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GOTTLIEB GROUPS AND SUBGROUPS OF THE GROUP OF SELF-HOMOTOPY EQUIVALENCES

Jae-Ryong Kim, Nobuyuki Oda,
Jianzhong Pan, and Moo Ha Woo

Abstract. Let $\mathcal{E}_{\#}(X)$ be the subgroups of $\mathcal{E}(X)$ consisting of homotopy classes of self-homotopy equivalences that fix homotopy groups through the dimension of X and $\mathcal{E}_*(X)$ be the subgroup of $\mathcal{E}(X)$ that fix homology groups for all dimension. In this paper, we establish some connections between the homotopy group of X and the subgroup $\mathcal{E}_{\#}(X) \cap \mathcal{E}_*(X)$ of $\mathcal{E}(X)$. We also give some relations between $\pi_n(W)$, as well as a generalized Gottlieb group $G_n^f(W, X)$, and a subset $\mathcal{M}^f_{\#_N}(X, W)$ of $[X, W]$. Finally we establish a connection between the coGottlieb group of X and the subgroup of $\mathcal{E}(X)$ consisting of homotopy classes of self-homotopy equivalences that fix cohomology groups.

1. Introduction and preliminaries

By a *space*, we mean a connected CW-complex of finite type. We mainly consider finite dimensional CW-complexes with base point. We begin this section with remarks about the set of all base point preserving continuous maps from a space X to a space W. This set of maps splits up into disjoint equivalence classes, called *homotopy classes*. We write $[X, W]$ for the set of all base point preserving homotopy classes of the maps from X to W; by keeping X fixed and varying W, this set is an invariant of the homotopy type of W in the sense that it is determined by the homotopy equivalence of spaces: the set $[X, W]$ can often be endowed, in a natural way, with some algebraic structure, and we obtain

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exactly the algebraic invariant. Alternatively, we can keep W fixed and vary X: once again a homotopy invariant results. Let W be a CW-complex of dimension N and $\mathcal{E}(W)$ the group of homotopy classes of self-homotopy equivalences of W. In this paper, we give results about some subgroups of $\mathcal{E}(W)$ and we will extend these results to more general cases.

We now review some standard material that we will use. A cofibration sequence $Z \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma Z$ where X is the mapping cone of γ, gives a homotopy coaction $c: X \to X \vee \Sigma Z$, obtained by pinching the 'equator' of the cone of Z to a point. This coaction induces an action of $[\Sigma Z, W]$ on $[X, W]$ for any space W. That is,

$$\mu : [X,W] \times [\Sigma Z, W] \longrightarrow [X,W], \quad \mu(f,\alpha) = f^{\alpha}$$

for any $\alpha \in [\Sigma Z, W]$ and $f \in [X, W]$, where

$$f^{\alpha} : X \xrightarrow{c} X \vee \Sigma Z \xrightarrow{f \vee \alpha} W \vee W \xrightarrow{\nabla} W.$$

The following properties of this action are mentioned on p. 174 of Hilton [6]:

(1) If $h: W \to W'$, then $h(f^{\alpha}) = (hf)^{h\alpha}$,

(2) If $\alpha, \beta \in [\Sigma Z, W]$, then $(f^{\alpha})^{\beta} = f^{(\alpha+\beta)}$.

We are interested in the effect that f^{α} has on homology and homotopy groups. This is described in the following results by Proposition 2.1 of [1].

PROPOSITION 1.1. *For the above cofibration sequence, suppose $f \in [X, W]$ and $\alpha \in [\Sigma Z, W]$. Then we have the following for any $i > 0$:*

(1) The induced homology homomorphism $(f^{\alpha})_* : H_i(X) \to H_i(W)$ is given by

$$(f^{\alpha})_*(x) = f_*(x) + \alpha_* q_*(x)$$

for each $x \in H_i(X)$.

(2) Suppose that (f, α): $X \vee \Sigma Z \to W$ factors through the product $X \times \Sigma Z$. Then the induced homotopy homomorphism $(f^{\alpha})_\sharp : \pi_i(X) \to \pi_i(W)$ is given by

$$(f^{\alpha})_\sharp(x) = f_\sharp(x) + \alpha_\sharp q_\sharp(x)$$

for each $x \in \pi_i(X)$.

We now specialize to a mapping cone sequence of the form

$$S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma S^{n-1} \equiv S^n,$$

i.e. $X = Y \cup_\gamma e^n$. Then we have an action of $\pi_n(W)$ on $[X, W]$. We will consider elements of the form $f^\alpha \in [X, W]$ for $f \in [X, W]$ and $\alpha \in \pi_n(W)$.

Let X be a CW-complex of dimension N and $\mathcal{E}(X)$ the group of homotopy classes of self-equivalences of X. If we consider elements of the form $\iota^\alpha \in [X, X]$ for the identity map ι of X and $\alpha \in \pi_n(X)$, these are not self-homotopy equivalences in general.

In [1], Arkowitz, Lupton and Murillo studied the subgroup $\mathcal{E}_\#(X)$ and $\mathcal{E}_*(X)$ of $\mathcal{E}(X)$ as follows (cf. Dror and Zabrodsky [4]; Maruyama [8] and [9]):

$$\mathcal{E}_\#(X) = \{f \in \mathcal{E}(X) \mid f_\sharp = 1 : \pi_i(X) \to \pi_i(X) \text{ for } i \leq N\},$$

$$\mathcal{E}_\#\infty(X) = \{f \in \mathcal{E}(X) \mid f_\sharp = 1 : \pi_i(X) \to \pi_i(X) \text{ for all } i\},$$

$$\mathcal{E}_*(X) = \{f \in \mathcal{E}(X) \mid f_* = 1 : H_i(X) \to H_i(X) \text{ for all } i\}.$$

Maruyama [10] introduces a subset of $[X, W]$ as follows:

$$Z^n_\#(X, W) = \{\alpha \in [X, W] \mid \alpha_\sharp = 0 : \pi_i(X) \to \pi_i(W) \text{ for } i \leq n\}$$

If we consider $\mathcal{E}_\#(X) \subset [X, X]$ and $Z^n_\#(X, X) \subset [X, X]$ as two special subsets in $[X, X]$, we can give some definitions which generalize them and which we will study in the next section.

Definition 1.1. Let $f \in [X, W]$. We define

$$\mathcal{M}^f_*(X, W) = \{g \in [X, W] \mid g_* = f_* : H_i(X) \to H_i(W) \text{ for all } i\}.$$

Similarly we denote

$$\mathcal{M}^f_{\#_N}(X, W) = \{g \in [X, W] \mid g_\sharp = f_\sharp : \pi_i(X) \to \pi_i(W) \text{ for all } i \leq N\}.$$

Especially, we denote $\mathcal{M}^f_\#(X, W) = \mathcal{M}^f_{\#_N}(X, W)$ if W is a N-dimensional CW-complex. Thus we can get

$$\mathcal{M}^0_{\#_n}(X, W) = Z^n_\#(X, W)$$

$$\mathcal{M}^\iota_\#(W, W) = \mathcal{E}_\#(W)$$

$$\mathcal{M}^\iota_{\#\infty}(W, W) = \mathcal{E}_{\#\infty}(W),$$

where 0 denotes the constant map.

Recall the n'th Gottlieb group [5] of $\pi_n(W)$, denoted by $G_n(W)$, consists of those $\alpha \in \pi_n(W)$ for which there is an associated map $F : W \times S^n \to W$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} W \times S^n & \xrightarrow{F} & W \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \nabla} \\ W \vee S^n & \xrightarrow{\iota \vee \alpha} & W \vee W \end{array}$$

2. Gottlieb groups and subgroups of self-homotopy equivalences

It was shown in Theorem 2.3 of [1] that there is a homomorphism between the Gottlieb group of X and $\mathcal{E}_{\#}(X)$ and $\mathcal{E}_*(X)$ as follows.

THEOREM 2.1. *Let $X = Y \cup_\gamma e^n$ be a 1-connected n-dimensional complex. Suppose that $q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$. Then there is a homomorphism*

$$\theta : G_n(X) \to \mathcal{E}_{\#}(X)$$

defined by $\theta(\alpha) = \iota^\alpha$ for $\alpha \in G_n(X)$. This homomorphism restricts to

$$\theta' : G_n(X) \cap \operatorname{Ker} h_n \to \mathcal{E}_*(X) \cap \mathcal{E}_{\#}(X),$$

where $h_n : \pi_n(X) \to H_n(X)$ denotes the Hurewicz homomorphism.

For the map $\theta : \pi_n(X) \to [X, X]$ given by $\theta(\alpha) = \iota^\alpha$ for $\alpha \in \pi_n(X)$ and the group $\mathcal{E}_{\#}(X) \subset [X, X]$, this theorem gives a condition to be $G_n(X) \subset \theta^{-1}(\mathcal{E}_{\#}(X))$. The objects of this section is to find conditions such that $\pi_n(X) \subset \theta^{-1}(\mathcal{E}_{\#}(X))$ and $\pi_n(X) \subset \zeta^{-1}(Z^n_{\#}(X, X))$ for a function $\zeta : \pi_n(X) \to [X, X]$ defined later.

First we will find some conditions for θ to be a homomorphism from the homotopy group $\pi_n(X)$ to $\mathcal{E}_{\#}(X)$. Consider the cofibration sequence

$$S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} S^n \qquad (n \geq 2).$$

THEOREM 2.2. *Let $n \geq 2$ and Y be 1-connected. Let $X = Y \cup_\gamma e^n$ with $X^{n-1} = Y$. If the Hurewicz homomorphism $h_n = 0 : \pi_n(X) \to H_n(X)$, then there is a homomorphism*

$$\theta : \pi_n(X) \longrightarrow \mathcal{E}_*(X) \cap \mathcal{E}_{\#}(X)$$

defined by $\theta(\alpha) = \iota^\alpha$ for any $\alpha \in \pi_n(X)$.

Proof. First we prove that $j_\sharp : \pi_k(Y) \to \pi_k(X)$ is surjective for all $k \leq n$.

For $n \geq 1$, the Whitehead's Γ-group $\Gamma_n(X)$ is defined by the following image group

$$\Gamma_n(X) = \text{image}\,(j_\sharp : \pi_n(X^{n-1}) \to \pi_n(X^n)),$$

where $j : Y = X^{n-1} \to X = X^n$ is the inclusion of the $(n-1)$-skeleton to the n-skeleton of X. Let $j_n : \pi_n(Y) \to \Gamma_n(X)$ be the map induced by $j_\sharp : \pi_n(Y) = \pi_n(X^{n-1}) \to \pi_n(X) = \pi_n(X^n)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} & \pi_n(Y) & & & & & \\ & \downarrow{\scriptstyle j_n} & \searrow^{j_\sharp} & & & & \\ \longrightarrow & \Gamma_n(X) & \xrightarrow{i_n} & \pi_n(X) & \xrightarrow{h_n} & H_n(X) & \longrightarrow \end{array}$$

where $n \geq 2$. The row in the above diagram is the exact sequence of J. H. C. Whitehead (see [3]). Then we see that $j_\sharp : \pi_n(Y) \to \pi_n(X)$ is surjective since $j_n : \pi_n(Y) \to \Gamma_n(X)$ is surjective and $h_n = 0 : \pi_n(X) \to H_n(X)$. It is clear that $j_\sharp : \pi_k(Y) \to \pi_k(X)$ is surjective for all $k < n$. Therefore $j_\sharp : \pi_k(Y) \to \pi_k(X)$ is surjective for all $k \leq n$.

Since $h_n = 0 : \pi_n(X) \to H_n(X)$, we have $h_n(\alpha) = 0$ in $H_n(X)$ for any $\alpha \in \pi_n(X)$. This implies $\alpha_* = 0 : H_n(S^n) \to H_n(X)$ and hence $\alpha_* q_* = 0 : H_k(X) \to H_k(X)$ for any $k > 0$. We remark that X is a 1-connected n-dimensional complex by the assumption. Hence by Proposition 1.1 (1), we have $\iota^\alpha \in \mathcal{E}_*(X)$ for any element α of $\pi_n(X)$.

Let $c : X \to X \vee S^n$ be the co-action and $i_1 : X \to X \vee S^n$ be the inclusion map to the first factor. Then we have

$$cj = i_1 j : Y \longrightarrow X \vee S^n.$$

Let $\iota : X \to X$ be the identity map. It follows that for any element $\delta \in \pi_k(X)$ for $k \leq n$, there exists an element $\beta \in \pi_k(Y)$ such that $j_\sharp(\beta) = \delta$ by the discussion above. Therefore we have

$$\begin{aligned} (\iota^\alpha)_\sharp(\delta) &= \nabla(\iota \vee \alpha)c\delta \\ &= \nabla(\iota \vee \alpha)cj\beta \\ &= \nabla(\iota \vee \alpha)i_1 j\beta \\ &= 1_X j\beta = \delta. \end{aligned}$$

Thus $\theta(\alpha) \in \mathcal{E}_\#(X)$. Suppose that α and β are two elements in $\pi_n(X)$. We remark that $(\iota^\alpha)_\sharp(\beta) = \beta$ since $\iota^\alpha = \theta(\alpha) \in \mathcal{E}_\#(X)$ as is shown above. It follows that

$$\theta(\alpha+\beta) = \iota^{\alpha+\beta} = \iota^{\alpha+(\iota^\alpha)_\sharp(\beta)} = (\iota^\alpha)^{(\iota^\alpha)_\sharp(\beta)} = (\iota^\alpha\iota)^{\iota^\alpha\beta} = \iota^\alpha\iota^\beta = \theta(\alpha)\theta(\beta)$$

by the formulas $f^{\alpha+\beta} = (f^{\alpha})^{\beta}$ and $(hf)^{h\alpha} = h(f)^{\alpha}$. This completes the proof. □

P. J. Kahn [7] has proved the following:

THEOREM 2.3. *Let X be a homotopy type of the mapping cone of a map $\gamma : S^{2n-1} \to Y = X_n$, $n \geq 2$, where X_n is the r-fold bouquet of n-spheres, $r = rank H_n(X) \geq 1$. Then there exists an exact sequence*

$$[\Sigma Y, X] \xrightarrow{(\Sigma\gamma)^* + \Psi} \pi_{2n}(X) \xrightarrow{\theta} \mathcal{E}_{\#}(X) \xrightarrow{R} \mathcal{E}_{\#}(Y),$$

where R is given by restriction to Y.

We will slightly extend above theorem.

THEOREM 2.4. *Let $n \geq 2$ and Y be 1-connected homotopy associative co-H-space. Let $X = Y \cup_{\gamma} e^n$ with finite n'th homotopy group and $X^{n-2} = Y$. Then there exists an exact sequence*

$$[\Sigma Y, X] \xrightarrow{\Gamma} \pi_n(X) \xrightarrow{\theta} \mathcal{E}_{\#}(X) \xrightarrow{R} \mathcal{E}_{\#}(Y),$$

where R is given by restriction to Y.

Proof. By Corollary 3.2.2 of Rutter [13] (cf. also Lemmas 2.7, 2.8 and 2.9 of [12]), we have the following exact sequence:

$$[\Sigma Y, X] \xrightarrow{\Gamma(j,\gamma)} [S^n, X] \xrightarrow{\theta} [X, X]_{1_X} \xrightarrow{j^*} [Y, X]_j \xrightarrow{\gamma^*} [S^{n-1}, X],$$

where $[X, X]_{1_X}$ and $[Y, X]_j$ show that 1_X and j are base points of each homotopy sets.

By Theorem 2.2, the image of θ is contained $\mathcal{E}_{\#}(X)$ because the Hurewicz map $h_n = 0 : \pi_n(X) \to H_n(X)$ by the assumption that $\pi_n(X)$ is a finite group. Next we show that the image of R $(= j^*)$ is contained in $\mathcal{E}_{\#}(Y)$. Consider the following commutative diagram

$$\begin{array}{ccc}
\pi_i(X) & \xrightarrow{f_\sharp = 1} & \pi_i(X) \\
\uparrow{\scriptstyle j_\sharp} & & \uparrow{\scriptstyle j_\sharp} \\
\pi_i(Y) & \xrightarrow{R(f)_\sharp} & \pi_i(Y) \\
\uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} \\
\pi_{i+1}(X, Y) = 0 & & \pi_{i+1}(X, Y) = 0
\end{array}$$

for any $f \in \mathcal{E}_{\#}(X)$ and each $k \leq n-2$. From the above diagram, we have

$$j_\sharp(\alpha) = f_\sharp j_\sharp(\alpha) = j_\sharp R(f)_\sharp(\alpha) = j_\sharp(R(f)_\sharp(\alpha)).$$

Since $j_\sharp$ is injective, we have $\alpha = R(f)_\sharp(\alpha)$ for any $k \leq \dim Y$.

Then the exactness of the sequence is obtained by the exact sequence of Rutter. □

Hence if we use the above Theorem and [7, Lemma 4], we have the following

COROLLARY 2.1. *Let X be a closed compact, oriented, C^∞, $(n-1)$-connected $2n$-manifold, $n \geq 2$. Then there is an exact sequence $[\Sigma Y, X] \to \pi_{2n}(X) \xrightarrow{\theta} \mathcal{E}_\#(X) \xrightarrow{R} 0$.*

Especially, if we let $X = S^n \times S^n$, then we have

$\mathcal{E}_\#(X)$ is isomorphic to $\pi_{2n}(X)$ if $n = 2, 6$ or $3(mod\ 4)$

$\mathcal{E}_\#(X)$ is isomorphic to $\pi_{2n}(X)/(Z_2 \oplus Z_2)$ if $n \neq 2, 6$ or $3(mod\ 4)$.

Proof. If we consider the following exact sequence

$$[\Sigma Y, X] \to \pi_{2n}(X) \xrightarrow{\theta} \mathcal{E}_\#(X) \xrightarrow{R} \mathcal{E}_\#(Y)$$

from the above theorem, it is sufficient to show that the image of R is trivial. It will be proved by the same method of [2, Proposition 6.1] that the kernel of R is $\mathcal{E}_\#(X)$. □

Next we will show a condition that there exists a morphism

$$\theta : \pi_n(X) \longrightarrow Z^n_\#(X, X).$$

Recall the *generalized Gottlieb group* [16] of $\pi_n(W)$, denoted by $G^f_n(W, X)$, consists of those $\alpha \in \pi_n(W)$ for which there is an associated map $F : X \times S^n \to W$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{F} & W \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ X \vee S^n & \xrightarrow{f \vee \alpha} & W \vee W \end{array}$$

Since $G_n(W) = G^\iota_n(W, W)$, this group is a generalization of Gottlieb group and it is clear that $G_n(W) \subset G^f_n(W, X) \subset \pi_n(W)$.

By the new notations and Proposition 1.1, we have the following:

COROLLARY 2.2. *Let $X = Y \cup_\gamma e^n$ be a 1-connected CW-complex, $\alpha \in \pi_n(W)$ and $f \in [X, W]$. Then the following results hold.*

(1) $f^\alpha \in \mathcal{M}^f_*(X, W)$ if and only if $\alpha_* q_* = 0 : H_n(X) \to H_n(W)$.

(2) Suppose that $\alpha \in G^f_n(W, X)$. Then $f^\alpha \in \mathcal{M}^f_\#(X, W)$ if and only if $\alpha q \in \mathcal{M}^0_\#(X, W)$.

Here we want to show a condition for $\pi_n(W)$ to be contained in $\theta^{-1}(\mathcal{M}^f_{\#}(X,W))$.

THEOREM 2.5. *Let $n \geq 2$ and Y be 1-connected. Let $X = Y \cup_\gamma e^n$ with $X^{n-1} = Y$ and $\dim W \leq n$ and $f \in [X,W]$. If $q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$, then there exists a morphism*

$$\Theta : \pi_n(W) \longrightarrow \mathcal{M}^f_{\#}(X,W)$$

which is defined by $\Theta(\alpha) = f^\alpha$ for any $\alpha \in \pi_n(W)$. This morphism restricts to

$$\Theta' : \mathrm{Ker}\,\{h_n : \pi_n(W) \longrightarrow H_n(W)\} \to \mathcal{M}^f_*(X,W) \cap \mathcal{M}^f_{\#}(X,W),$$

where $h_n : \pi_n(W) \longrightarrow H_n(W)$ is the Hurewicz homomorphism.

Proof. By Blaker-Massey Theorem (7.12) on p. 368 (Chapter VII) of [15], we see that

$$p_\sharp : \pi_n(X,Y) \longrightarrow \pi_n(X/Y) = \pi_n(S^n)$$

is an isomorphism when Y is 1-connected. We consider the long homotopy exact sequence

$$\cdots \longrightarrow \pi_n(Y) \xrightarrow{j_\sharp} \pi_n(X) \xrightarrow{k_\sharp} \pi_n(X,Y) \longrightarrow \cdots$$

Since $p_\sharp k_\sharp = q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$ and $p_\sharp$ is an isomorphism, we see that $k_\sharp = 0 : \pi_n(X) \to \pi_n(X,Y)$ and hence $j_\sharp : \pi_n(Y) \to \pi_n(X)$ is surjective and hence $j_\sharp : \pi_i(Y) \to \pi_i(X)$ is surjective for any $i \leq n$. Hence for any element $\delta \in \pi_i(X)$ for $i \leq \dim W \leq n$, there exists an element $\beta \in \pi_i(Y)$ such that $j_\sharp(\beta) = \delta$. Let $c : X \to X \vee S^n$ be the co-action and $i_1 : X \to X \vee S^n$ be the inclusion map to the first factor. Then as in the proof of Theorem 2.2 we have

$$cj = i_1 j : Y \longrightarrow X \vee S^n.$$

Hence we have

$$\begin{aligned} (f^\alpha)_\sharp(\delta) &= \nabla(f \vee \alpha)c\delta \\ &= \nabla(f \vee \alpha)cj\beta \\ &= \nabla(f \vee \alpha)i_1 j\beta \\ &= fj\beta = f_\sharp(\delta). \end{aligned}$$

It follows that $\Theta(\alpha) = f^\alpha \in \mathcal{M}^f_{\#}(X,W)$.

Now suppose that $h_n(\alpha) = 0$ for an element $\alpha \in \pi_n(W)$. Then $\alpha_* = 0 : H_n(S^n) \to H_n(W)$. It follows that $\alpha_* q_* = 0 : H_i(X) \to H_i(S^n) \to H_i(W)$ for any $i > 0$ and hence $f^\alpha \in \mathcal{M}^f_*(X,W)$ by Proposition 1.1 (1).

The following result is (partly) a generalization of Theorem 2.3 of [1]. (In Theorem 2.3 of [1], the condition $X^{n-1} = Y$ is not assumed.) □

COROLLARY 2.3. *Let* $n \geq 2$ *and* Y *be* 1*-connected. Let* $X = Y \cup_\gamma e^n$ *with* $X^{n-1} = Y$. *If* $q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$, *then there exists a homomorphism*

$$\theta : \pi_n(X) \longrightarrow \mathcal{E}_*(X) \cap \mathcal{E}_\#(X)$$

defined by $\theta(\alpha) = \iota^\alpha$ *for any* $\alpha \in \pi_n(X)$.

Proof. If $q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$, then $j_\sharp : \pi_n(Y) \to \pi_n(X)$ is surjective as is shown in the proof of Theorem 2.5, and hence we see that $i_n : \Gamma_n(X) \to \pi_n(X)$ is surjective by the assumption that $X^{n-1} = Y$. It follows that the Hurewicz homomorphism $h_n = 0 : \pi_n(X) \to H_n(X)$, and hence Theorem 2.2 implies the result. □

COROLLARY 2.4. *Let* $n \geq 2$ *and* Y *be* 1*-connected. Let* $X = Y \cup_\gamma e^n$ *with* $X^{n-1} = Y$. *If* $q_\sharp = 0 : \pi_n(X) \to \pi_n(S^n)$, *then there exists a morphism*

$$\zeta : \pi_n(X) \longrightarrow Z^n_\#(X, X) = \mathcal{M}^0_\#(X, X)$$

defined by $\zeta(\alpha) = 0^\alpha = \alpha q$ *for any* $\alpha \in \pi_n(X)$. *This morphism restricts to*

$$\zeta' : \mathrm{Ker}\,\{h_n : \pi_n(X) \longrightarrow H_n(X)\} \longrightarrow \mathcal{M}^0_*(X, X) \cap Z^n_\#(X, X).$$

The function ζ *defined above satisfies* $\zeta(\alpha)\zeta(\beta) = 0$ *for any* $\alpha, \beta \in \pi_n(X)$.

Proof. By Proposition 2.6 of [11], we see that $\zeta(\alpha) = 0^\alpha = * \dotplus \alpha = q^*(\alpha) = \alpha q$. Then the result follows by putting $X = W$ in Theorem 2.5. □

Arkowitz, Lupton and Murillo [1] showed a space X with the homomorphism $\theta : G_n(X) \neq 0 \to \mathcal{E}_\#(X)$ but they didn't show that the homomorphism is nontrivial. Next theorem gives a condition for X which does not have a nontrivial homomorphism given by $\theta(\alpha) = \iota^\alpha$.

THEOREM 2.6. *Let* $X = Y \cup_\gamma e^n$ *be a* 1*-connected* n*-dimensional CW-complex. Suppose that* $q : X \to S^n$ *has a right homotopy inverse. If* $\theta : G_n(X) \to \mathcal{E}_\#(X)$ *given by* $\theta(\alpha) = \iota^\alpha$ *is a well-defined homomorphism, then* $G_n(X)$ *is trivial.*

Proof. Let $S^{n-1} \to Y \to X \xrightarrow{q} \Sigma S^{n-1} = S^n$ be the mapping cone sequence for $X = Y \cup_\gamma e^n$. For any $\alpha \in G_n(X)$, we have $\iota^\alpha \in \mathcal{E}_\#(X)$ if and only if $\alpha_\sharp q_\sharp = 0 : \pi_i(X) \to \pi_i(X)$ for $i \leq n$ by Corollary 2.2

(2) of [1]. Let $s \in \pi_n(X)$ be a right homotopy inverse of q. Then for $\iota_{S^n} \in \pi_n(S^n)$, we have

$$\alpha = \alpha\iota_{S^n} = \alpha_\sharp(\iota_{S^n}) = (\alpha q s)_\sharp(\iota_{S^n}) = \alpha_\sharp q_\sharp(s) = 0.$$

Therefore $G_n(X)$ is trivial. □

COROLLARY 2.5. *Let $X = Y \cup_\gamma e^n$ be a 1-connected n-dimensional CW-complex. If $\theta : G_n(X) \to \mathcal{E}_\#(X)$ given by $\theta(\alpha) = \iota^\alpha$ is a well-defined homomorphism and nontrivial, then $G_n(X) \neq 0$ and consequently the map $q : X \to S^n$ does not have a right homotopy inverse.*

The following example has been considered in [1], but here we extend the domain from $G_n(X)$ to $\pi_n(X)$ and reduce the codomain from $\mathcal{E}_\#(X)$ to $\mathcal{E}_*(X) \cap \mathcal{E}_\#(X)$.

EXAMPLE. Let $X = S^2 \times S^3 = S^2 \vee S^3 \cup_{[i_1,i_2]} e^5$. Since $H_5(X)$ is infinite cyclic, the Hurewicz homomorphism $h_5 : \pi_5(X) \to H_5(X)$ is zero. Now $Y = S^2 \vee S^3 = X^4$ and hence X satisfies all the hypothesis in Theorem 2.2. Therefore we have a homomorphism

$$\theta : \pi_5(X) \longrightarrow \mathcal{E}_*(X) \cap \mathcal{E}_\#(X).$$

Let $S^4 \xrightarrow{[i_1,i_2]} S^2 \vee S^3 \to X \xrightarrow{q} \Sigma S^4 = S^5$ be the mapping cone sequence for $X = S^2 \times S^3 = S^2 \vee S^3 \cup_{[i_1,i_2]} e^5$ and F_q be the homotopy fibre of $q : X \to S^5$. Since $G_5(X)$ is nontrivial, $G_4(F_q)$ is nontrivial by Theorem 2.6. Because if $G_4(F_q) = 0$, the map $q : X \to S^5$ has a right homotopy inverse (see [5, Corollary 2.7]).

3. CoGottlieb groups and a subgroup of self-homotopy equivalences

Let $(f, \alpha) : W \xrightarrow{\Delta} W \times W \xrightarrow{f \times \alpha} X \times \Omega Z$ be the composite map of the maps $f \in [W, X]$ and $\alpha \in [W, \Omega Z]$. A map $f : W \to X$ is said to be *cocyclic* [14] if there exists a map $\Phi : W \to W \vee X$ such that the following diagram is homotopy commutative.

$$\begin{array}{ccc} W & \xrightarrow{\Phi} & W \vee X \\ & \searrow^{(\iota, f)} & \downarrow j \\ & & W \times X \end{array}$$

We denote by $H^n(X;G)$ the cohomology group of X with coefficient group G, and we simply write $H^n(X) = H^n(X;Z)$ for the integral cohomology group of X.

DEFINITION 3.1. $G^n(X) = \{\alpha \in H^n(X) \mid \alpha \text{ is cocyclic}\}$ is said to be a *coGottlieb group*.

Let $\Omega Z \xrightarrow{q} X \xrightarrow{j} Y \xrightarrow{\gamma} Z$ be a fibration sequence. This sequence gives an action of ΩZ on X by $\mu : X \times \Omega Z \to X$. The induced action of $[W, \Omega Z]$ on $[W, X]$ is given by

$$f^\alpha = \mu(f \times \alpha)\Delta : W \xrightarrow{\Delta} W \times W \xrightarrow{f\times\alpha} X \times \Omega Z \xrightarrow{\mu} X$$

for any $\alpha \in [W, \Omega Z]$ and $f \in [W, X]$.

REMARK. We will use the same notation f^α for the composite maps $f^\alpha = \mu(f \times \alpha)\Delta$ and the f^α used in the previous sections, for the convenience, because we can easily distinguish them.

LEMMA 3.1. *Let $\alpha \in [W, \Omega Z]$ and $f \in [W, X]$. Then this action satisfies the following*:

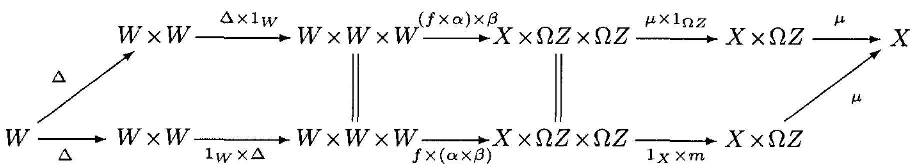
(1) *If $h : W' \to W$, then $(f^\alpha)h = (fh)^{\alpha h}$.*

(2) *If α, $\beta \in [W, \Omega Z]$, then $(f^\alpha)^\beta = f^{(\alpha+\beta)}$.*

Proof. The first is clear, so we will prove the second case: (2) We have the following relation

$$\begin{aligned}(f^\alpha)^\beta &= \mu(f^\alpha \times \beta)\Delta = \mu((\mu(f \times \alpha)\Delta) \times \beta)\Delta \\ &= \mu(\mu \times 1_{\Omega Z})[(f \times \alpha) \times \beta](\Delta \times 1_W)\Delta \\ &= \mu(1_X \times m)[f \times (\alpha \times \beta)](1_W \times \Delta)\Delta \\ &= \mu(f \times (\alpha + \beta))\Delta = f^{(\alpha+\beta)}\end{aligned}$$

by using the following homotopy commutative diagram:



THEOREM 3.1. *Let $\Omega Z \xrightarrow{q} X \xrightarrow{h} Y \xrightarrow{\gamma} Z$ be a fibration sequence. Let $f \in [W, X]$ and $\alpha \in [W, \Omega Z]$. Then the following formulas hold for any $i > 0$.*

(1) *The induced homotopy homomorphism* ${f^\alpha}_\sharp : \pi_i(W) \to \pi_i(X)$ *satisfies*

$${f^\alpha}_\sharp(x) = f_\sharp(x) + q_\sharp \alpha_\sharp(x)$$

for any $x \in \pi_i(W)$.

(2) *If* $(f, \alpha) : W \to X \times \Omega Z$ *factors through* $X \vee \Omega Z$, *then the induced cohomology homomorphism* $f^{\alpha *} : H^i(X; G) \to H^i(W; G)$ *satisfies*

$$f^{\alpha *}(x) = f^*(x) + \alpha^* q^*(x)$$

for any $x \in H^i(X; G)$ *and any abelian group* G.

Proof. (1) Consider the following diagram:

$$\begin{array}{ccccccc}
\pi_i(W) & \xrightarrow{\Delta_\sharp} & \pi_i(W \times W) & \xrightarrow{(f\times\alpha)_\sharp} & \pi_i(X \times \Omega Z) & \xrightarrow{\mu_\sharp} & \pi_i(X) \\
& \searrow^{\Delta} & \big\downarrow (p_{1\sharp}, p_{2\sharp}) & & \big\downarrow (p_{1\sharp}, p_{2\sharp}) & \nearrow_{(1, q_\sharp)} & \\
& & \pi_i(W) \oplus \pi_i(W) & \xrightarrow[f_\sharp \oplus \alpha_\sharp]{} & \pi_i(X) \oplus \pi_i(\Omega Z) & &
\end{array}$$

The right-most triangle commutes since $\mu | X \times \{*\} \simeq 1_X$ and $\mu | \{*\} \times \Omega Z \simeq q$ (cf. [11, Proposition 3.4 (2)]). The fact that the other two parts of the diagram commute is obvious. By definition, the composite of the homomorphisms on the top line is the homomorphism induced by f^α. The other way to go around the diagram gives the desired formula.

(2) By definition, $(f^\alpha)^*(x)$ is represented by

$$W \xrightarrow{(f,\alpha)} X \times \Omega Z \xrightarrow{\mu} X \xrightarrow{x} K(G, i)$$

On the other hand, by definition of the sum of cohomology classes in $H^*(W; Z)$, $f^*(x) + \alpha^* q^*(x)$ is represented by

$$W \xrightarrow{(f,\alpha)} X \times \Omega Z \xrightarrow{x \times xq} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i).$$

Now, by the given condition, (f, α) is the composite $W \xrightarrow{\Phi} X \vee \Omega Z \xrightarrow{j} X \times \Omega Z$, where $j : X \vee \Omega Z \to X \times \Omega Z$ is the natural inclusion. The second part follows from the fact that the following two maps are homotopic:

$$X \vee \Omega Z \xrightarrow{j} X \times \Omega Z \xrightarrow{\mu} X \xrightarrow{x} K(G, i)$$

$$X \vee \Omega Z \xrightarrow{j} X \times \Omega Z \xrightarrow{x \times xq} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i).$$

□

Let $\mathcal{E}^{*\prime}(X) = \{f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X) \to H^i(X) \text{ for } i \leq N\}$, where N is the homotopical dimension of X denoted by h-dim X. One use this subgroup instead of requiring $f^* = 1$ for all i because only the present subgroup is nilpotent and commutes with the rationalization operation when X has finite h-dim X.

Let $N = \text{h-dim}\, X = \max\{\, i \mid \pi_i(X) \neq 0 \,\}$. We define

$$\mathcal{E}^{*\prime}(X) = \{\, f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X) \to H^i(X) \text{ for any } i \le N \,\}.$$

COROLLARY 3.1. *Let X be the homotopy fibre of a map $\gamma : Y \to K(Z, n+1)$ and $\alpha \in H^n(X)$. Then*

(1) *$\iota^\alpha \in \mathcal{E}_{\#\infty}(X)$ if and only if $q_\sharp \alpha_\sharp = 0 : \pi_n(X) \to \pi_n(K(Z,n)) \to \pi_n(X)$.*

(2) *Let $\alpha \in G^n(X) \subset H^n(X)$ and $N = \text{h-dim}\, X$. Then $\iota^\alpha \in \mathcal{E}^{*\prime}(X)$ if and only if $\alpha^* q^* = 0 : H^i(X) \to H^i(K(Z,n)) \to H^i(X)$ for any $i \le N$.*

Proof. (1) is obtained by Theorem 3.1 (1).

(2) By Theorem 3.1 (2), we see that the condition $\iota^\alpha \in \mathcal{E}^{*\prime}(X)$ implies $\alpha^* q^* = 0 : H^i(X) \to H^i(X)$ for any $i \le N$.

To prove the converse, we consider two cases separately:

The case $N < n$: Consider the following composite of homomorphisms.

$$q_\sharp \alpha_\sharp : \pi_i(X) \longrightarrow \pi_i(K(Z,n)) \longrightarrow \pi_i(X).$$

Since $\pi_i(X) = 0$ for any $i > N$, we see that $q_\sharp \alpha_\sharp = 0 : \pi_i(X) \to \pi_i(X)$ for any $i > 0$. Then $(\iota^\alpha)_\# = \text{id} : \pi_i(X) \to \pi_i(X)$ for any $i > 0$ by Theorem 3.1 (1) and hence $(\iota^\alpha)_\#$ is a homotopy equivalence.

The case $N \ge n$: By the assumption we see

$$\alpha^* q^* = 0 : H^n(X) \longrightarrow H^n(K(Z,n)) \longrightarrow H^n(X),$$

and hence $(\iota^\alpha)^*(x) = x$ for any $x \in H^n(X)$ by Theorem 3.1 (2). It follows that

$$\iota^{-\alpha}\iota^\alpha = (\iota^\alpha)^{(-\alpha)\iota^\alpha} = (\iota^\alpha)^{(\iota^\alpha)^*(-\alpha)} = (\iota^\alpha)^{-\alpha} = \iota^{\alpha-\alpha} = \iota^0 = \iota\,;$$

$$\iota^\alpha\iota^{-\alpha} = (\iota^{-\alpha})^{(\alpha)\iota^{-\alpha}} = (\iota^{-\alpha})^{(\iota^{-\alpha})^*(\alpha)} = (\iota^{-\alpha})^{\alpha} = \iota^{-\alpha+\alpha} = \iota^0 = \iota$$

by Lemma 3.1. Hence ι^α is a homotopy equivalence. □

THEOREM 3.2. *Let X be the homotopy fiber of a map $\gamma : Y \to K(Z, n+1)$ and n is h-dim X. Suppose $q^* = 0 : H^n(X) \to H^n(K(Z,n))$. Then there is a homomorphism $\theta : G^n(X) \to \mathcal{E}^{*\prime}(X)$.*

Proof. The condition $q^* = 0 : H^n(X) \to H^n(K(Z,n))$ implies that $q^* = 0 : H^i(X) \to H^i(K(Z,n))$ for $i \le n$. It follows that $\iota^\alpha \in \mathcal{E}^{*\prime}(X)$ by Corollary 3.1. Let α, $\beta \in G^n(X)$ be any elements. We see

$$\alpha\iota^\beta = (\iota^\beta)^*(\alpha) = \iota^*(\alpha) + \beta^* q^*(\alpha) = \alpha$$

by Theorem 3.1 (2). Since

$$(\iota^\alpha)^\beta = \iota^{\alpha+\beta}$$

and

$$\iota^\alpha \iota^\beta = (\iota^\beta)^{\alpha\iota^\beta} = \iota^{\beta+\alpha\iota^\beta} = \iota^{\beta+\alpha},$$

we have

$$\theta(\alpha+\beta) = \iota^{\alpha+\beta} = \iota^{\beta+\alpha} = \iota^\alpha \iota^\beta = \theta(\alpha)\,\theta(\beta).$$

Therefore θ is a homomorphism. □

We define a map

$$DH : H^n(X) = [X, K(Z,n)] \longrightarrow \mathrm{Hom}(\pi_n(X), Z)$$

by $DH(x) = x_\sharp : \pi_n(X) \to \pi_n(K(Z,n))$.

COROLLARY 3.2. *Let X be the homotopy fiber of a map $\gamma : Y \to K(Z, n+1)$ and n is* h-dim X. *Suppose $q^* = 0 : H^n(X) \to H^n(K(Z,n))$. Then there is a homomorphism*

$$\theta' : \mathrm{Ker}\,(DH) \cap G^n(X) \to \mathcal{E}^{*\prime}(X) \cap \mathcal{E}_{\#\infty}(X)$$

given by $\theta'(\alpha) = \iota^\alpha$.

Proof. Let α be an element of Ker (DH). Then $\alpha_\sharp = 0 : \pi_n(X) \to \pi_n(K(Z,n))$. Since $\pi_i(K(Z,n)) = 0$ for all $i \neq n$, we see $q_\sharp \alpha_\sharp = 0 : \pi_i(X) \to \pi_i(X)$ for all i. Therefore $\theta(\alpha) = \iota^\alpha \in \mathcal{E}_{\#\infty}(X)$ by Corollary 3.1 and the proof is completed. □

THEOREM 3.3. *Let X be the homotopy fiber of a map $\gamma : Y \to K(Z, n+1)$ and n is* h-dim X. *Suppose $q : K(Z,n) \to X$ has a left homotopy inverse. If $\theta : G^n(X) \to \mathcal{E}^{*\prime}(X)$ given by $\theta(\alpha) = \iota^\alpha$ is a well-defined homomorphism, then $G^n(X) = 0$.*

Proof. Suppose that $\theta : G^n(X) \to \mathcal{E}^{*\prime}(X)$ given by $\theta(\alpha) = \iota^\alpha$ is a well-defined homomorphism. Then for any $\alpha \in G^n(X)$, we see $\alpha^* q^* = 0 : H^i(X) \to H^i(X)$ for $i \leq n$ by Corollary 3.1. Let $s \in H^n(X)$ be a left homotopy inverse of q. Then for $\iota_{K(Z,n)} \in H^n(K(Z,n))$, we have

$$\alpha = \iota_{K(Z,n)}\alpha = \alpha^*(\iota_{K(Z,n)}) = \alpha^*(sq)^*(\iota_{K(Z,n)}) = \alpha^* q^*(s) = 0.$$

Therefore $G^n(X)$ is trivial. □

COROLLARY 3.3. *Let X be the homotopy fiber of a map $\gamma : Y \to K(Z, n+1)$ and n is* h-dim X. *Suppose the homomorphism $\theta : G^n(X) \to \mathcal{E}^{*\prime}(X)$ given by $\theta(\alpha) = \iota^\alpha$ is well-defined and nontrivial. Then $G^n(X) \neq 0$ and consequently the map $q : K(Z,n) \to X$ does not have a left homotopy inverse .*

To work rationally, one has to define DH_Q instead of DH and one needs the following rational version of Corollary 3.1.

We now assume that all the spaces are *1-connected rational spaces.* We define

$$DH_Q : H^n(X;Q) \longrightarrow \mathrm{Hom}\,(\pi_n(X), Q)$$

by $DH_Q(x) = x_\sharp : \pi_n(X) \to \pi_n(K(Q,n))$ for any $x \in H^n(X;Q) = [X, K(Q,n)]$. Moreover we define

$$G^n_Q(X) = \{\, \alpha \in H^n(X;Q) \mid \alpha \text{ is cocylic} \,\},$$

$$\mathcal{E}^*_Q{}'(X) = \{\, f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X;Q) \to H^i(X;Q) \text{ for any } i \leq N = \text{h-dim}\,(X) \,\}.$$

Corollary 3.1′. (rational case) *Let X be the homotopy fibre of a map $\gamma : Y \to K(Q, n+1)$ and $\alpha \in H^n(X;Q)$. Then*

(1) *$\iota^\alpha \in \mathcal{E}_{\#\infty}(X)$ if and only if $q_\sharp \alpha_\sharp = 0 : \pi_n(X) \to \pi_n(K(Q,n)) \to \pi_n(X)$.*

(2) *Let $\alpha \in G^n_Q(X) \subset H^n(X)$ and $N = \text{h-dim}\,(X)$. Then $\iota^\alpha \in \mathcal{E}^*_Q{}'(X)$ if and only if $\alpha^* q^* = 0 : H^i(X;Q) \to H^i(K(Q,n);Q) \to H^i(X;Q)$ for any $i \leq N$.*

Proof. The proof of Corollary 3.1 can be applied changing the coefficient group Z to G. □

Lemma 3.2. *Suppose that we have a fibration sequence*

$$X \xrightarrow{\ j\ } Y \xrightarrow{\ \gamma\ } K(Q, n+1)$$

with X and Y 1-connected. If $\gamma : Y \to K(Q, n+1)$ is nontrivial, then $j^* : H^n(Y;Q) \to H^n(X;Q)$ is surjective.

Proof. The Lemma is a direct consequence of the "Wang" sequence of the given fibration. The original Wang sequence is an infinite long exact sequence and applies only to fibrations with sphere as a base space. A usual argument using the Serre spectral sequence gives a part of Wang sequence which is enough for our purpose. □

Theorem 3.4. *Let X be the homotopy fibre of a map $\gamma : Y \to K(Q, n+1)$ and assume that $n \equiv 1 \pmod 2$. If γ is nontrivial, then there exists a homomorphism*

$$\theta : G^n_Q(X) \longrightarrow \mathcal{E}^*_Q{}'(X).$$

The homomorphism θ defined above restricts to

$$\theta' : \mathrm{Ker}\,(DH_Q) \cap G^n_Q(X) \to \mathcal{E}^*_Q{}'(X) \cap \mathcal{E}_{\#\infty}(X).$$

Proof. By Lemma 3.2, we see that $q^* = 0 : H^n(X;Q) \to H^n(K(Q,n);Q)$. Since $K(Q,n) \simeq S_Q^n$ when $n \equiv 1 \pmod 2$, we see that $q^* = 0 : H^i(X;Q) \to H^i(K(Q,n);Q)$ for any $i > 0$. Hence by Theorem 3.1 (2), we have $\iota^\alpha \in \mathcal{E}_Q^{*\prime}(X)$ for any $\alpha \in G_Q^n(X)$.

If $\alpha \in \operatorname{Ker}\{DH_Q : H^n(X;Q) \to \operatorname{Hom}(\pi_n(X), Q)\}$, then we see that

$$q_\sharp \alpha_\sharp = 0 : \pi_n(X) \longrightarrow \pi_n(K(Q,n)) \longrightarrow \pi_n(X),$$

and hence $\iota^\alpha \in \mathcal{E}_{\#\infty}(X)$ by Theorem 3.1 (1). □

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Jae-Ryong Kim
Department of Mathematics
Kookmin University
Seoul 136-702, Korea
E-mail: kimjr@kookmin.ac.kr

Nobuyuki Oda
Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180, Japan
E-mail: odanobu@cis.fukuoka-u.ac.jp

Jianzhong Pan
Institute of Mathematics
Academy Sciences of China
Beijing, China
E-mail: pjz@mail.amss.ac.cn

Moo Ha Woo
Department of Mathematics Education
Korea University
Seoul 136-701, Korea
E-mail: woomh@korea.ac.kr