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CONDITIONAL FOURIER-FEYNMAN TRANSFORMS OF VARIATIONS OVER WIENER PATHS IN ABSTRACT WIENER SPACE

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ABSTRACT. In this paper, we evaluate first variations, conditional first variations and conditional Fourier-Feynman transforms of cylinder type functions over Wiener paths in abstract Wiener space and then, investigate relationships among first variation, conditional first variation, Fourier-Feynman transform and conditional Fourier-Feynman transform of those functions. Finally, we derive the conditional Fourier-Feynman transform for the product of cylinder type function which defines the functions in a Banach algebra introduced by Yoo, with n linear factors.

1. Introduction

As mentioned in the earlier paper [9], Fourier-Feynman transforms, first variations and their relationships were developed on the classical and abstract Wiener spaces via several papers [1, 2, 3, 8, 17]. In particular, using a recurrence relation, Chang, Song and Yoo derived the Fourier-Feynman transform for the product of a function in the Fresnel class, with n linear factors ([8]).

On the other hand, in [19], Yoo introduced a Banach algebra $\mathcal{S}''_{\mathbb{B}}$ which corresponds to the Banach algebra $\mathcal{S}''$ introduced by Cameron and Storvick in [4]. In [6], Chang, Cho and Yoo introduced a concept of conditional analytic Feynman integral over Wiener paths in abstract Wiener space and in [7], they evaluated the integrals for the functions in the Banach algebra $\mathcal{S}''_{\mathbb{B}}$. In [5], Chang, Cho, Kim, Song and Yoo introduced a kind of conditional Fourier-Feynman transform and in [9], Cho

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evaluated the conditional Fourier-Feynman transform of functions in the Banach algebra $S''_{\mathbb{B}}$. And also, in [10], he introduced conditional first variation over Wiener paths in abstract Wiener space and in [11], derived relationships between first variations and Fourier-Feynman transforms of several functions on the space.

In this paper, we evaluate first variations, conditional first variations and conditional Fourier-Feynman transforms of cylinder type functions and the functions in $S''_{\mathbb{B}}$ over Wiener paths in abstract Wiener space. And then, we investigate relationships among first variation, conditional first variation, Fourier-Feynman transform and conditional Fourier-Feynman transform of the functions. Finally, we derive the conditional Fourier-Feynman transform for the product of cylinder type function which defines the functions in $S''_{\mathbb{B}}$, with n linear factors.

2. Wiener paths in abstract Wiener space and preliminaries

In this section, we introduce analytic and conditional analytic Feynman integrals over Wiener paths in abstract Wiener space.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and B be a real normed linear space with the Borel σ-field $\mathcal{B}(B)$. For a random variable $X : \Omega \to B$ and integrable function $F : \Omega \to \mathbb{C}$, we have the conditional expectation $E[F|X]$ of F given X on Ω from a well-known probability theory. But there exists a P_X-integrable function ψ on B, which is unique up to P_X-a.e., such that $E[F|X](\omega) = (\psi \circ X)(\omega)$ for $P_{\mathcal{D}}$-a.e. ω in Ω, where P_X denotes the probability distribution of X on $(B, \mathcal{B}(B))$ and $P_{\mathcal{D}} = P|_{\mathcal{D}}$, the restriction of P on $\mathcal{D} \equiv \{X^{-1}(A) : A \in \mathcal{B}(B)\}$. Throughout this paper, we regard the function ψ as the conditional expectation of F given X and without loss of generality, it is also denoted by $E[F|X](\xi)$ for $\xi \in B$.

Let $(\mathcal{H}, \mathbb{B}, m)$ denote an abstract Wiener space ([16]) and $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that e_j's are in $\mathbb{B}^*$, the dual space of real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ of h and y by

$$(h, y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^n \langle h, e_j \rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$. Note that for each $h(\neq 0)$ in $\mathcal{H}$, $(h, \cdot)^\sim$ is mean zero Gaussian with variance $|h|^2$ and $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda (h, y)^\sim$ for all $\lambda \in \mathbb{R}$. Moreover, if

$\{h_1, h_2, \ldots, h_n\}$ is an orthogonal set, then the random variables $(h_j, \cdot)^{\sim}$'s are independent and further, if both h and y are in $\mathcal{H}$, then $(h, y)^{\sim} = \langle h, y\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$ ([14]). Now, we introduce a useful integral formula which appears in the proofs of several results. The proof immediately follows from the above comments.

LEMMA 1. *Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space and let $h \in \mathcal{H}$. Suppose that α is pure imaginary or real. Then we have*

$$\int_{\mathbb{B}} \exp\{\alpha(h, x_1)^{\sim}\} dm(x_1) = \exp\left\{\frac{\alpha^2 |h|^2}{2}\right\}.$$

Let $C_0(\mathbb{B})$ be the space of all continuous paths $x : [0, T] \to \mathbb{B}$ with $x(0) = 0$. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0 \le t \le T} \|x(t)\|_{\mathbb{B}}$ and the Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean zero Gaussian ([18]). A complex-valued measurable function on $C_0(\mathbb{B})$ is said to be Wiener integrable if it is integrable with respect to $m_{\mathbb{B}}$.

DEFINITION 2. Let $F : C_0(\mathbb{B}) \to \mathbb{C}$ be Wiener integrable and X: $C_0(\mathbb{B}) \to B$ be a random variable, where B is a real normed linear space. The conditional expectation $E[F|X]$ of F given X on B is called the conditional Wiener integral of F given X.

Now, we introduce the Wiener integration theorem without proof. For the proof, see [18].

THEOREM 3. (Wiener integration theorem) *Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ and let $f : \mathbb{B}^k \to \mathbb{C}$ be a Borel measurable function. Then, we have*

$$\int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \ldots, x(t_k))\ dm_{\mathbb{B}}(x)$$
$$\stackrel{*}{=} \int_{\mathbb{B}^k} (f \circ T_{\vec{t}})(x_1, x_2, \ldots, x_k)\ d\left(\prod_{j=1}^{k} m\right)(x_1, x_2, \ldots, x_k),$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal, and $T_{\vec{t}}$ is a function from $\mathbb{B}^k$ into itself with

$$T_{\vec{t}}(x_1, x_2, \ldots, x_k)$$
$$= \left(\sqrt{t_1 - t_0}x_1, \sqrt{t_1 - t_0}x_1 + \sqrt{t_2 - t_1}x_2, \ldots, \sum_{j=1}^{k} \sqrt{t_j - t_{j-1}}x_j\right).$$

Let $\tau : 0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0,T]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$(1) \qquad [x](t) = \sum_{j=1}^{k} \chi_{(t_{j-1},t_j]}(t) \left[x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} (x(t_j) - x(t_{j-1})) \right]$$

for $t \in [0,T]$. For $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{B}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given by (1) with replacing $x(t_j)$ by ξ_j for $j = 0, 1, \ldots, k$ ($\xi_0 = 0$). The following lemmas are useful to define and evaluate conditional analytic Wiener and Feynman integrals. For the detailed proof, see [6].

LEMMA 4. *If $\{x(t) : 0 \le t \le T\}$ is the Wiener process on $C_0(\mathbb{B}) \times [0,T]$, then $\{x(t) - [x](t) : t_{j-1} \le t \le t_j\}$, where $j = 1, \ldots, k$, are stochastically independent.*

LEMMA 5. *Let F be integrable on $C_0(\mathbb{B})$ and $X_\tau : C_0(\mathbb{B}) \to \mathbb{B}^k$ be a random variable given by $X_\tau(x) = (x(t_1), \ldots, x(t_k))$. Then for every Borel measurable subset B of $\mathbb{B}^k$, we have*

$$(2) \qquad \int_{X_\tau^{-1}(B)} F(x)\, dm_{\mathbb{B}}(x) = \int_B E[F(x - [x] + [\vec{\xi}])]\, dP_{X_\tau}(\vec{\xi}),$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$.

A subset E of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$ and a property is said to hold scale-invariant almost everywhere (in abbreviation, s-a.e.) if it holds except for a scale-invariant null set. For a function $F : C_0(\mathbb{B}) \to \mathbb{C}$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}} x)$ and let $X_\tau^\lambda(x) = X_\tau(\lambda^{-\frac{1}{2}} x)$ for $\lambda > 0$. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$ and let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. By the definition of conditional Wiener integral (Definition 2) and the equation (2), we have

$$E[F^\lambda | X_\tau^\lambda](\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$$

for $P_{X_\tau^\lambda}$-a.e. $\vec{\xi} \in \mathbb{B}^k$, where $P_{X_\tau^\lambda}$ is the probability distribution of X_τ^λ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$. If, as functions of λ, $E[F^\lambda]$ and $E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$ have analytic extensions $J_\lambda^*(F)$ and $J_\lambda^{**}(F)(\vec{\xi})$, respectively, on $\mathbb{C}_+$, then they are called the analytic Wiener integral and conditional analytic Wiener integral, respectively, of F over $C_0(\mathbb{B})$ with parameter λ, and denoted by

$$E^{anw_\lambda}[F] = J_\lambda^*(F) \quad \text{and} \quad E^{anw_\lambda}[F | X_\tau](\vec{\xi}) = J_\lambda^{**}(F)(\vec{\xi})$$

for $\vec{\xi} \in \mathbb{B}^k$. Moreover, if for non-zero real q,

$$E^{anw_\lambda}[F] \quad \text{and} \quad E^{anw_\lambda}[F|X_\tau](\vec{\xi})$$

have limits as λ approaches to $-iq$ through $\mathbb{C}_+$, then they are called the analytic Feynman integral and conditional analytic Feynman integral, respectively, of F over $C_0(\mathbb{B})$ with parameter q and denoted by

$$E^{anf_q}[F] = \lim_{\lambda \to -iq} E^{anw_\lambda}[F]$$

and

$$E^{anf_q}[F|X_\tau](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_\tau](\vec{\xi}).$$

3. First and conditional first variations over Wiener paths in abstract Wiener space

In this section, we investigate several properties of first and conditional first variations of cylinder type functions over Wiener paths in abstract Wiener space and then, evaluate the analytic Feynman integrals of the variations of the functions.

DEFINITION 6. Let F be a Wiener measurable function on $C_0(\mathbb{B})$ and let $w \in C_0(\mathbb{B})$. The derivative

$$\frac{\partial}{\partial t} F(x+tw)\bigg|_{t=0}$$

for $x \in C_0(\mathbb{B})$, if it exists, is called the first variation of F at x in the direction of w and denoted by

$$\delta_w F(x) = \frac{\partial}{\partial t} F(x+tw)\bigg|_{t=0}.$$

DEFINITION 7. Let F be a Wiener measurable function on $C_0(\mathbb{B})$ and let $F(\cdot + x)$ be integrable for $x \in C_0(\mathbb{B})$. Let $w \in C_0(\mathbb{B})$ and let $X : C_0(\mathbb{B}) \to B$ be a random variable, where B is a real linear normed space. Further, let P_X be the probability distribution of X on $(B, \mathcal{B}(B))$. For $x \in C_0(\mathbb{B})$, if the derivative

$$\frac{\partial}{\partial t} E[F(\cdot + x + tw)|X](\xi)\bigg|_{t=0}$$

exists for P_X-a.e. $\xi \in B$, then it is called the conditional first variation of F given X at x in the direction of w and denoted by

$$\delta_w E[F|X](x,\xi) = \frac{\partial}{\partial t} E[F(\cdot + x + tw)|X](\xi)\bigg|_{t=0}.$$

Let $\mathcal{H}$ be an infinite dimensional separable real Hilbert space and let $\Delta_n = \{(s_1,\ldots,s_n) : 0 < s_1 < \cdots < s_n \le T\}$ for $n \in \mathbb{N}$. For

$$\vec{s} = (s_1,\ldots,s_n) \in \Delta_n \text{ and } \vec{h} = (h_1,\ldots,h_n) \in \mathcal{H}^n, \tag{3}$$

let

$$F_n(x,\vec{s},\vec{h}) = \exp\left\{ i \sum_{j=1}^{n} (h_j, x(s_j))^{\sim} \right\} \tag{4}$$

for $x \in C_0(\mathbb{B})$. Further, for $w \in C_0(\mathbb{B})$ with $w(s_j) \in \mathcal{H}(j = 1,2,\ldots,n)$, let

$$G_n(x,w,\vec{s},\vec{h}) = F_n(x,\vec{s},\vec{h}) \sum_{j=1}^{n} (w(s_j), x(s_j))^{\sim} \tag{5}$$

for $x \in C_0(\mathbb{B})$, where F_n is given by (4).

Let $\mathcal{M}''_n = \mathcal{M}''_n(\Delta_n \times \mathcal{H}^n)$ be the class of all complex Borel measures on $\Delta_n \times \mathcal{H}^n$ and let $\|\mu_n\| = \operatorname{var}\mu_n$, the total variation of μ_n in $\mathcal{M}''_n$. Let $S''_{n,\mathbb{B}} = S''_{n,\mathbb{B}}(\Delta_n \times \mathcal{H}^n)$ be the space of functions of the form

$$H_n(x) = \int_{\Delta_n \times \mathcal{H}^n} F_n(x,\vec{s},\vec{h}) d\mu_n(\vec{s},\vec{h}) \tag{6}$$

for s-a.e. $x \in C_0(\mathbb{B})$, where μ_n is in the class $\mathcal{M}''_n$ and F_n is given by (4). Here, we take $\|H_n\|''_n = \inf\{\|\mu_n\|\}$, where the infimum is taken over all μ_n's so that H_n and μ_n are related by (6). Let $\mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times \mathcal{H}^n)$ be the class of all sequences $\{\mu_n\}$ of measures such that each μ_n is in $\mathcal{M}''_n$ with $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$. Let $S''_{\mathbb{B}} = S''_{\mathbb{B}}(\sum \Delta_n \times \mathcal{H}^n)$ be the space of functions on $C_0(\mathbb{B})$ of the form

$$H(x) = \sum_{n=1}^{\infty} H_n(x), \tag{7}$$

where each H_n is in $S''_{n,\mathbb{B}}$ with $\sum_{n=1}^{\infty} \|H_n\|''_n < \infty$. The norm of H is defined by $\|H\|'' = \inf\{\sum_{n=1}^{\infty} \|H_n\|''_n\}$, where the infimum is taken over all representations of H given by (7)([19]).

In convenience, for $j_1 + j_2 + \cdots + j_k = n$ and $\alpha = 1,\ldots,k$, let

$$\vec{s}_\alpha = (s_{\alpha,1},\ldots,s_{\alpha,j_\alpha}), \quad \vec{h}_\alpha = (h_{\alpha,1},\ldots,h_{\alpha,j_\alpha}) \tag{8}$$

and

$$\vec{s}_n = (\vec{s}_1, \dots, \vec{s}_k) \in \Delta_n, \tag{9}$$

$$\vec{h}_n = (\vec{h}_1, \dots, \vec{h}_k) \in \mathcal{H}^n. \tag{10}$$

LEMMA 8. *Let F_n, $\vec{s}_n$, $\vec{h}_n$ be given by* (4), (9), (10), *respectively, and let w be in $C_0(\mathbb{B})$. Then, for s-a.e. $x \in C_0(\mathbb{B})$, $\delta_w F_n(x, \vec{s}_n, \vec{h}_n)$ exists and it is given by*

$$\delta_w F_n(x, \vec{s}_n, \vec{h}_n) = iF_n(x, \vec{s}_n, \vec{h}_n) \sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (h_{\alpha,\beta}, w(s_{\alpha,\beta}))^\sim. \tag{11}$$

Proof. For t in $\mathbb{R}$, we have

$$\begin{aligned} &\frac{\partial}{\partial t} F_n(x + tw, \vec{s}_n, \vec{h}_n)\Big|_{t=0} \\ &= \frac{\partial}{\partial t} \exp\Big\{ i \sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (h_{\alpha,\beta}, x(s_{\alpha,\beta}) + tw(s_{\alpha,\beta}))^\sim \Big\}\Big|_{t=0} \\ &= iF_n(x, \vec{s}_n, \vec{h}_n) \sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (h_{\alpha,\beta}, w(s_{\alpha,\beta}))^\sim \end{aligned}$$

which is the desired result. □

LEMMA 9. *Let F_n, $\vec{s}_n$, $\vec{h}_n$ be given by* (4), (9), (10), *respectively, and let X_τ be given as in Lemma 5. Moreover, let $t \in \mathbb{R}$ and suppose that*

$$\begin{aligned} 0 < s_{1,1} < \cdots < s_{1,j_1} \le t_1 < s_{2,1} < \cdots < s_{2,j_2} \le t_2 \\ < \cdots \le t_{k-1} < s_{k,1} < \cdots < s_{k,j_k} \le T. \end{aligned}$$

Then, for s-a.e. $x \in C_0(\mathbb{B})$ and $w \in C_0(\mathbb{B})$, $E[F_n(\cdot + x + tw, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and it is given by

$$E[F_n(\cdot + x + tw, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi}) = F_n(x + tw + [\vec{\xi}], \vec{s}_n, \vec{h}_n)\Gamma_n(1, \vec{s}_n, \vec{h}_n),$$

where, for $\lambda \in \mathbb{C}_+^\sim \equiv \{z \in \mathbb{C} : \mathrm{Re}\, z \ge 0\} - \{0\}$, Γ_n is given by

$$\Gamma_n(\lambda, \vec{s}_n, \vec{h}_n) = \exp\Big\{ -\frac{1}{2\lambda} \sum_{\alpha=1}^{k} \sum_{v=1}^{j_\alpha+1} l_{\alpha,v} |A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha)|^2 \Big\} \tag{12}$$

with

$$s_{1,0} = t_0 = 0, \quad s_{k,j_k+1} = t_k = T, \quad s_{\alpha+1,0} = s_{\alpha,j_\alpha+1} = t_\alpha$$

for $\alpha=1,\dots,k-1$ *and*

$$l_{\alpha,v}=s_{\alpha,v}-s_{\alpha,v-1}, \tag{13}$$

$$A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha)=\sum_{\beta=1}^{v-1}\frac{t_{\alpha-1}-s_{\alpha,\beta}}{t_\alpha-t_{\alpha-1}}h_{\alpha,\beta}+\sum_{\beta=v}^{j_\alpha}\frac{t_\alpha-s_{\alpha,\beta}}{t_\alpha-t_{\alpha-1}}h_{\alpha,\beta} \tag{14}$$

for $\alpha=1,\dots,k;\ \ v=1,\dots,j_\alpha+1$.

Proof. For a.e. $\vec{\xi}\in\mathbb{B}^k$, by Lemmas 4 and 5, we have

$$\begin{aligned}
&E[F_n(\cdot+x+tw,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})\\
&=\int_{C_0(\mathbb{B})}\exp\bigg\{i\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(h_{\alpha,\beta},\ y(s_{\alpha,\beta})-[y](s_{\alpha,\beta})+[\vec{\xi}](s_{\alpha,\beta})\\
&\quad+x(s_{\alpha,\beta})+tw(s_{\alpha,\beta}))^{\sim}\bigg\}dm_{\mathbb{B}}(y)\\
&=F_n(x+tw+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\\
&\quad\times\prod_{\alpha=1}^{k}\bigg[\int_{C_0(\mathbb{B})}\exp\bigg\{i\sum_{\beta=1}^{j_\alpha}(h_{\alpha,\beta},\ y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\bigg\}dm_{\mathbb{B}}(y)\bigg].
\end{aligned}$$

Let $l_{\alpha,v}$ be given by (13) and $\vec{y}_\alpha=(y_1,\dots,y_{j_\alpha+1})$ for $\alpha=1,\dots,k$. Using the same process in the proof of [9, Theorem 14], we have

$$\begin{aligned}
&E[F_n(\cdot+x+tw,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})\\
&=F_n(x+tw+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\\
&\quad\times\prod_{\alpha=1}^{k}\bigg[\int_{\mathbb{B}^{j_\alpha+1}}\exp\bigg\{i\sum_{v=1}^{j_\alpha+1}\sqrt{l_{\alpha,v}}(A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),y_v)^{\sim}\bigg\}dm^{j_\alpha+1}(\vec{y}_\alpha)\bigg]\\
&=F_n(x+tw+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)
\end{aligned}$$

by Lemma 1 and Theorem 3, where A_v is given by (14). This completes the proof. □

LEMMA 10. *Let the assumptions and notations be given as in Lemma 9. Then, for s-a.e.* $x\in C_0(\mathbb{B})$ *and* $w\in C_0(\mathbb{B})$, $\delta_wE[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](x,\vec{\xi})$ *exists for a.e.* $\vec{\xi}\in\mathbb{B}^k$ *and it is given by*

$$\delta_wE[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](x,\vec{\xi})=\delta_wF_n(x+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n), \tag{15}$$

where $\delta_wF_n(\cdot,\vec{s}_n,\vec{h}_n)$ *is given by* (11).

Proof. For $t \in \mathbb{R}$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$E[F_n(\cdot + x + tw, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi}) = F_n(x + [\vec{\xi}] + tw, \vec{s}_n, \vec{h}_n)\Gamma_n(1, \vec{s}_n, \vec{h}_n)$$

by Lemma 9. Differentiating both sides of the equality with respect to t and letting $t = 0$, we have the result by Lemma 8. □

THEOREM 11. *Let H_n be given by (6). Moreover, let X_τ be given as in Lemma 5 and suppose that, for s-a.e $w \in C_0(\mathbb{B})$,*

$$\int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^{n} (h_j, w(s_j))^{\sim} \right| d\|\mu_n\|(\vec{s}, \vec{h}) < \infty, \tag{16}$$

where both $\vec{s}$ and $\vec{h}$ are given by (3). Then, for s-a.e. $x \in C_0(\mathbb{B})$ and $w \in C_0(\mathbb{B})$, we have

$$\begin{aligned} &\delta_w E[H_n|X_\tau](x, \vec{\xi}) \\ &= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} \delta_w E[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](x, \vec{\xi}) d\mu_n(\vec{s}_n, \vec{h}_n) \end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$, where each integrand is given by (15) and

$$\begin{aligned} &\Delta'_{n;j_1,\ldots,j_k} \\ &= \{\vec{s}_n \in \Delta_n : 0 < s_{1,1} < \cdots < s_{1,j_1} \le t_1 < s_{2,1} < \cdots < s_{2,j_2} \le t_2 \\ &\qquad < \cdots \le t_{k-1} < s_{k,1} < \cdots < s_{k,j_k} \le T\} \end{aligned} \tag{17}$$

with $\vec{s}_n$ being given by (9).

Proof. For $t \in \mathbb{R}$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, by Lemma 5 and Fubini's theorem, we have

$$\begin{aligned} &E[H_n(\cdot + x + tw)|X_\tau](\vec{\xi}) \\ &= \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathcal{H}^n} F_n(y - [y] + [\vec{\xi}] + x + tw, \vec{s}, \vec{h}) d\mu_n(\vec{s}, \vec{h}) dm_{\mathbb{B}}(y) \\ &= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} F_n(y - [y] + [\vec{\xi}] + x + tw, \vec{s}_n, \vec{h}_n) \end{aligned}$$

$$dm_{\mathbb{B}}(y)d\mu_n(\vec{s}_n,\vec{h}_n)$$
$$= \sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} E[F_n(\cdot+x+tw,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})d\mu_n(\vec{s}_n,\vec{h}_n)$$
$$= \sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} F_n(x+[\vec{\xi}]+tw,\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)$$
$$d\mu_n(\vec{s}_n,\vec{h}_n),$$

where the last equality follows from Lemma 9. Differentiating with respect to t and letting $t=0$, by (16) and [12, Theorem 2.27], we have the result from Lemma 10. □

Now, by Theorem 11 and the dominated convergence theorem, we have the following theorem.

THEOREM 12. *Let H be given by (7). Moreover, let X_τ be given as in Lemma 5. Suppose that, for s-a.e. $w\in C_0(\mathbb{B})$,*

$$\sum_{n=1}^{\infty}\int_{\Delta_n\times\mathcal{H}^n}\left|\sum_{j=1}^{n}(h_j,w(s_j))^\sim\right|d\|\mu_n\|(\vec{s},\vec{h})<\infty, \tag{18}$$

where $\vec{s}$ and $\vec{h}$ are given by (3). Then, for s-a.e. $x,w\in C_0(\mathbb{B})$, we have

$$\delta_w E[H|X_\tau](x,\vec{\xi}) = \sum_{n=1}^{\infty}\delta_w E[H_n|X_\tau](x,\vec{\xi})$$

for a.e. $\vec{\xi}\in\mathbb{B}^k$, where each summand is given as in Theorem 11.

THEOREM 13. *Let the assumptions and notations be given as in Theorem 11. Moreover, let $q\in\mathbb{R}-\{0\}$. Then, for s-a.e. $w\in C_0(\mathbb{B})$, we have*

$$E^{anf_q}[\delta_w E[H_n|X_\tau](\cdot,\vec{\xi})]$$
$$= \sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n}\delta_w F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)$$
$$\times\exp\left\{-\frac{i}{2q}\sum_{u=1}^{k}\sum_{v=1}^{j_u}\gamma_{u,v}\left|\sum_{\beta=v}^{j_u}h_{u,\beta}+\sum_{\alpha=u+1}^{k}\sum_{\beta=1}^{j_\alpha}h_{\alpha,\beta}\right|^2\right\}d\mu_n(\vec{s}_n,\vec{h}_n)$$

for a.e. $\vec{\xi}\in\mathbb{B}^k$, where $\vec{s}_n$, $\vec{h}_n$, $\delta_wF_n(\cdot,\vec{s}_n,\vec{h}_n)$, Γ_n and $\Delta'_{n;j_1,\ldots,j_k}$ are given by (9), (10), (11), (12) and (17), respectively, and for $u=1,\ldots,k-1$,

$$s_{1,0}=0,\quad s_{u+1,0}=s_{u,j_u},$$

and for $u=1,\dots,k\,;\ v=1,\dots,j_u$,

$$\gamma_{u,v}=s_{u,v}-s_{u,v-1}. \tag{19}$$

Proof. For $\lambda>0$ and a.e. $\vec{\xi}\in\mathbb{B}^k$, by Lemma 10, Theorem 11 and Fubini's theorem, we have

$$\begin{aligned}
&\int_{C_0(\mathbb{B})}\delta_w E[H_n|X_\tau](\lambda^{-\frac12}x,\vec{\xi})dm_{\mathbb{B}}(x)\\
&=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}\int_{C_0(\mathbb{B})}\delta_w F_n(\lambda^{-\frac12}x+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\\
&\quad\times\Gamma_n(1,\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(x)d\mu_n(\vec{s}_n,\vec{h}_n)\\
&=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}\delta_w F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)\\
&\quad\times\int_{C_0(\mathbb{B})}\exp\Big\{i\lambda^{-\frac12}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(h_{\alpha,\beta},x(s_{\alpha,\beta}))^{\sim}\Big\}dm_{\mathbb{B}}(x)d\mu_n(\vec{s}_n,\vec{h}_n)\\
&=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}\delta_w F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)\\
&\quad\times\int_{\mathbb{B}^n}\exp\Big\{i\lambda^{-\frac12}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}\Big(h_{\alpha,\beta},\sum_{u=1}^{\alpha-1}\sum_{v=1}^{j_u}\sqrt{\gamma_{u,v}}\ x_{u,v}\\
&\quad+\sum_{v=1}^{\beta}\sqrt{\gamma_{\alpha,v}}\ x_{\alpha,v}\Big)^{\sim}\Big\}dm^n(\vec{x}_n)d\mu_n(\vec{s}_n,\vec{h}_n)
\end{aligned}$$

by Theorem 3, where $\vec{x}_n=(x_{1,1},\dots,x_{1,j_1},x_{2,1},\dots,x_{2,j_2},\dots,x_{k,1},\dots,x_{k,j_k})$ and $\gamma_{u,v}$ is given by (19). Then we have

$$\begin{aligned}
&\int_{C_0(\mathbb{B})}\delta_w E[H_n|X_\tau](\lambda^{-\frac12}x,\vec{\xi})dm_{\mathbb{B}}(x)\\
&=\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}\delta_w F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)\\
&\quad\times\int_{\mathbb{B}^n}\exp\Big\{i\lambda^{-\frac12}\sum_{u=1}^{k}\sum_{v=1}^{j_u}\sqrt{\gamma_{u,v}}\Big(\sum_{\beta=v}^{j_u}h_{u,\beta}+\sum_{\alpha=u+1}^{k}\sum_{\beta=1}^{j_\alpha}h_{\alpha,\beta},\ x_{u,v}\Big)^{\sim}\Big\}
\end{aligned}$$

$$dm^n(\vec{x}_n)d\mu_n(\vec{s}_n,\vec{h}_n)$$
$$= \sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\dots,j_k}\times\mathcal{H}^n}\delta_w F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(1,\vec{s}_n,\vec{h}_n)$$
$$\times\exp\bigg\{-\frac{1}{2\lambda}\sum_{u=1}^{k}\sum_{v=1}^{j_u}\gamma_{u,v}\bigg|\sum_{\beta=v}^{j_u}h_{u,\beta}+\sum_{\alpha=u+1}^{k}\sum_{\beta=1}^{j_\alpha}h_{\alpha,\beta}\bigg|^2\bigg\}d\mu_n(\vec{s}_n,\vec{h}_n)$$

by Lemma 1. By Morera's theorem, we have the analytic extension for $\lambda\in\mathbb{C}_+$. Now, letting $\lambda\to -iq$ through $\mathbb{C}_+$, we have the theorem by the dominated convergence theorem. □

By Theorems 12 and 13, we have the following theorem.

THEOREM 14. *Let the assumptions and notations be given as in Theorem 12. Let $q\in\mathbb{R}-\{0\}$. Then, for s-a.e. $w\in C_0(\mathbb{B})$, we have*

$$E^{anf_q}[\delta_w E[H|X_\tau](\cdot,\vec{\xi})]=\sum_{n=1}^{\infty}E^{anf_q}[\delta_w E[H_n|X_\tau](\cdot,\vec{\xi})]$$

for a.e. $\vec{\xi}\in\mathbb{B}^k$, where each summand is given as in Theorem 13.

THEOREM 15. *Let the assumptions and notations be given as in Lemma 9. Moreover, let G_n be given by (5) and let q be a non-zero real number. Then, for a.e. $\vec{\xi}\in\mathbb{B}^k$, we have*

$$E^{anf_q}[G_n(\cdot,w,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})$$
$$= E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})\bigg[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),[\vec{\xi}](s_{\alpha,\beta}))^\sim$$
$$-\frac{1}{q}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1}l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\bigg],$$

where $\vec{w}_\alpha=(w(s_{\alpha,1}),\dots,w(s_{\alpha,j_\alpha}))$ for $\alpha=1,\dots,k$ and

$$E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})=F_n([\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(-iq,\vec{s}_n,\vec{h}_n). \tag{20}$$

Proof. Using similar method given as in the proof Lemma 9, for $\lambda>0$ and for a.e. $\vec{\xi}\in\mathbb{B}^k$, we have

$$E[F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)] \tag{21}$$

$$= \int_{C_0(\mathbb{B})} F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}], \vec{s}_n, \vec{h}_n)dm_{\mathbb{B}}(y)$$
$$= F_n([\vec{\xi}], \vec{s}_n, \vec{h}_n)$$
$$\times \prod_{\alpha=1}^{k}\left[\int_{\mathbb{B}^{j_\alpha+1}} \exp\left\{i\lambda^{-\frac{1}{2}}\sum_{v=1}^{j_\alpha+1}\sqrt{l_{\alpha,v}}(A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),y_v)^{\sim}\right\}dm^{j_\alpha+1}(\vec{y}_\alpha)\right],$$

where $\vec{y}_\alpha = (y_1, \ldots, y_{j_\alpha+1})$ for $\alpha = 1, \ldots, k$. By Cameron-Martin translation theorem ([15]), for any real t, we also have

$$E[F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}], \vec{s}_n, \vec{h}_n)]$$
$$= F_n([\vec{\xi}], \vec{s}_n, \vec{h}_n)\prod_{\alpha=1}^{k}\left[\exp\left\{-\frac{t^2}{2}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2\right\}\times\right.$$
$$\int_{\mathbb{B}^{j_\alpha+1}}\exp\left\{i\lambda^{-\frac{1}{2}}\sum_{v=1}^{j_\alpha+1}(\sqrt{l_{\alpha,v}}A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ y_v+t\sqrt{l_{\alpha,v}}A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha))^{\sim}\right\}$$
$$\left.\times\exp\left\{-t\sum_{v=1}^{j_\alpha+1}(\sqrt{l_{\alpha,v}}A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha),\ y_v)^{\sim}\right\}dm^{j_\alpha+1}(\vec{y}_\alpha)\right]$$
$$= F_n([\vec{\xi}], \vec{s}_n, \vec{h}_n)\prod_{\alpha=1}^{k}\left[\exp\left\{-\frac{t^2}{2}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2\right\}\right.$$
$$\times\exp\left\{it\lambda^{-\frac{1}{2}}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\right\}$$
$$\times\int_{\mathbb{B}^{j_\alpha+1}}\exp\left\{i\lambda^{-\frac{1}{2}}\sum_{v=1}^{j_\alpha+1}(\sqrt{l_{\alpha,v}}A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ y_v)^{\sim}\right\}$$
$$\left.\times\exp\left\{-t\sum_{v=1}^{j_\alpha+1}(\sqrt{l_{\alpha,v}}A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha),\ y_v)^{\sim}\right\}dm^{j_\alpha+1}(\vec{y}_\alpha)\right].$$

By Theorem 3 (Wiener integration theorem) and Lemma 4, we have

$$E[F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}], \vec{s}_n, \vec{h}_n)] \tag{22}$$
$$=\exp\left\{-\frac{t^2}{2}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2\right\}$$
$$\times\exp\left\{it\lambda^{-\frac{1}{2}}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\right\}$$

$$\times \int_{C_0(\mathbb{B})} F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)$$
$$\times \exp\Big\{-t\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\, y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\Big\}dm_{\mathbb{B}}(y).$$

Now, using similar method given as in the proof of Lemma 9, for $\delta \in \mathbb{R}$, we have

$$\int_{C_0(\mathbb{B})}\Big\{\delta\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\, y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\Big\}dm_{\mathbb{B}}(y) \tag{23}$$
$$= \exp\Big\{\frac{\delta^2}{2}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2\Big\} < \infty$$

by Lemma 1. Take $\epsilon > 0$, arbitrarily. Since $|\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),y(s_{\alpha,\beta}) -[y](s_{\alpha,\beta}))^{\sim}| < \exp\{|\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}|\}$, both $\exp\{-t\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\}$ and $[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}]\exp\{-t\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\}$ are integrable functions of y for $|t| < \epsilon$ by (23). Moreover, the integrals are independent of t. Differentiating both sides of (22) with respect to t, by [12, Theorem 2.27], we have

$$0 = \exp\Big\{-\frac{t^2}{2}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2\Big\}$$
$$\times \exp\Big\{it\lambda^{-\frac{1}{2}}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\Big\}$$
$$\times \int_{C_0(\mathbb{B})}\Big[-t\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}|A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)|^2$$
$$+ i\lambda^{-\frac{1}{2}}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle$$
$$-\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\, y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\Big]$$
$$\times F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)$$

$$\times \exp\left\{-t\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\ y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\right\}dm_{\mathbb{B}}(y)$$

and letting $t=0$, we also have

$$\int_{C_0(\mathbb{B})}\left[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\ y(s_{\alpha,\beta})-[y](s_{\alpha,\beta}))^{\sim}\right]$$
$$\times F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(y)$$

$$= i\lambda^{-\frac{1}{2}}\left[\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1}l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\right]$$
$$\times\int_{C_0(\mathbb{B})}F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(y).$$

Multiplying $\lambda^{-\frac{1}{2}}$ to both sides of the equality, we have

$$\int_{C_0(\mathbb{B})}\left[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\ \lambda^{-\frac{1}{2}}(y(s_{\alpha,\beta})-[y](s_{\alpha,\beta})))^{\sim}\right] \tag{24}$$
$$\times F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(y)$$
$$= i\lambda^{-1}\left[\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1}l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\right]$$
$$\times\int_{C_0(\mathbb{B})}F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(y)$$

and hence

$$E[G_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],w,\vec{s}_n,\vec{h}_n)]$$
$$= E[F_n(\lambda^{-\frac{1}{2}}(y-[y])+[\vec{\xi}],\vec{s}_n,\vec{h}_n)]\left[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),[\vec{\xi}](s_{\alpha,\beta}))^{\sim}\right.$$
$$\left.+\,i\lambda^{-1}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1}l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\right].$$

By Morera's theorem, we have the analytic extension for $\lambda\in\mathbb{C}_+$. Letting $\lambda\to -iq$ through $\mathbb{C}_+$, we have the result. (20) follows from (21), Lemma 1 and Morera's theorem. □

In the proof of Theorem 15, taking $\lambda = 1$, we have the following corollary by (2).

COROLLARY 16. *Let the assumptions and notations be given as in Theorem 15. Then, for a.e.* $\vec{\xi} \in \mathbb{B}^k$, *we have*

$$\begin{aligned}
&E[G_n(\cdot, w, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi}) \\
&= E[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi})\Big[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}), [\vec{\xi}](s_{\alpha,\beta}))^{\sim} \\
&\quad + i\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha),\ A_v(\tau, \vec{s}_\alpha, \vec{w}_\alpha)\rangle\Big].
\end{aligned}$$

The following theorem is an immediate consequence of Theorem 15.

THEOREM 17. *Let the assumptions and notations be given as in Theorem 15. Let* $\mu_n \in \mathcal{M}''_n$ *and* $w(s) \in \mathcal{H}$ *for all* $s \in [0, T]$. *Moreover, suppose that*

$$\begin{aligned}
&\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n}\Big[\Big|\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha), A_v(\tau, \vec{s}_\alpha, \vec{w}_\alpha)\rangle\Big| \\
&+\Big|\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}), [\vec{\xi}](s_{\alpha,\beta}))^{\sim}\Big|\Big]d\|\mu_n\|(\vec{s}_n, \vec{h}_n) < \infty
\end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$, *where* $\Delta'_{n;j_1,\ldots,j_k}$ *is given by (17). Then, for a.e.* $\vec{\xi} \in \mathbb{B}^k$, *we have*

$$\begin{aligned}
&\sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} E^{anf_q}[G_n(\cdot, w, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi})d\mu_n(\vec{s}_n, \vec{h}_n) \\
&= \sum_{j_1+\cdots+j_k=n}\int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} E^{anf_q}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](\vec{\xi})\Big[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}), \\
&[\vec{\xi}](s_{\alpha,\beta}))^{\sim} - \frac{1}{q}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha),\ A_v(\tau, \vec{s}_\alpha, \vec{w}_\alpha)\rangle\Big] \\
&d\mu_n(\vec{s}_n, \vec{h}_n)
\end{aligned}$$

with the existence of each integral of both sides of the equality.

4. Conditional Fourier-Feynman transforms of variations

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let K_n and K be measurable functions such that, for each $\gamma > 0$,

$$\lim_{n\to\infty} \int_{C_0(\mathbb{B})} |K_n(\gamma x) - K(\gamma x)|^{p'} dm_{\mathbb{B}}(x) = 0.$$

Then we write

$$\underset{n\to\infty}{\text{l.i.m.}}(w_s^{p'})(K_n) \approx K$$

and call K the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Now, we define Fourier-Feynman transform and conditional Fourier-Feynman transform of functions on $C_0(\mathbb{B})$.

DEFINITION 18. Let F be defined on $C_0(\mathbb{B})$ and let X_τ be given as in Lemma 5. For $\lambda \in \mathbb{C}_+$ and for s-a.e. $y \in C_0(\mathbb{B})$, let

$$T_\lambda(F)(y) = E^{anw_\lambda}[F(\cdot + y)]$$

and

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = E^{anw_\lambda}[F(\cdot + y)|X_\tau](\vec{\xi})$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ if they exist. For a non-zero real q and for s-a.e. $y \in C_0(\mathbb{B})$, we define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ and L_1 analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_\tau]$ of F by the formulas

$$T_q^{(1)}(F)(y) = E^{anf_q}[F(\cdot + y)]$$

and

$$T_q^{(1)}[F|X_\tau](y, \vec{\xi}) = E^{anf_q}[F(\cdot + y)|X_\tau](\vec{\xi})$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ if they exist. Moreover, for $1 < p \le \infty$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ and L_p analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_\tau]$ of F by the formulas

$$T_q^{(p)}(F) \approx \underset{\lambda\to -iq}{\text{l.i.m.}}(w_s^{p'})(T_\lambda(F))$$

and

$$T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}) \approx \underset{\lambda\to -iq}{\text{l.i.m.}}(w_s^{p'})(T_\lambda[F|X_\tau](\cdot, \vec{\xi}))$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$, where λ approaches to $-iq$ through $\mathbb{C}_+$.

LEMMA 19. *Let the assumptions and notations be given as in Lemma 9. Let q be a non-zero real number and let $1 \le p \le \infty$. Then, for s-a.e. $y \in C_0(\mathbb{B})$, we have*

$$\begin{aligned}(25)\qquad & T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](y,\vec{\xi}) \\ &= F_n(y+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(-iq,\vec{s}_n,\vec{h}_n) \\ &= F_n(y,\vec{s}_n,\vec{h}_n)E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})\end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ with the existence of the conditional Fourier-Feynman transform.

Proof. Using similar method given as in the proof of Lemma 9, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$T_\lambda[F_n|X_\tau](y,\vec{\xi}) = F_n(y+[\vec{\xi}],\vec{s}_n,\vec{h}_n)\Gamma_n(\lambda,\vec{s}_n,\vec{h}_n)$$

by Morera's theorem. Let $T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](y,\vec{\xi})$ be given by (25). For $p=1$, we have the result, letting $\lambda \to -iq$ through $\mathbb{C}_+$. For $1 < p \le \infty$ with $\frac{1}{p}+\frac{1}{p'}=1$ and for $\gamma > 0$, we have

$$\begin{aligned}&\int_{C_0(\mathbb{B})} |T_\lambda[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\gamma y,\vec{\xi}) \\ &\qquad - T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\gamma y,\vec{\xi})|^{p'} dm_{\mathbb{B}}(y) \\ &\le \int_{C_0(\mathbb{B})} |\Gamma_n(\lambda,\vec{s}_n,\vec{h}_n) - \Gamma_n(-iq,\vec{s}_n,\vec{h}_n)|^{p'} dm_{\mathbb{B}}(y)\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$. This proves the first equality in (25). The second equality follows from (20). □

By Lemmas 8, 10 and 19, we have the following theorem and corollary.

THEOREM 20. *Let the assumptions and notations be given as in Lemma 19. Then, for s-a.e. $w, y \in C_0(\mathbb{B})$, we have*

$$\begin{aligned}& T_q^{(p)}[\delta_w E[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2) \\ &= \delta_w E[T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\cdot,\vec{\xi}_2)|X_\tau](y,\vec{\xi}_1) \\ &= \delta_w E[T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2) \\ &= iF_n(y+[\vec{\xi}_1]+[\vec{\xi}_2],\vec{s}_n,\vec{h}_n)\Gamma_n\left(\frac{-iq}{1-iq},\vec{s}_n,\vec{h}_n\right) \\ &\quad \times \sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(h_{\alpha,\beta}, w(s_{\alpha,\beta}))^\sim\end{aligned}$$

for a.e. $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$.

COROLLARY 21. *Let the assumptions and notations be given as in Lemma 19. Then, for s-a.e.* $w, y \in C_0(\mathbb{B})$, *we have*

$$T_q^{(p)}[\delta_w F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi}) = \delta_w(T_q^{(p)}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](\cdot, \vec{\xi}))(y)$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$.

COROLLARY 22. *Let the assumptions and notations be given as in Lemma 19. Then, for s-a.e.* $w, y \in C_0(\mathbb{B})$ *and for a.e.* $\vec{\xi} \in \mathbb{B}^k$, *we have*

$$T_q^{(p)}(\delta_w E[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](\cdot, \vec{\xi}))(y) = \delta_w E[T_q^{(p)}(F_n(\cdot, \vec{s}_n, \vec{h}_n))|X_\tau](y, \vec{\xi}).$$

Proof. It is not difficult to show

$$(26)\quad T_q^{(p)}(F_n(\cdot, \vec{s}_n, \vec{h}_n))(y) = F_n(y, \vec{s}_n, \vec{h}_n) \exp\left\{-\frac{i}{2q}\sum_{u=1}^{k}\sum_{v=1}^{j_u}\gamma_{u,v}\left|\sum_{\beta=v}^{j_u} h_{u,\beta} + \sum_{\alpha=u+1}^{k}\sum_{\beta=1}^{j_\alpha} h_{\alpha,\beta}\right|^2\right\}$$

for s-a.e. $y \in C_0(\mathbb{B})$, where $\gamma_{u,v}$ is given by (19). Now, for a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$\begin{aligned}
&\delta_w E[T_q^{(p)}(F_n(\cdot, \vec{s}_n, \vec{h}_n))|X_\tau](y, \vec{\xi}) \\
&= \delta_w F_n(y + [\vec{\xi}], \vec{s}_n, \vec{h}_n)\Gamma_n(1, \vec{s}_n, \vec{h}_n) \\
&\quad \times \exp\left\{-\frac{i}{2q}\sum_{u=1}^{k}\sum_{v=1}^{j_u}\gamma_{u,v}\left|\sum_{\beta=v}^{j_u} h_{u,\beta} + \sum_{\alpha=u+1}^{k}\sum_{\beta=1}^{j_\alpha} h_{\alpha,\beta}\right|^2\right\} \\
&= T_q^{(p)}(\delta_w E[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](\cdot, \vec{\xi}))(y)
\end{aligned}$$

by (15), where Γ_n is given by (12). □

From [9, Theorems 14, 15], we have the following theorem.

THEOREM 23. *Let* H_n *and* H *be given by* (6) *and* (7), *respectively. Let the assumptions and notations be given as in Lemma 19. Then, for s-a.e.* $y \in C_0(\mathbb{B})$, *we have*

$$\begin{aligned}
&T_q^{(p)}[H_n|X_\tau](y, \vec{\xi}) \\
&= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} T_q^{(p)}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi})d\mu_n(\vec{s}_n, \vec{h}_n)
\end{aligned}$$

and

$$T_q^{(p)}[H|X_\tau](y, \vec{\xi}) = \sum_{n=1}^{\infty} T_q^{(p)}[H_n|X_\tau](y, \vec{\xi})$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ *with the existences of the conditional Fourier-Feynman transforms, where* $\Delta'_{n;j_1,\ldots,j_k}$ *and* $T_q^{(p)}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau]$ *are given by* (17) *and* (25), *respectively.*

By Theorems 11, 12, 20 and 23, we have the following theorem.

THEOREM 24. *Let the assumptions and notations be given as in Theorem 23. Moreover, suppose that* (18) *holds. Then, for s-a.e.* $w, y \in C_0(\mathbb{B})$, *we have*

$$\begin{aligned}(27)\qquad & T_q^{(p)}[\delta_w E[H_n|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2)\\ &= \delta_w E[T_q^{(p)}[H_n|X_\tau](\cdot,\vec{\xi}_2)|X_\tau](y,\vec{\xi}_1)\\ &= \delta_w E[T_q^{(p)}[H_n|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2)\end{aligned}$$

and

$$\begin{aligned}& T_q^{(p)}[\delta_w E[H|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2)\\ &= \delta_w E[T_q^{(p)}[H|X_\tau](\cdot,\vec{\xi}_2)|X_\tau](y,\vec{\xi}_1)\\ &= \delta_w E[T_q^{(p)}[H|X_\tau](\cdot,\vec{\xi}_1)|X_\tau](y,\vec{\xi}_2)\end{aligned}$$

for a.e. $(\vec{\xi}_1,\vec{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k$.

REMARK 1. For the result in (27), the assumption (18) is surplus. In fact, it suffices to assume that (16) holds.

COROLLARY 25. *Let the assumptions and notations be given as in Theorem 24. Then, for s-a.e.* $w, y \in C_0(\mathbb{B})$, *we have*

$$T_q^{(p)}[\delta_w H_n|X_\tau](y,\vec{\xi}) = \delta_w(T_q^{(p)}[H_n|X_\tau](\cdot,\vec{\xi}))(y)$$

and

$$T_q^{(p)}[\delta_w H|X_\tau](y,\vec{\xi}) = \delta_w(T_q^{(p)}[H|X_\tau](\cdot,\vec{\xi}))(y)$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$.

Proof. It is not difficult to show

$$\delta_w H_n(y) = \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k}\times\mathcal{H}^n} \delta_w F_n(y,\vec{s}_n,\vec{h}_n)\,d\mu_n(\vec{s}_n,\vec{h}_n)$$

and

$$\delta_w H(y) = \sum_{n=1}^{\infty} \delta_w H_n(y)$$

for $y \in C_0(\mathbb{B})$, where $\delta_w F_n(y, \vec{s}_n, \vec{h}_n)$ is given by (11). The first result follows from Lemmas 8 and 19, Corollary 21 and the first part in Theorem 23. The second result follows from the first result and the second part in Theorem 23. □

COROLLARY 26. *Let the assumptions and notations be given as in Theorem 24. Then, for s-a.e. $w, y \in C_0(\mathbb{B})$ and for a.e. $\vec{\xi} \in \mathbb{B}^k$, we have*

$$T_q^{(p)}(\delta_w E[H_n|X_\tau](\cdot, \vec{\xi}))(y) = \delta_w E[T_q^{(p)}(H_n)|X_\tau](y, \vec{\xi})$$

and

$$T_q^{(p)}(\delta_w E[H|X_\tau](\cdot, \vec{\xi}))(y) = \delta_w E[T_q^{(p)}(H)|X_\tau](y, \vec{\xi}).$$

Proof. We can easily show

$$T_q^{(p)}(H_n)(y) = \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} T_q^{(p)}(F_n(\cdot, \vec{s}_n, \vec{h}_n))(y) d\mu_n(\vec{s}_n, \vec{h}_n)$$

and

$$T_q^{(p)}(H)(y) = \sum_{n=1}^{\infty} T_q^{(p)}(H_n)(y)$$

for s-a.e. $y \in C_0(\mathbb{B})$, where $T_q^{(p)}(F_n(\cdot, \vec{s}_n, \vec{h}_n))(y)$ is given by (26). The first result follows from Theorem 11 and Corollary 22. The second result follows from the first result and Theorem 12. □

THEOREM 27. *Let the assumptions and notations be given as in Theorem 15. Let $1 \le p \le \infty$. Then, for s-a.e $y \in C_0(\mathbb{B})$, we have*

$$\begin{aligned} &T_q^{(p)}[G_n(\cdot, w, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi}) \\ &= T_q^{(p)}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi}) \Bigg[\sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (w(s_{\alpha,\beta}),\ y(s_{\alpha,\beta}) + [\vec{\xi}](s_{\alpha,\beta}))^\sim \\ &\quad - \frac{1}{q} \sum_{\alpha=1}^{k} \sum_{v=1}^{j_\alpha+1} l_{\alpha,v} \langle A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha),\ A_v(\tau, \vec{s}_\alpha, \vec{w}_\alpha) \rangle \Bigg] \end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$, where $T_q^{(p)}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi})$ is given by (25).

Proof. With replacing y by x in (24) and then, multiplying $F_n(y,\vec{s}_n,\vec{h}_n)$ to both sides of the equality, we have

$$\begin{aligned}
&\int_{C_0(\mathbb{B})}\Big[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\ \lambda^{-\frac{1}{2}}(x(s_{\alpha,\beta})-[x](s_{\alpha,\beta})))^\sim\Big]\\
&\times F_n(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}]+y,\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(x)\\
&= i\lambda^{-1}\Big[\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\Big]\\
&\times\int_{C_0(\mathbb{B})}F_n(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}]+y,\vec{s}_n,\vec{h}_n)dm_{\mathbb{B}}(x)
\end{aligned}$$

and hence

$$\begin{aligned}
&E[G_n(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}]+y,w,\vec{s}_n,\vec{h}_n)]\\
&= E[F_n(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}]+y,\vec{s}_n,\vec{h}_n)]\Big[\sum_{\alpha=1}^{k}\sum_{\beta=1}^{j_\alpha}(w(s_{\alpha,\beta}),\ y(s_{\alpha,\beta})\\
&+[\vec{\xi}](s_{\alpha,\beta}))^\sim+i\lambda^{-1}\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),\ A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\Big].
\end{aligned}$$

By Morera's theorem, we have the analytic extension for $\lambda\in\mathbb{C}_+$. Now, letting $\lambda\to -iq$ through $\mathbb{C}_+$, we have the theorem from Lemma 19. □

REMARK 2. The formula in Theorem 27 can be rewritten by

$$\begin{aligned}
&T_q^{(p)}[G_n(\cdot,w,\vec{s}_n,\vec{h}_n)|X_\tau](y,\vec{\xi})\\
&= G_n(y,w,\vec{s}_n,\vec{h}_n)E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})+G_n([\vec{\xi}],w,\vec{s}_n,\vec{h}_n)\\
&\times F_n(y,\vec{s}_n,\vec{h}_n)\Gamma_n(-iq,\vec{s}_n,\vec{h}_n)\\
&-\frac{1}{q}\Big[\sum_{\alpha=1}^{k}\sum_{v=1}^{j_\alpha+1} l_{\alpha,v}\langle A_v(\tau,\vec{s}_\alpha,\vec{h}_\alpha),A_v(\tau,\vec{s}_\alpha,\vec{w}_\alpha)\rangle\Big]\\
&\times F_n(y,\vec{s}_n,\vec{h}_n)E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi}),
\end{aligned}$$

where Γ_n and $E^{anf_q}[F_n(\cdot,\vec{s}_n,\vec{h}_n)|X_\tau](\vec{\xi})$ are given by (12) and (20), respectively.

By Theorems 17 and 27, we have the final result.

THEOREM 28. *Let the assumptions and notations be given as in Theorem 17. Let $1 \le p \le \infty$ and suppose that*

$$\int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^{n} (w(s_j), y(s_j))^{\sim} \right| d\|\mu_n\|(\vec{s}, \vec{h}) < \infty$$

for s-a.e. $y \in C_0(\mathbb{B})$, where $\vec{s}$ and $\vec{h}$ are given by (3). Then, for s-a.e $y \in C_0(\mathbb{B})$, we have

$$\begin{aligned}
&\sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} T_q^{(p)}[G_n(\cdot, w, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi}) d\mu_n(\vec{s}_n, \vec{h}_n) \\
&= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n;j_1,\ldots,j_k} \times \mathcal{H}^n} T_q^{(p)}[F_n(\cdot, \vec{s}_n, \vec{h}_n)|X_\tau](y, \vec{\xi}) \\
&\quad \times \left[\sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (w(s_{\alpha,\beta}),\ y(s_{\alpha,\beta}) + [\vec{\xi}](s_{\alpha,\beta}))^{\sim} \right. \\
&\quad \left. - \frac{1}{q} \sum_{\alpha=1}^{k} \sum_{v=1}^{j_\alpha+1} l_{\alpha,v} \langle A_v(\tau, \vec{s}_\alpha, \vec{h}_\alpha),\ A_v(\tau, \vec{s}_\alpha, \vec{w}_\alpha) \rangle \right] d\mu_n(\vec{s}_n, \vec{h}_n)
\end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ with the existence of each integral of both sides of the equality.

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