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PL FIBRATORS AMONG PRODUCTS OF HOPFIAN MANIFOLDS

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ABSTRACT. Suppose that F is a closed t-aspherical PL n-manifold with finite, sparsely abelian $\pi_1(F)$ and A is a closed aspherical PL m-manifold with hopfian, normally cohopfian $\pi_1(A)$. If $\chi(F) \neq 0 \neq \chi(A)$, then $F \times A$ is a codimension-$(t+1)$ PL fibrator.

1. Introduction

Approximate fibrations form a useful class of maps, in part, because they provide computable relationships involving the domain, image and homotopy fiber.

Fix a closed, connected n-manifold N. A proper PL map $p : M \to B$ from an $(n+k)$-manifold M into a polyhedron B is *N-like* if each fiber collapses to an n-complex homotopy equivalent to N; N is a *codimension-k PL fibrator* if, for every N-like map $p : M \to B$, where M is a PL $(n+k)$-manifold, p is an approximate fibration. Codimension-k PL fibrators are necessarily codimension-$(k-1)$ PL fibrators as well. Codimension-k PL fibrators are abundant. In this note we find new codimension-k PL fibrators among products of hopfian manifolds.

2. Preliminaries

A group Γ is *hopfian* if every epimorphism $\Gamma \to \Gamma$ is an automorphism. A group Γ is *hyperhopfian* if every endomorphism $\psi : \Gamma \to \Gamma$ with normal image and cyclic cokernel is necessarily an automorphism. A group Γ is called *normally co-hopfian* if every monomorphism of Γ that image of Γ is a normal subgroup of Γ is an automorphism, i.e., it is not isomorphic

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to any of its proper normal subgroups. The question whether the direct product of normally cohopfian groups are again normally cohopfian is still open. Say that a group Γ is *sparsely abelian* if it contains no nontrivial abelian normal subgroup A such that Γ/A is isomorphic to a normal subgroup of Γ. Groups Γ that are both sparsely abelian and normally cohopfian have the useful feature that every homomorphism $\Gamma \to \Gamma$ with, at worst, abelian kernel necessarily is an automorphism. For brevity a group Γ which is both normally cohopfian and sparsely abelian will be said to have Property NCSA. The fundamental groups of most connected sums of manifolds have Property NCSA (See [5]).

So far the best well-known fact for codimension-2 PL fibrators can be described as follows;

PROPOSITION 2.1. [6, 7] *Let N be a closed PL n-manifold. If either $\pi_1(N)$ is hopfian and $\chi(N) \neq 0$ or $\pi_1(N)$ is hyperhopfian, then N is a codimension-2 PL fibrator.*

Any closed manifold that cyclically cover itself (nontrivially) fails to be a codimension-2 PL fibrator, for example, S^1 and $RP^n \# RP^n$ [2, Theorem 4.2]. Hence the mapping torus of a periodic self homeomorphism of any closed manifold fails to be a codimension-2 PL fibrator.

Whether the product of fibrators is again a fibrator is still open. There are some cases that have affirmative answers (See [4]).

PROPOSITION 2.2. [4, Theorem 5.7] *Let F be a closed PL n-manifold with finite $\pi_1(F)$ and $\chi(F) \neq 0$ and let A be a closed aspherical PL m-manifold with hopfian $\pi_1(A)$ and $\chi(A) \neq 0$. Then $F \times A$ is a codimension-2 PL fibrator.*

The following proposition is useful when we determine whether a product of two manifolds is not a codimension-k PL fibrator.

PROPOSITION 2.3. *Let N_1^n and N_2^m be closed manifolds. If N_1 is not a codimension-k PL fibrator, then $N_1 \times N_2$ is not a codimension-k PL fibrator.*

Proof. Take an N_1–like map $p : M^{n+k} \to B^k$ which fails to be an approximate fibration. Then the composition map

$$M^{n+k} \times N_2^m \xrightarrow{\text{projection}} M^{n+k} \xrightarrow{p} B^k$$

fails to be an approximate fibration. □

DEFINITION. An ANR Y is said to be t-aspherical if $\pi_i(Y) = 0$ whenever $1 < i \leq t$.

PROPOSITION 2.4. [5, Corollary 2.6] *Suppose N is a closed, hopfian PL manifold satisfying:* (i) *N is a codimension-2 PL fibrator,* (ii) *N is t-aspherical, and* (iii) *$\pi_1(N)$ has Property NCSA. Then N is a codimension-$(t+1)$ PL fibrator.*

3. Fibrator properties of products of hopfian manifolds

LEMMA 3.1 *For any group homomorphism $f : G \to H$ and any normal subgroup N of G, the homomorphic image $f(N)$ of N is normal in $f(G)$.*

Proof. Given $x \in f(N)$ and $z \in f(G)$, $f(n) = x$ and $f(g) = z$ for some $n \in N$ and $g \in G$. Then $z^{-1}xz = f(g)^{-1}f(n)f(g) = f(g^{-1}ng) \in f(N)$, since $N \lhd G$. □

DEFINITION. A group G is *incommensurable* with another group H if for every homomorphism $\varphi : G \to H$ is trivial.

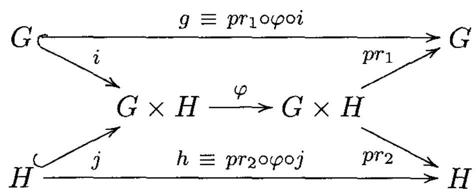
Any finite group G is incommensurable with any torsion free group H, for any $g \in G$, the homomorphic image of g must have a finite order; and perfect groups are incommensurable with all abelian groups.

Given two groups G and H, we denote elements of the direct product $G \times H$ by ordered pairs (a, b), where $a \in G$ and $b \in H$. The structure of subgroups of the direct product $G \times H$ wasn't known to the general public until Jacques Thévenaz described the subgroups of $G \times H$ in 1997 [9].

The elementary lemmas that follow expose the role of incommensurability here.

LEMMA 3.2. *Suppose that a group G is incommensurable with a group H and that $\varphi : G \times H \to G \times H$ is a group homomorphism. Consider the following diagram:*

(†)



where i and j are the inclusion maps, pr_1 and pr_2 are the projection maps. Then we have the followings:

1. [4, Lemma 3.2] $\varphi(G \times 1) \subset G \times 1$.

2. $h(H) = pr_2(\varphi(G \times H))$. *Moreover, if* $\varphi(G \times H) \lhd G \times H$, *then* $h(H) \lhd H$.
3. $\ker g \times pr_2(\ker\varphi \cap 1 \times H) \subset pr_1(\ker\varphi \cap G \times 1) \times pr_2(\ker\varphi \cap 1 \times H) \subset \ker\varphi \subset pr_1(\ker\varphi) \times pr_2(\ker\varphi) \subset pr_1(\ker\varphi) \times \ker h$. *In particular,* φ *is a monomorphism, then* g *is a monomorphism.*

Proof. (1) The homomorphism

$$pr_2 \circ \varphi \circ i : G \xhookrightarrow{i} G \times H \xrightarrow{\varphi} G \times H \xrightarrow{pr_2} H$$

is trivial, for G is incommensurable with H. Hence $pr_2 \circ \varphi \circ i(G) = pr_2(\varphi(G \times 1)) = 1$, i.e., $\varphi(G \times 1)$ doesn't have a factor of H, so $\varphi(G \times 1) \subset G \times 1$.

(2) Clearly $h(H) \subset pr_2(\varphi(G \times H))$. Conversely, given an element y of $pr_2(\varphi(G \times H))$, take an x in G such that $(x, y) \in \varphi(G \times H)$. Then $(x, y) = \varphi(a, b)$ for some (a, b) in $G \times H$. By (1) $\varphi(a, 1) = (\alpha, 1)$ for some α in G. Put $\varphi(1, b) := (\beta, \gamma) \in G \times H$. Then $(x, y) = \varphi(a, b) = \varphi((a, 1)(1, b)) = \varphi(a, 1)\varphi(1, b) = (\alpha, 1)(\beta, \gamma) = (\alpha\beta, \gamma)$. Hence $\gamma = y$. Then,

$$b \xmapsto{j} (1, b) \xmapsto{\varphi} (\beta, y) \xmapsto{pr_2} y, \qquad h = pr_2 \circ \varphi \circ j$$

i.e., $h(b) = y$ for some b in H, so the latter element belongs to $h(H)$, as desired.

Moreover, if $\varphi(G \times H) \lhd G \times H$, then since pr_2 is onto, by Lemma 3.1 $h(H) = pr_2(\varphi(G \times H)) \lhd H$.

(3) First we claim that $\ker g \subset pr_1(\ker\varphi \cap (G \times 1))$. Given any $x \in \ker g$, $(x, 1) \in G \times 1$. By (1), $\varphi(x, 1) = (a, 1)$ for some a in G. Then $x \xmapsto{i} (x, 1) \xmapsto{\varphi} (a, 1) \xmapsto{pr_1} a = 1$ and so $\varphi(x, 1) = (1, 1)$, i.e., $(x, 1) \in \ker\varphi \cap (G \times 1)$, whence we have $x \in pr_1(\ker\varphi \cap (G \times 1))$.

Next, we show that $pr_2(\ker\varphi) \subset \ker h$. Given any $y \in pr_2(\ker\varphi)$, there exists an x in G such that $(x, y) \in \ker\varphi$. By (1), $\varphi(x, 1) = (a, 1)$ for some a in G. Since $(1, 1) = \varphi(x, y) = \varphi(x, 1)\varphi(1, y) = (a, 1)\varphi(1, y)$, $\varphi(1, y)$ must be $(a^{-1}, 1)$. Then,

$$y \xmapsto{j} (1, y) \xmapsto{\varphi} (a^{-1}, 1) \xmapsto{pr_2} 1, \qquad h = pr_2 \circ \varphi \circ j$$

Therefore, $y \in \ker h$. □

COROLLARY 3.3. *If a finite group* G *is incommensurable with a group* H *and* $\varphi : G \times H \to G \times H$ *is a monomorphism, then* $\varphi(G \times 1) = G \times 1$ *and* $h \equiv pr_2 \circ \varphi \circ j$ *is a monomorphism.*

Proof. Since φ is a monomorphism, by Lemma 3.2 (3), we have a injective endomorphism $g \equiv pr_1 \circ \varphi \circ i : G \to G$. It follows from the finiteness of G that g is an isomorphism. But by Lemma 3.2 (1), $\varphi(G \times 1) \subseteq G \times 1$. Then $\varphi(G \times 1) = G \times 1$.

Now, consider the sequence of homomorphisms

$$H \xrightarrow{j} G \times H \xrightarrow[mono]{\varphi} G \times H \xrightarrow{pr_2} H, \qquad h = pr_2 \circ \varphi \circ j .$$

Given any $z \in \ker(pr_2 \circ \varphi \circ j)$, $z \longmapsto (1, z) \overset{\varphi}{\longmapsto} (a, b) \overset{pr_2}{\longmapsto} b = 1$. Since $(a, b) = (a, 1) \in G \times 1$ and $\varphi(G \times 1) = G \times 1$, we have $\varphi(\gamma, 1) = (a, 1)$ for some $\gamma \in G$. Since $\varphi(\gamma, 1) = \varphi(1, z)$ and φ is a monomorphism, $(1, z) = (\gamma, 1)$ whence we have $z = 1$. Consequently, we have $\ker(pr_2 \circ \varphi \circ j) = 1$. □

LEMMA 3.4. *Suppose that a finite group G is incommensurable with a group H. If H is a normally co-hopfian group, so is $G \times H$.*

Proof. Let $\varphi : G \times H \to G \times H$ be a monomorphism with $\varphi(G \times H) \lhd G \times H$. By Lemma 3.2 (2), $(pr_2 \circ \varphi \circ j)(H) \lhd H$. It follows from the normally co-hopficity of H and Lemma 3.2 (3) that $pr_2 \circ \varphi \circ j$ is an isomorphism.

Now, we show that φ is a surjective map. Given any $(x, y) \in G \times H$, we divide (x, y) into $(x, 1)(1, y)$. Since $\varphi(G \times 1) = G \times 1$, there exists an $\alpha \in G$ such that $\varphi(\alpha, 1) = (x, 1)$. And since $pr_2(1, y) = y \in H$ and $pr_2 \circ \varphi \circ j$ is an isomorphism, $(1, \beta) \overset{\varphi}{\longmapsto} (z, y) \longmapsto y$ for some $(z, y) \in \varphi(1 \times H)$ and $(1, \beta) \in 1 \times H$. We again divide (z, y) into $(z, 1)(1, y)$. Then, $\varphi(\omega, 1) = (z, 1)$ for some $(\omega, 1) \in G \times 1$ and since φ is a homomorphism, the preimage of $(1, y)$ must be (ω^{-1}, β). Hence we have

$$\varphi(\alpha\omega^{-1}, \beta) = \varphi(\alpha, 1)\varphi(\omega^{-1}, \beta) = (x, 1)(1, y) = (x, y).$$

□

Now, we state the main result.

THEOREM 3.5. *Suppose that F is a closed t-aspherical PL n-manifold with finite, sparsely abelian $\pi_1(F)$ and A is a closed aspherical PL m-manifold with hopfian, normally cohopfian $\pi_1(A)$. If $\chi(F) \neq 0 \neq \chi(A)$, then $F \times A$ is a codimension-$(t + 1)$ PL fibrator.*

Proof. First, we note that $F \times A$ is a codimension-2 PL fibrator according to Proposition 2.2, and is t-aspherical, for $\pi_i(F \times A) = \pi_i(F) \times \pi_i(A)$.

Now we show that $\pi_1(F \times A)$ has Property NCSA. Set $G := \pi_1(F)$ and $H := \pi_1(A)$. Then H is torsion free, for A is aspherical. Hence G is incommensurable with H. Since H is normally cohopfian, by Lemma 3.4, $G \times H$ is normally co-hopfian. Moreover, we show that $G \times H$ is sparsely abelian. Let $\varphi : G \times H \to G \times H$ be a homomorphism with $\varphi(G \times H) \lhd G \times H$ and abelian $\ker \varphi$. But since $\ker \varphi$ is abelian, by Lemma 3.2 (3), $\ker(g \equiv pr_1 \circ \varphi \circ i)$ is also abelian. Since A is aspherical, by work of Rosset [8], $\chi(A) \neq 0$ implies that H has no nontrivial abelian normal subgroup. Therefore, we have $\ker \varphi \subset G \times 1$. But since G is incommensurable with H, by Lemma 3.2 (3), we have $\ker g \times 1 = \ker \varphi$. Then $\ker \varphi$ should be trivial so that $G \times H$ is sparsely abelian. Consequently, the conclusion follows from Proposition 2.4. □

References

[1] G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[2] R. J. Daverman, *Submanifold decompositions that induce approximate fibrations*, Topology Appl. **33** (1989), no. 2, 173–184

[3] ———, *Hyperbolic groups are hyper-Hopfian*, J. Austral. Math. Soc. Ser. A **68** (2000), no. 1, 126–130.

[4] R. J. Daverman, Y. H. Im, and Y. Kim, *Products of Hopfian manifold and codimension-2 fibrators*, Topology Appl. **103** (2000), no. 3, 323–338.

[5] ———, *PL fibrator properties of partially aspherical manifolds*, Topology Appl. **140** (2004), no. 2-3, 181–195.

[6] Y. Kim, *Strongly Hopfian manifolds as codimension-2 fibrators*, Topology Appl. **92** (1999), no. 3, 237–245.

[7] ———, *Manifolds with hyperHopfian fundamental group as codimension-2 fibrators*, Topology Appl. **96** (1999), no. 3, 241–248.

[8] S. Rosset, *A vanishing theorem for Euler characteristics*, Math. Z. **185** (1984), no. 2, 211–215.

[9] J. Thévenaz, *Maximal subgroups of direct products*, J. Algebra **198** (1997), no. 2, 352–361.

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