\pi;\mathcal{I}) = \mathbb{R}^+ \times (\backslash\, \mathrm{O}(2)\mathrm{GL}(2,\mathbb{R}) \approx \mathbb{R}^+ \times \mathbb{R}^3 \approx \mathbb{R}^4$.*
3. *For $\pi = \mathfrak{G}_3$, $\mathfrak{G}_4$, and $\mathfrak{G}_5$, $\mathcal{T}(\pi;\mathcal{I}) = (\mathbb{R}^+)^2 \approx \mathbb{R}^2$.*
4. *For $\pi = \mathfrak{G}_6$, $\mathcal{T}(\pi;\mathcal{I}) = (\mathbb{R}^*)^3/(\mathbb{Z}_2)^3 = (\mathbb{R}^+)^3 \approx \mathbb{R}^3$.*
5. *For $\pi = \mathfrak{B}_1$ and $\mathfrak{B}_2$, then $\mathcal{T}(\pi;\mathcal{I}) = (\mathrm{O}(2) \backslash \mathrm{GL}(2,\mathbb{R})) \times \mathbb{R}^+ \approx \mathbb{R}^3 \times \mathbb{R}^+ \approx \mathbb{R}^4$.*
6. *For $\pi = \mathfrak{B}_3$ and $\mathfrak{B}_4$, then $\mathcal{T}(\pi;\mathcal{I}) = (\mathbb{Z}_2)^3 \backslash (\mathbb{R}^*)^3 = (\mathbb{R}^+)^3 \approx \mathbb{R}^3$.*

Now we investigate the moduli spaces $\mathcal{M}(\pi;\mathcal{I})$'s. We will show that if the holonomy group Φ of a 3-dimensional Bieberbach group has order greater than two, then the moduli spaces are homeomorphic to the Euclidean spaces. The moduli space is homeomorphic to $\mathbb{R}^2$ if Φ is a cyclic group of order $>$ 2 and $\mathbb{R}^3$ if $\Phi \cong \mathbb{Z} \times \mathbb{Z}$.

THEOREM 4.5. *Let M be a 3-dimensional flat manifold with $\pi_1(M) = \pi$. Then the moduli spaces are as follow:*

1. *If π is of type $\mathfrak{G}_1$, $\mathcal{M}(\pi;\mathcal{I}) = \mathrm{O}(3) \backslash \Big(\mathrm{GL}(3,\mathbb{R})/\,\mathrm{GL}(3,\mathbb{Z})\Big)$.*
2. *If π is of type $\mathfrak{G}_2$, $\mathcal{M}(\pi;\mathcal{I}) = \mathbb{R}^+ \times \Big(\mathrm{O}(2) \backslash (\mathrm{GL}(2,\mathbb{R})/\,\mathrm{GL}(2,\mathbb{Z}))\Big)$.*
3. *If π is of type $\mathfrak{G}_3$, $\mathfrak{G}_4$ or $\mathfrak{G}_5$, $\mathcal{M}(\pi;\mathcal{I}) = (\mathbb{R}^+)^2$.*
4. *If π is of type $\mathfrak{G}_6$, $\mathcal{M}(\pi;\mathcal{I}) = (\mathbb{R}^+)^3$.*
5. *If π is of type $\mathfrak{B}_1$ or $\mathfrak{B}_2$,*
$$\mathcal{M}(\pi;\mathcal{I}) = \Big(\mathrm{O}(2) \backslash \mathrm{GL}(2,\mathbb{R})/\widehat{\mathrm{GL}}(2,\mathbb{R})\Big) \times \mathbb{R}^+.$$
6. *If π is of type $\mathfrak{B}_3$ or $\mathfrak{B}_4$, $\mathcal{M}(\pi;\mathcal{I}) = (\mathbb{R}^+)^3$.*

Proof. The moduli space is obtained by

$$\begin{aligned}\mathcal{M}(\pi;\mathcal{I}) &= \mathcal{I} \setminus (\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathcal{N}(\pi)\\ &= \mathrm{O}(3) \setminus \mathcal{X}(\Phi)/\mathcal{M},\end{aligned}$$

where $\mathcal{M}$ is the matrix part of $\mathcal{N}(\pi)$. □

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