Bull. Korean Math. Soc. **43** (2006), No. 4, pp. 831–839

BOEHMIANS ON THE TORUS

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ABSTRACT. By relaxing the requirements for a sequence of functions to be a delta sequence, a space of Boehmians on the torus $\beta(T^d)$ is constructed and studied. The space $\beta(T^d)$ contains the space of distributions as well as the space of hyperfunctions on the torus. The Fourier transform is a continuous mapping from $\beta(T^d)$ onto a subspace of Schwartz distributions. The range of the Fourier transform is characterized. A necessary and sufficient condition for a sequence of Boehmians to converge is that the corresponding sequence of Fourier transforms converges in $\mathcal{D}'(\mathbb{R}^d)$.

1. Introduction

Boehmians are classes of generalized functions whose construction is algebraic. In [3], Mikusiński constructs a space of Boehmians $\beta_{\mathcal{T}}$ in which each element has a Fourier transform which is a distribution. Moreover, the Fourier transform is a continuous bijection from $\beta_{\mathcal{T}}$ onto the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$.

In this note, we will investigate a subspace $\beta(T^d)$ of $\beta_{\mathcal{T}}$. The space $\beta(T^d)$ can be thought of as the space of Boehmians on the torus T^d.

By using a slightly different construction, the space $\beta(\Gamma)$ of Boehmians on the unit circle has been studied in [1, 5, 6]. The space $\beta(\Gamma)$ is quite general. It contains a subspace which can be identified with the space of periodic Schwartz distributions as well as some elements which are not hyperfunctions. However, the space of hyperfunctions cannot be identified with any subspace of $\beta(\Gamma)$.

One of the motivating factors for this paper was to construct a space of Boehmians which contains the space of periodic hyperfunctions. By using a more general definition for delta sequences, we will see that both $\beta(\Gamma)$ and the space of hyperfunctions are properly contained in $\beta(T^1)$.

Received January 11, 2006.
2000 Mathematics Subject Classification: 44A40, 46F12, 42B05.
Key words and phrases: Boehmian, Fourier transform, distribution.

This article is organized as follows. Section 2 contains notation and the construction of the space of tempered Boehmians $\beta_{\mathcal{T}}$. Section 3 contains the construction of the space of Boehmians on the torus $\beta(T^d)$ as well as the investigation of the Fourier transform on $\beta(T^d)$. The range of the Fourier transform is characterized. In Section 4, we study the convergence structure of δ-convergence on $\beta(T^d)$. An inversion formula for the Fourier transform is given. It is shown that there is a locally convex metric topology which gives the identical convergent sequences in $\beta(T^d)$ as does δ-convergence.

2. Notation and the space $\beta_{\mathcal{T}}$

In this section, the space of tempered Boehmians $\beta_{\mathcal{T}}$ [3] is introduced.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, where α_j is a nonnegative integer, be a multi-index. Then, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. If $x, y \in \mathbb{R}^d$, then $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d)$, $x \cdot y = x_1 y_1 + \cdots + x_d y_d$, and $||x|| = \sqrt{x \cdot x}$.

A complex-valued infinitely differentiable function f is called *rapidly decreasing* if

$$\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^d} (1 + x_1^2 + \cdots + x_d^2)^m |D^\alpha f(x)| < \infty \tag{2.1}$$

for every nonnegative integer m. The space of all rapidly decreasing functions on $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d)$. Elements of the dual space $\mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{S}(\mathbb{R}^d)$ are called tempered distributions.

A sequence $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ is called a *delta sequence* provided:

(i) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(ii) $\int |\varphi_n| \leq M$ for some constant M and all $n \in \mathbb{N}$,

(iii) $\lim_{n \to \infty} \int_{||x|| \geq \varepsilon} ||x||^k |\varphi_n(x)| dx = 0$ for all $k \in \mathbb{N}$ and $\varepsilon > 0$.

A complex-valued function f is called *slowly increasing* if there exists a polynomial p on $\mathbb{R}^d$ such that $\frac{f(x)}{p(x)}$ is bounded. The space of all slowly increasing continuous functions on $\mathbb{R}^d$ is denoted by $\mathcal{T}$. Notice that $\mathcal{T}$ may be viewed as a subspace of $\mathcal{S}'(\mathbb{R}^d)$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in \mathcal{T}$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(x - u)\varphi(u) du, \tag{2.2}$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called tempered Boehmians. The space of all tempered Boehmians will be denoted by $\beta_{\mathcal{T}}$ and a typical element of $\beta_{\mathcal{T}}$ will be written as $F = \left[\frac{f_n}{\varphi_n}\right]$.

A function $f \in \mathcal{T}$ can be identified with the Boehmian $\left[\frac{f * \varphi_n}{\varphi_n}\right]$. The identification is independent of the delta sequence.

The operations of addition, scalar multiplication, and differentiation are defined as follows:

$$\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}\right] \tag{2.3}$$

$$\alpha \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\alpha f_n}{\varphi_n}\right], \text{where } \alpha \in \mathbb{C} \tag{2.4}$$

$$D^\alpha \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * D^\alpha \varphi_n}{\varphi_n * \varphi_n}\right] \tag{2.5}$$

The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is denoted by $\widehat{f}$, i.e.,

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(u) e^{-i(u \cdot x)} du. \tag{2.6}$$

The space of infinitely differentiable functions on $\mathbb{R}^d$ having compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$, and the dual space of $\mathcal{D}(\mathbb{R}^d)$ is denoted by $\mathcal{D}'(\mathbb{R}^d)$. The space $\mathcal{D}'(\mathbb{R}^d)$ is the space of distributions. The Fourier transform $\widehat{f}$ for a member $f \in \mathcal{S}'(\mathbb{R}^d)$ is the tempered distribution defined by $\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle$, where $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Let $\{\varphi_n\}_{n=1}^\infty$ be a delta sequence. A useful property that will be used throughout the sequel is that $\widehat{\varphi}_n \to 1$ uniformly on compact subsets of $\mathbb{R}^d$ as $n \to \infty$.

The existence of a delta sequence $\{\varphi_n\}_{n=1}^\infty$ such that the support of each $\widehat{\varphi}_n$ is compact will also be useful. This can be shown as follows.

Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi(0) = 1$. Since the Fourier transform maps the space $\mathcal{S}(\mathbb{R}^d)$ onto itself, there exists a $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\varphi} = \psi$. Now, put $\varphi_n(x) = n^d \varphi(nx), n \in \mathbb{N}$. It is not difficult to show that $\{\varphi_n\}_{n=1}^\infty$ is a delta sequence and that $\widehat{\varphi}_n$ has compact support for all $n \in \mathbb{N}$.

DEFINITION 2.1. The Fourier transform of a tempered Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is given by $\widehat{F} = \lim_{n\to\infty} \widehat{f_n}$, where the limit is taken in $\mathcal{D}'(\mathbb{R}^d)$.

The above limit exists and is well-defined. Moreover, Mikusiński [3] has shown that the range of the Fourier transform is $\mathcal{D}'(\mathbb{R}^d)$.

3. The space $\beta(T^d)$

In this section we will construct and study the space of Boehmians on the torus

$$T^d = \{(e^{ix_1}, \ldots, e^{ix_d}) : x_j \text{ real}\}. \tag{3.1}$$

We make no distinction between a function on T^d and a function on $\mathbb{R}^d$ that is 2π-periodic in each variable.

For $f \in \mathcal{T}$, define $\tau_{2\pi} f(x) = f(x_1 + 2\pi, \ldots, x_d + 2\pi)$. The translation operator $\tau_{2\pi}$ can be extended to $\beta_{\mathcal{T}}$ as follows:

$$\text{For } F = \left[\frac{f_n}{\varphi_n}\right], \text{ put } \tau_{2\pi} F = \left[\frac{\tau_{2\pi} f_n}{\varphi_n}\right].$$

It is easy to check that $\tau_{2\pi} F$ is a tempered Boehmian.

The space of Boehmians on the torus $\beta(T^d)$ is defined as follows.

$$\beta(T^d) = \{F \in \beta_{\mathcal{T}} : \tau_{2\pi} F = F\}. \tag{3.2}$$

The proof of the next lemma is routine.

LEMMA 3.1. *Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{\mathcal{T}}$. Then $F \in \beta(T^d)$ if and only if for all $n \in \mathbb{N}$, f_n is 2π-periodic in each variable .*

The *Fourier coefficients* for a locally integrable function f on T^d are given by

$$c_k(f) = \frac{1}{(2\pi)^d} \int_{T^d} f(x) e^{-i(k\cdot x)} dx, \quad k \in \mathbb{Z}^d. \tag{3.3}$$

LEMMA 3.2. *Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. For each k, the sequence $\{c_k(f_n)\}_{n=1}^{\infty}$ converges.*

Proof. Let $k \in \mathbb{Z}^d$. Since $\{\varphi_p\}_{p=1}^{\infty}$ is a delta sequence, there exists a $p \in \mathbb{N}$ such that $\widehat{\varphi}_p(k) \neq 0$.

Now,

$$c_k(f_n) = c_k(f_n)\frac{\widehat{\varphi}_p(k)}{\widehat{\varphi}_p(k)} = \frac{c_k(f_n * \varphi_p)}{\widehat{\varphi}_p(k)} = \frac{c_k(f_p * \varphi_n)}{\widehat{\varphi}_p(k)}$$
$$= \frac{c_k(f_p)}{\widehat{\varphi}_p(k)}\widehat{\varphi}_n(k) \to \frac{c_k(f_p)}{\widehat{\varphi}_p(k)}$$

as $n \to \infty$. □

Suppose that $\left[\frac{f_n}{\varphi_n}\right], \left[\frac{g_n}{\sigma_n}\right] \in \beta(T^d)$ such that $\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{g_n}{\sigma_n}\right]$. Then $c_p(f_n)\widehat{\sigma}_k(p) = c_p(f_n * \sigma_k) = c_p(g_k * \varphi_n) = c_p(g_k)\widehat{\varphi}_n(p)$ for all p, n, and k. Thus the following definition is well-defined.

DEFINITION 3.3. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. Define the k-th Fourier coefficient of F as

$$c_k(F) = \lim_{n\to\infty} c_k(f_n). \tag{3.4}$$

Let $\delta \in \mathcal{D}'(\mathbb{R}^d)$ denote the Dirac measure on $\mathbb{R}^d$. Thus, $\langle \delta(x-k), \varphi\rangle = \varphi(k)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $k \in \mathbb{Z}^d$.

THEOREM 3.4. *Let $F \in \beta(T^d)$. Then $\widehat{F} = \sum_{k\in\mathbb{Z}^d} c_k(F)\delta(x-k)$.*

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. Then, for each n, $\sum_{|k|\le m} c_k(f_n)e^{i(k\cdot x)} \to f_n$ in $\mathcal{S}'(\mathbb{R}^d)$ as $m \to \infty$. By the continuity of the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$, we obtain $\sum_{|k|\le m} c_k(f_n)\delta(x-k) \to \widehat{f}_n$ in $\mathcal{S}'(\mathbb{R}^d)$ as $m \to \infty$. Thus, $\widehat{f}_n = \sum_{k\in\mathbb{Z}^d} c_k(f_n)\delta(x-k), n \in \mathbb{N}$. Now, $\sum_{k\in\mathbb{Z}^d} c_k(f_n)\delta(x-k) \to \sum_{k\in\mathbb{Z}^d} c_k(F)\delta(x-k)$ in $\mathcal{D}'(\mathbb{R}^d)$ as $n \to \infty$, and, by definition, $\widehat{f}_n \to \widehat{F}$ in $\mathcal{D}'(\mathbb{R}^d)$ as $n \to \infty$. Therefore $\widehat{F} = \sum_{k\in\mathbb{Z}^d} c_k(F)\delta(x-k)$. □

The collection of all distributions of the form $\sum_{k\in\mathbb{Z}^d} \alpha_k\delta(x-k)$ (where $\alpha_k \in \mathbb{C}$) will be denoted by $\mathcal{D}'_\delta(\mathbb{R}^d)$.

The previous theorem shows that the Fourier transform maps $\beta(T^d)$ into $\mathcal{D}'_\delta(\mathbb{R}^d)$. The next theorem shows that the mapping is actually onto $\mathcal{D}'_\delta(\mathbb{R}^d)$.

THEOREM 3.5. *The Fourier transform is a bijection from $\beta(T^d)$ onto $\mathcal{D}'_\delta(\mathbb{R}^d)$.*

Proof. Let $\{\alpha_k\}_{k\in\mathbb{Z}^d}$ be a matrix of complex numbers. Let $\{\varphi_n\}_{n=1}^\infty$ be a delta sequence such that supp $\widehat{\varphi}_n$ is compact. Put

$$f_n(x) = \sum_{k\in\mathbb{Z}^d} \alpha_k\widehat{\varphi}_n(k)e^{i(k\cdot x)}$$

for $n = 1, 2, \ldots$. It is routine to show that $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. Moreover, for each $k \in \mathbb{Z}^d$, $c_k(F) = \lim_{n\to\infty} c_k(f_n) = \lim_{n\to\infty} \alpha_k \widehat{\varphi}_n(k) = \alpha_k$. Thus, $\widehat{F} = \sum_{k\in\mathbb{Z}^d} \alpha_k \delta(x-k)$. Therefore the Fourier transform maps $\beta(T^d)$ onto $\mathcal{D}'_\delta(\mathbb{R}^d)$.

Now to show that the Fourier transform is an injection, let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. It is not difficult to show that $c_k(f_p) = c_k(F)\widehat{\varphi}_p(k)$ for all $k \in \mathbb{Z}^d$ and $p \in \mathbb{N}$. Suppose that $\widehat{F} = 0$. Then $c_k(F) = 0$ for all $k \in \mathbb{Z}^d$. Thus $c_k(f_p) = 0$ for all $k \in \mathbb{Z}^d$ and $p \in \mathbb{N}$. Therefore, $F = 0$. This completes the proof. $\square$

It is known that f is a periodic hyperfunction if and only if $\widehat{f} = \sum_{k\in\mathbb{Z}^d} \alpha_k \delta(x-k)$, where $\limsup_{|n|\to\infty} \sqrt[|n|]{|\alpha_n|} \leq 1$. Thus the previous theorem shows that the space of hyperfunctions on T^d can be identified with a proper subspace of $\beta(T^d)$.

4. Convergence

Let $\mathcal{U}$ be a class of sequences on a space $\mathcal{X}$ (with or without a topology). We say that $x_n \xrightarrow{\mathcal{U}} x$ if $(x, x_1, x_2, \ldots)$ is in $\mathcal{U}$. $\mathcal{U}$ is called topological if there exists a topology $\mathcal{O}$ for $\mathcal{X}$ such that $x_n \xrightarrow{\mathcal{U}} x$ if and only if $x_n \xrightarrow{\mathcal{O}} x$.

The space $\beta(\Gamma)$ of Boehmians on the unit circle with a convergence structure known as Δ-convergence is topological. Indeed, $\beta(\Gamma)$ is an F-space. That is, it is a complete topological vector space where the topology is given by an invariant metric. However, $\beta(\Gamma)$ is not locally convex [1].

In this section, we introduce a convergence structure on $\beta(T^d)$ known as δ-convergence. For the space $\beta(T^d)$, δ-convergence is equivalent to Δ-convergence. We will show that $\beta(T^d)$ with δ-convergence is topological. In fact, δ-convergence is equivalent to a locally convex metric topology.

Definition 4.1. A sequence of functions $f_n \in \mathcal{T}$ is said to be convergent to $f \in \mathcal{T}$ if there exists a polynomial p such that $\frac{f_n - f}{p} \to 0$ uniformly on $\mathbb{R}^d$ as $n \to \infty$.

Define the map $\iota : \mathcal{T} \to \beta_{\mathcal{T}}$ by

$$\iota(f) = \left[\frac{f * \varphi_n}{\varphi_n}\right],$$

where $\{\varphi_n\}_{n=1}^\infty$ is any fixed delta sequence.

It is not difficult to show that the mapping ι is an injection which preserves the algebraic properties of $\mathcal{T}$. Thus, $\mathcal{T}$ can be identified with a proper subspace of $\beta_{\mathcal{T}}$.

For $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$, $F * \psi$ is defined as $F * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]$. It is straightforward to verify that $F * \psi \in \beta(T^d)$. Moreover, by a routine calculation we see that $c_k(F * \psi) = c_k(F)\widehat{\psi}(k)$ for all $k \in \mathbb{Z}^d$.

Definition 4.2. A sequence of tempered Boehmians $\{F_n\}_{n=1}^{\infty}$ is said to be *δ-convergent* to a tempered Boehmian F, denoted $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that $F_n * \varphi_k, F * \varphi_k \in \mathcal{T}$ for all $k, n \in \mathbb{N}$, and for each $k \in \mathbb{N}, F_n * \varphi_k \to F * \varphi_k$ in $\mathcal{T}$ as $n \to \infty$.

The proof of the following theorem may be found in [3].

Theorem 4.3. *Suppose* $F_n, F \in \beta_{\mathcal{T}}$ *for* $n = 1, 2, \dots$ *and* $\delta\text{-}\lim_{n\to\infty} F_n = F$. *Then* $\lim_{n\to\infty} \widehat{F_n} = \widehat{F}$, *where the limit is taken in* $\mathcal{D}'(\mathbb{R}^d)$.

Theorem 4.4. (*Inversion*) *Let* $F \in \beta(T^d)$. *Then,*

$$F = \delta\text{-} \lim_{n\to\infty} \sum_{|k|\le n} c_k(F) e^{i(k\cdot x)}.$$

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T^d)$. We may assume that $f_n \in C^{\infty}(T^d)$ for all $n \in \mathbb{N}$. For if not, replace f_n with $f_n * \varphi_n$. Then $F = \left[\frac{f_n * \varphi_n}{\varphi_n * \varphi_n}\right]$ and $f_n * \varphi_n \in C^{\infty}(T^d)$ for all $n \in \mathbb{N}$. Let $F_n = \sum_{|k|\le n} c_k(F) e^{i(k\cdot x)}$ for $n = 1, 2, \dots$. Then for each p, $F_n * \varphi_p = \sum_{|k|\le n} c_k(F)\widehat{\varphi}_p(k) e^{i(k\cdot x)} = \sum_{|k|\le n} c_k(F * \varphi_p) e^{i(k\cdot x)} \in \mathcal{T}$ for $n = 1, 2, \dots$. Also, for each p, $F * \varphi_p = f_p \in \mathcal{T}$. Since for each p, $F * \varphi_p \in C^{\infty}(T^d)$, $\sum_{|k|\le n} c_k(F * \varphi_p) e^{i(k\cdot x)} \to F * \varphi_p$ uniformly on $\mathbb{R}^d$ as $n \to \infty$. This implies that for each p, $F_n * \varphi_p \to F * \varphi_p$ in $\mathcal{T}$ as $n \to \infty$. Hence, $\delta\text{-}\lim_{n\to\infty} F_n = F$. This completes the proof of the theorem. □

Let $\gamma_p(F) = \sum_{|k|\le p} |c_k(F)|$ for $p = 1, 2, \dots$. Then $\{\gamma_p\}_{p=1}^{\infty}$ is a countable separating family of seminorms on $\beta(T^d)$. It is not difficult to show that $\left(\beta(T^d), \{\gamma_p\}_{p=1}^{\infty}\right)$ is complete, and hence, $\left(\beta(T^d), \{\gamma_p\}_{p=1}^{\infty}\right)$ is a Fréchet space.

Theorem 4.5. *Let* $F_n, F \in \beta(T^d)$ *for* $n = 1, 2, \dots$. *Then* $\delta\text{-}\lim_{n\to\infty} F_n = F$ *if and only if for each* p, $\gamma_p(F_n - F) \to 0$ *as* $n \to \infty$.

Proof. Let $F_n \in \beta(T^d)$ for $n = 1, 2, \ldots$ such that for each p, $\gamma_p(F_n) \to 0$ as $n \to \infty$. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a delta sequence such that supp $\widehat{\varphi}_j$ is compact.

Let $n, j \in \mathbb{N}$ be fixed. Then, for each $m \in \mathbb{N}$ there exists a constant M_m such that $|c_k(F_n * \varphi_j)| = |c_k(F_n)||\widehat{\varphi}_j(k)| \leq \frac{M_m}{(1+|k|)^m}$ for all $k \in \mathbb{Z}^d$. Thus $F_n * \varphi_j \in C^{\infty}(T^d)$ for all $n, j \in \mathbb{N}$. Therefore, $F_n * \varphi_j \in \mathcal{T}$ for all $n, j \in \mathbb{N}$.

Now, for each $n, j \in \mathbb{N}$,

$$(4.1) \quad |(F_n * \varphi_j)(x)| = \left| \sum_{k \in \mathbb{Z}^d} c_k(F_n * \varphi_j) e^{i(k \cdot x)} \right| \leq \sum_{k \in \mathbb{Z}^d} |c_k(F_n)||\widehat{\varphi}_j(k)|,$$

for all $x \in \mathbb{R}^d$. Since $\widehat{\varphi}_j$ has compact support, for a fixed j, the above sum has only a finite number of nonzero terms. Therefore, since, for each p, $\gamma_p(F_n) \to 0$ as $n \to \infty$, $(F_n * \varphi_j)(x) \to 0$ uniformly on $\mathbb{R}^d$ as $n \to \infty$. Thus, for each $j \in \mathbb{N}$, $F_n * \varphi_j \to 0$ in $\mathcal{T}$ as $n \to \infty$, and hence $\delta\text{-}\lim_{n\to\infty} F_n = 0$.

For the other direction, suppose that $\delta\text{-}\lim_{n\to\infty} F_n = F$. Now $\widehat{F}_n = \sum_{k \in \mathbb{Z}^d} c_k(F_n)\delta(x-k)$ for $n = 1, 2, \ldots$ and $\widehat{F} = \sum_{k \in \mathbb{Z}^d} c_k(F)\delta(x-k)$. By Theorem 4.3, $\widehat{F}_n \to \widehat{F}$ in $\mathcal{D}'(\mathbb{R}^d)$ as $n \to \infty$. This and the above give for each $k \in \mathbb{Z}^d$, $c_k(F_n) \to c_k(F)$ as $n \to \infty$. Thus, for all $p \in \mathbb{N}$, $\gamma_p(F_n - F) \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

The previous theorem shows that a sequence in $\beta(T^d)$ is δ-convergent if and only if it is convergent in the locally convex metric topology generated by the family of seminorms $\{\gamma_p\}_{p=1}^{\infty}$.

The following corollary shows that $\beta(T^d)$ is isomorphic to the space $\mathcal{D}'_{\mathcal{S}}(\mathbb{R}^d)$.

COROLLARY 4.6. *Let $F_n, F \in \beta(T^d)$ for $n = 1, 2, \ldots$. Then, δ-$\lim_{n\to\infty} F_n = F$ if and only if $\lim_{n\to\infty} \widehat{F}_n = \widehat{F}$ (in $\mathcal{D}'(\mathbb{R}^d)$).*

As a final note, in [4], Mikusiński describes a construction for Boehmians on manifolds. He indicates how this may be used to construct Boehmians on a torus. However, the delta sequences used in Mikusiński's construction are less general than the one used in our construction.

ACKNOWLEDGEMENT. The author would like to thank Piotr Mikusiński for his helpful comments.

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