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# ZETA FUNCTIONS OF GRAPH BUNDLES

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ABSTRACT. As a continuation of computing the zeta function of a regular covering graph by Mizuno and Sato in [9], we derive in this paper computational formulae for the zeta functions of a graph bundle and of any (regular or irregular) covering of a graph. If the voltages to derive them lie in an abelian or dihedral group and its fibre is a regular graph, those formulae can be simplified. As a by-product, the zeta function of the cartesian product of a graph and a regular graph is obtained. The same work is also done for a discrete torus and for a discrete Klein bottle.

## 1. Introduction

In this paper we consider an undirected finite simple graph. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* $\deg_G(v)$ of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$. An *automorphism* of $G$ is a permutation of the vertex set $V(G)$ that preserves adjacency. The set of automorphisms forms a permutation group, called the *automorphism group* $\mathrm{Aut}(G)$ of $G$.

A $(v_0, v_n)$-path $P$ of length $n$ in $G$ is a sequence $P = (v_0, v_1, \ldots, v_{n-1}, v_n)$ of $n+1$ vertices and $n$ edges such that consecutive vertices share an edge (we do not require that all vertices are distinct). Sometimes, the path $P$ is also considered as a subgraph of $G$. We say that a path has a *backtracking* if $v_{i-1} = v_{i+1}$ for some $i$, $1 \le i \le n-1$. A $(v_0, v_n)$-path is called a *cycle* if $v_0 = v_n$. The *inverse cycle* of a cycle $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is the cycle $C^{-1} = (v_0, v_{n-1}, \ldots, v_1, v_0)$.

A subpath $(v_1, \ldots, v_{m-1}, v_m)$ of a cycle $C = (v_1, \ldots, v_m, \ldots, v_1)$ is called a *tail* if $\deg_C(v_1) = 1$, $\deg_C(v_i) = 2$, $2 \le i \le m-1$, and

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$\deg_C(v_m) \geq 3$, where $\deg_C(v)$ is the degree of $v$ in the subgraph $C$. Each cycle $C$ without backtracking determines a unique tail-less, backtracking-less cycle $C^*$ by removing all tails of $C$. Note that any backtracking-less tail-less cycle $C$ is just a cycle such that both $C$ and $C^2$ have no backtracking (see [4], [11]). Two backtracking-less, tail-less cycles $C_1 = (v_1, \ldots, v_m)$ and $C_2 = (w_1, \ldots, w_m)$ are called *equivalent* if there is $k$ such that $w_j = v_{j+k}$ for all $j$, where the subscripts are module $m$. Let $[C]$ denote the equivalence class which contains a cycle $C$. A backtracking-less, tail-less cycle $C$ is *primitive* if $C$ is not obtained by going $r$ times around some other cycle $B$, i.e., $C \neq B^r$ for $r \geq 2$. Note that each equivalence class of primitive, backtracking-less and tail-less cycles of a graph $G$ corresponds to a conjugacy class of the fundamental group $\pi_1(G, v)$ of $G$ for a vertex $v \in V(G)$.

The *(Ihara) zeta function* [13] of a graph $G$ is defined to be the function of $u \in \mathbb{C}$ with $u$ sufficiently small, given by

$$Z(G, u) = Z_G(u) = \prod_{[C]} (1 - u^{\nu(C)})^{-1},$$

where $[C]$ runs over all equivalence classes of primitive, backtracking-less and tail-less cycles of $G$ and $\nu(C)$ denotes the length of $C$. Clearly, the zeta function of a disconnected graph is the product of the zeta functions of its connected components. Zeta functions of graphs were originated from zeta functions of regular graphs by Ihara [5], where their reciprocals are expressed as explicit polynomials. A zeta function of a regular graph $G$ associated to a unitary representation of the fundamental group of $G$ was developed by Sunada [14]. Hashimoto [4] treated multivariable zeta functions of bipartite graphs. Northshield [11] proved that the number of spanning trees in a graph $G$ can be expressed in terms of the zeta function $Z_G(u)$.

Let $G$ be a connected graph with $V(G) = \{v_1, \ldots, v_m\}$. The *adjacency matrix* $A(G) = (a_{ij})$ is an $m \times m$ matrix with $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. Let $D_G$ denote the diagonal matrix with diagonal entries $d_i^G = \deg_G(v_i)$, $1 \leq i \leq m$, and let $Q_G = D_G - I$. The Ihara's result on zeta functions of regular graphs is generalized as follows.

THEOREM 1. (Bass [1]) *Let $G$ be a connected graph with $m$ vertices and $s$ edges. Then the reciprocal of the zeta function of $G$ is given by*

$$Z_G(u)^{-1} = (1 - u^2)^{s-m} \det \left( I - A(G)u + Q_G u^2 \right).$$

Note that Theorem 1 is still true for a disconnected graph $G$.

Later, Stark and Terras [13] gave an elementary proof of Theorem 1 and discussed three different zeta functions of a graph. Mizuno and Sato [9] gave a decomposition formula for the zeta function of a regular covering of a graph. In this paper, we compute the zeta function of a graph bundle. In section 2, a formula for the zeta function of a graph bundle is derived by Theorem 1 and a decomposition formula for the zeta function of any (regular or irregular) covering graph is given. In Sections 3 and 4, we compute the zeta function of a graph bundle when its voltages lie in an abelian group or in a dihedral group and whose fibre is a regular graph. The zeta function of a cartesian product of a graph with a regular graph is given in Section 3 too. As an application, in Section 5, the zeta functions of a discrete torus and of a discrete Klein bottle are computed.

## 2. Computing zeta functions of graph bundles

Let $G$ be a connected graph and let $\vec{G}$ be the digraph obtained from $G$ by replacing each edge of $G$ with a pair of oppositely directed edges. The set of directed edges of $\vec{G}$ is denoted by $E(\vec{G})$. By $e^{-1}$, we mean the reverse edge to an edge $e \in E(\vec{G})$. We denote the directed edge $e$ of $\vec{G}$ by $uv$ if the initial and terminal vertices of $e$ are $u$ and $v$, respectively. For a finite group $\Gamma$, a $\Gamma$-*voltage assignment* on $G$ is a function $\phi : E(\vec{G}) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\vec{G})$. We denote the set of all $\Gamma$-voltage assignments on $G$ by $C^1(G; \Gamma)$.

Let $F$ be another graph and let $\phi \in C^1(G; \mathrm{Aut}(F))$. Now, we construct a new graph $G \times^{\phi} F$ with the vertex set $V(G \times^{\phi} F) = V(G) \times V(F)$, and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times^{\phi} F$ if either $u_1 u_2 \in E(\vec{G})$ and $v_2 = v_1^{\phi(u_1 u_2)}$ or $u_1 = u_2$ and $v_1 v_2 \in E(F)$. We call $G \times^{\phi} F$ the $F$-*bundle over* $G$ *associated with* $\phi$ (or, simply a *graph bundle*) and the first coordinate projection induces the bundle projection $p^{\phi} : G \times^{\phi} F \to G$. The graphs $G$ and $F$ are called the *base* and the *fibre* of the graph bundle $G \times^{\phi} F$, respectively. Note that the map $p^{\phi}$ maps vertices to vertices, but the image of an edge can be either an edge or a vertex. If $F = \overline{K_n}$, the complement of the complete graph $K_n$ of $n$ vertices, then an $F$-bundle over $G$ is just an $n$-fold graph covering of $G$. If $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in E(\vec{G})$, then $G \times^{\phi} F$ is just the cartesian product of $G$ and $F$. (See [6]).

Let $\phi$ be an $\mathrm{Aut}(F)$-voltage assignment on $G$. For each $\gamma \in \mathrm{Aut}(F)$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$. Thus the digraph $\vec{G}$ is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\phi,\gamma)}$, $\gamma \in \mathrm{Aut}(F)$. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(F) = \{v_1, v_2, \ldots, v_n\}$. We define an order relation $\leq$ on $V(G \times^\phi F)$ as follows: for $(u_i, v_k), (u_j, v_\ell) \in V(G \times^\phi F)$, $(u_i, v_k) \leq (u_j, v_\ell)$ if and only if either $k < \ell$ or $k = \ell$ and $i \leq j$. Let $P(\gamma)$ denote the $n \times n$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$, i.e., its $(i,j)$-entry $P(\gamma)_{ij} = 1$ if $v_i^\gamma = v_j$ and $P(\gamma)_{ij} = 0$ otherwise. Then for any $\gamma, \delta \in \mathrm{Aut}(F)$, $P(\delta\gamma) = P(\delta)P(\gamma)$. The tensor product $A \otimes B$ of the matrices $A$ and $B$ is considered as the matrix $B$ having the element $b_{ij}$ replaced by the matrix $Ab_{ij}$. Kwak and Lee formulated the adjacency matrix $A(G \times^\phi F)$ of a graph bundle $G \times^\phi F$ as follows.

THEOREM 2. ([7])

$$A(G \times^\phi F) = \left( \sum_{\gamma \in \mathrm{Aut}(F)} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) \right) + I_m \otimes A(F),$$

*where $P(\gamma)$ is the $n \times n$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$, and $I_m$ is the identity matrix of order $m = |V(G)|$.*

For any vertex $(u_i, v_k) \in V(G \times^\phi F)$, its degree is $d_i^G + d_k^F$, where $d_i^G = \deg_G(u_i)$ and $d_k^F = \deg_F(v_k)$. Therefore, $D_{G \times^\phi F} = D_G \otimes I_n + I_m \otimes D_F$ and then $Q_{G \times^\phi F} = D_{G \times^\phi F} - I_{mn} = (D_G - I_m) \otimes I_n + I_m \otimes D_F = Q_G \otimes I_n + I_m \otimes D_F$. Furthermore, if we set $|E(G)| = s$ and $|E(F)| = t$, then

$$\begin{aligned} |E(G \times^\phi F)| &= \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{n} (d_i^G + d_k^F) \\ &= \frac{1}{2} \left( n \sum_{i=1}^{m} d_i^G + m \sum_{k=1}^{n} d_k^F \right) = ns + mt. \end{aligned}$$

The following theorem follows immediately from Theorem 1.

THEOREM 3. *Let $G$ be a connected graph with $m$ vertices and $s$ edges, $F$ a graph with $n$ vertices and $t$ edges and let $\phi$ be an $\mathrm{Aut}(F)$-voltage assignment on $G$. Then the reciprocal of the zeta function of*

*$G \times^{\phi} F$ is*

$$\begin{aligned}
&Z_{G\times^{\phi}F}(u)^{-1}\\
&= (1-u^2)^{(ns+mt)-mn}\\
&\quad\times \det\Big(I_{mn} - \big(\sum_{\gamma\in\mathrm{Aut}(F)} A(\vec{G}_{(\phi,\gamma)})\otimes P(\gamma) + I_m\otimes A(F)\big)u\\
&\quad + (Q_G\otimes I_n + I_m\otimes D_F)u^2\Big).
\end{aligned}$$

In the following, we consider three particular cases of Theorem 3: (i) $F = \overline{K}_n$, (ii) $\phi = 1$ is trivial, or more generally all voltages lie in an abelian group, and (iii) all voltages lie in a dihedral group. The last two cases will be treated in Sections 3 and 4, respectively.

As the first case, let $F = \overline{K}_n$ be $n$ isolated vertices. Then any $\mathrm{Aut}(\overline{K}_n)$-voltage assignment is just a permutation voltage assignment defined in [3], and $G \times^{\phi} \overline{K}_n = G^{\phi}$ is just an $n$-fold covering graph of $G$. In this case, it may not be a regular covering. Hence,

$$\begin{aligned}
&Z_{G^{\phi}}(u)^{-1}\\
&= (1-u^2)^{(s-m)n}\\
&\quad\times \det\left(I_{mn} - \left(\sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)})\otimes P(\gamma)\right)u + (Q_G\otimes I_n)u^2\right).
\end{aligned}$$

A *representation* $\rho$ of a group $\Gamma$ over the complex field $\mathbb{C}$ is a group homomorphism from $\Gamma$ to the general linear group $\mathrm{GL}(r,\mathbb{C})$ of all $r\times r$ invertible matrices over $\mathbb{C}$. The number $r$ is called the *degree* of the representation $\rho$ (see [15]). Let $\phi \in C^1(G;\mathrm{Aut}(\overline{K}_n))$ be a permutation voltage assignment on $G$, and let $\Gamma = \langle \phi(e) \mid e \in E(\vec{G})\rangle$ be the subgroup of $S_n = \mathrm{Aut}(\overline{K}_n)$ generated by the voltages $\phi(e)$. Then, it is clear that the homomorphism $P : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ defined by $\gamma \mapsto P(\gamma)$ is a representation of $\Gamma$, which is called the *permutation representation* of $\Gamma$. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the nonequivalent irreducible representations of $\Gamma$ and let $f_i$ be the degree of $\rho_i$ for $1 \le i \le \ell$, so that $\sum_{i=1}^{\ell} f_i^2 = |\Gamma|$. It is well-known [15] that the permutation representation $P$ can be decomposed as the direct sum of irreducible representations: Say $P = \bigoplus_{i=1}^{\ell} m_i\rho_i$ with multiplicities $m_i$. Then, there exists an invertible matrix $M$ such that

$$M^{-1}P(\gamma)M = \bigoplus_{i=1}^{\ell}(\rho_i(\gamma)\otimes I_{m_i})$$

for any $\gamma \in \Gamma$, where $A_1 \oplus \cdots \oplus A_k$ denotes the block diagonal matrix with diagonal blocks $A_1, \ldots, A_k$ consecutively. Noting that $A(G) = \sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)})$ and $\sum_{i=1}^{\ell} m_i f_i = n$, we have

$$
\begin{aligned}
&(I_m \otimes M)^{-1} \left( I_{mn} - \left( \sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) \right) u + (Q_G \otimes I_n) u^2 \right) \\
&\times (I_m \otimes M) \\
&= I_{mn} - \left( \bigoplus_{i=1}^{\ell} \left( \sum_{\gamma\in\Gamma} (A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma)) \otimes I_{m_i} \right) \right) u + (Q_G \otimes I_n) u^2 \\
&= \bigoplus_{i=1}^{\ell} \left( I_{mf_i} - \left( \sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \right) u + (Q_G \otimes I_{f_i}) u^2 \right) \otimes I_{m_i}.
\end{aligned}
$$

Furthermore, it is known [12] that $m_1$ is the number of orbits under the action of the group $\Gamma$, so that $m_1 \geq 1$. Therefore, the zeta function of a (regular or irregular) covering $G^\phi$ is

$$
\begin{aligned}
&Z_{G^\phi}(u)^{-1} \\
&= ((1-u^2)^{s-m} \det(I_m - A(G)u + Q_G u^2))^{m_1} \\
&\times \prod_{i=2}^{\ell} ((1-u^2)^{(s-m)f_i} \det(I_{mf_i} - (\sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma))u \\
&+ (Q_G \otimes I_{f_i})u^2))^{m_i}.
\end{aligned}
$$

Since $Z_G(u)^{-1} = (1-u^2)^{s-m} \det(I_m - A(G)u + Q_G u^2)$, we have the following theorem.

THEOREM 4. *Let $G$ be a connected graph with $m$ vertices and $s$ edges, $\phi \in C^1(G; \mathrm{Aut}(\overline{K}_n))$ a permutation voltage assignment on $G$, and let $\Gamma = \langle \phi(e) \mid e \in E(\vec{G}) \rangle$ be the subgroup generated by the voltages $\phi(e)$. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ with degrees $f_1, f_2, \ldots, f_\ell$, respectively. Then the reciprocal of the zeta function of the $n$-fold covering $G^\phi$ of $G$ derived from the voltage assignment $\phi$ is*

$$
Z_{G^\phi}(u)^{-1} = \left(Z_G(u)^{-1}\right)^{m_1} \prod_{i=2}^{\ell} \left( (1-u^2)^{(s-m)f_i} \; T_i(u) \right)^{m_i},
$$

*where*

$$(1)\quad T_i(u) = \det\left(I_{mf_i} - \left(\sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma)\right) u + (Q_G \otimes I_{f_i})u^2\right)$$

*and $m_i$ is the multiplicity of $\rho_i$ in the permutation representation $P$ of $\Gamma$.*

It gives another proof of Corollary 1 in Section 2 in [13].

COROLLARY 5. (Stark and Terras [13]) *For every connected covering graph $G^\phi$ of the graph $G$, the inverse zeta function $Z_G(u)^{-1}$ divides $Z_{G^\phi}(u)^{-1}$.*

If $\Gamma$ acts on itself by right multiplication, then $\Gamma$ can be identified as a regular subgroup of $S_\Gamma$ and the covering $G^\phi$ is a regular covering of $G$. In this case, the multiplicity $m_i$ is equal to the degree $f_i$ of the irreducible representation $\rho_i$. Therefore, we have Theorem 2 in [9] as a corollary.

COROLLARY 6. (Mizuno and Sato [9]) *The reciprocal of the zeta function of the connected regular covering $G^\phi$ of $G$ derived from an ordinary voltage assignment $\phi : E(\vec{G}) \to \Gamma$ is*

$$Z_{G^\phi}(u)^{-1} = Z_G(u)^{-1} \prod_{i=2}^{\ell} \left((1-u^2)^{(s-m)f_i}\, T_i(u)\right)^{f_i},$$

*where $T_i(u)$ is given in Eq (1).*

For a voltage assignment $\phi : E(\vec{G}) \to \Gamma$ and an irreducible representation $\rho$ of $\Gamma$, the *L-function* of $G$ associated to $\rho$ and $\phi$ is defined by

$$Z_G(u,\rho,\phi) = \prod_{[C]} \det\left(I_f - \rho(\phi(C))u^{\nu(C)}\right)^{-1},$$

where $f$ is the degree of $\rho$, $\phi(C)$ is the net voltage on $C$ and $[C]$ runs over all equivalence classes of primitive, backtracking-less and tail-less cycles of $G$ (see [4], [5] and [14]). It was proved in [9] that, for the irreducible representations $\rho_i$ of $\Gamma$,

$$Z_G(u,\rho_i,\phi)^{-1} = (1-u^2)^{(s-m)f_i}\, T_i(u),$$

where $T_i(u)$ is given in Eq (1). Therefore we have the following corollary.

COROLLARY 7. *Under the same assumptions as in Theorem 4, we have*

$$Z_{G^{\phi}}(u) = \prod_{i=1}^{\ell} Z_G(u, \rho_i, \phi)^{m_i},$$

*where $m_i$ is the multiplicity of $\rho_i$ in the permutation representation $P$ of $\Gamma$.*

EXAMPLE 1. Let $n = 3$ and $\Gamma = S_3$, so that $G^{\phi}$ is a 3-fold covering of $G$. The symmetric group $S_3$ has three irreducible representations $\rho_1 = 1$, $\rho_2$ the sign representation, with degrees $f_1 = f_2 = 1$ and $\rho_3$ with degree $f_3 = 2$ defined as follows:

$$\rho_3(1) = I_2, \quad \rho_3((123)) = \begin{bmatrix} \mu & 0 \\ 0 & \mu^2 \end{bmatrix}, \quad \rho_3((132)) = \begin{bmatrix} \mu^2 & 0 \\ 0 & \mu \end{bmatrix},$$
$$\rho_3((12)) = \begin{bmatrix} 0 & \mu \\ \mu^2 & 0 \end{bmatrix}, \quad \rho_3((13)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_3((23)) = \begin{bmatrix} 0 & \mu^2 \\ \mu & 0 \end{bmatrix},$$

where $\mu = \exp(2\pi i/3)$. The permutation representation can be decomposed as $P = \rho_1 \oplus \rho_3$. That is $m_1 = 1$, $m_2 = 0$ and $m_3 = 1$. By Theorem 4, one can get

$$\begin{aligned} & Z_{G^{\phi}}(u)^{-1} \\ &= Z_G(u)^{-1}(1-u^2)^{2(s-m)} \\ &\quad \times \det\left(I_{2m} - \left(\sum_{\gamma \in S_3} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_3(\gamma)\right) u + (Q_G \otimes I_2)u^2\right), \end{aligned}$$

where $m$ is the number of vertices of $G$. Note that $S_3$ is the dihedral group of order 6. This result can also be obtained from Theorem 9 in Section 4 later in a different way.

In particular, if $G$ is the diamond graph, i.e., the complete graph $K_4$ minus an edge, then an easy computation gives that

$$Z_G(u)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3).$$

Consider an $S_3$-voltage assignment $\phi$ which is defined as in Figure 1.

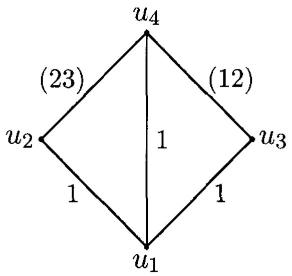


Figure 1: An $S_3$-voltage assignment $\phi$ on the diamond graph

Then the covering $G^\phi$ is connected. Moreover,

$$A(\vec{G}_{(\phi,(13))}) = A(\vec{G}_{(\phi,(123))}) = A(\vec{G}_{(\phi,(132))}) = \mathbf{0}$$

and

$$A(\vec{G}_{(\phi,1)}) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A(\vec{G}_{(\phi,(12))}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A(\vec{G}_{(\phi,(23))}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Thus one can have

$$\begin{aligned} Z_{G^\phi}(u)^{-1} &= Z_G(u)^{-1}(1-u^2)^2(1-u-u^3+2u^4) \\ &\quad \times(1-u+2u^2-u^3+2u^4)(1+u+u^3+2u^4) \\ &\quad \times(1+u+2u^2+u^3+2u^4). \end{aligned}$$

Note that this formula is given in Example 5(3) in [13] with an aid of Mathematica. However we compute it here with a permutation voltage assignment.

## 3. Graph bundles having voltages in an abelian group

In this section, we consider the zeta function of a graph bundle $G\times^\phi F$ when the images of $\phi$ lie in an abelian subgroup $\Gamma$ of $\mathrm{Aut}(F)$ and the graph $F$ is regular. In this case, for any $\gamma_1, \gamma_2 \in \Gamma$, the permutation matrices $P(\gamma_1)$ and $P(\gamma_2)$ are commutative and $D_F = k_F I_n$, where $k_F$ denotes the degree of the graph $F$.

It is well-known [2] that every permutation matrix $P(\gamma)$ commutes with the adjacency matrix $A(F)$ of $F$ for all $\gamma \in \mathrm{Aut}(F)$. Since the matrices $P(\gamma)$, $\gamma \in \Gamma$, and $A(F)$ are all diagonalizable and commute with each other, they are simultaneously diagonalizable. That is, there exists an invertible matrix $M_\Gamma$ such that $M_\Gamma^{-1}P(\gamma)M_\Gamma$ and $M_\Gamma^{-1}A(F)M_\Gamma$ are diagonal matrices for all $\gamma \in \Gamma$. Let $\lambda_{(\gamma,1)}, \ldots, \lambda_{(\gamma,n)}$ be the eigenvalues of the permutation matrix $P(\gamma)$ and let $\lambda_{(F,1)}, \ldots, \lambda_{(F,n)}$ be the eigenvalues of the adjacency matrix $A(F)$. Then

$$M_\Gamma^{-1}P(\gamma)M_\Gamma = \begin{bmatrix} \lambda_{(\gamma,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(\gamma,n)} \end{bmatrix} \quad \text{and}$$

$$M_\Gamma^{-1}A(F)M_\Gamma = \begin{bmatrix} \lambda_{(F,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(F,n)} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &(I_m \otimes M_\Gamma)^{-1}\left(\sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) + I_m \otimes A(F)\right)(I_m \otimes M_\Gamma) \\ = & \sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \begin{bmatrix} \lambda_{(\gamma,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(\gamma,n)} \end{bmatrix} \\ & + I_m \otimes \begin{bmatrix} \lambda_{(F,1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{(F,n)} \end{bmatrix} \\ = & \bigoplus_{i=1}^{n}\left(\sum_{\gamma\in\Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_m\right) \end{aligned}$$

and

$$(I_m \otimes M_\Gamma)^{-1}(Q_G \otimes I_n + I_m \otimes D_F)(I_m \otimes M_\Gamma) = \bigoplus_{i=1}^{n}(Q_G + k_F I_m).$$

Thus

$$\begin{aligned} &\det(I_{mn} - (\sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) + I_m \otimes A(F)\ )u \\ &+ (Q_G \otimes I_n + I_m \otimes D_F)u^2\ ) \end{aligned}$$

$$= \det \left(\bigoplus_{i=1}^{n} (I_m - (\sum_{\gamma\in\Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_m )u - (Q_G + k_F I_m)u^2)\right),$$

and then

$$\begin{aligned} & Z_{G\times^{\phi}F}(u)^{-1} \\ = \; & (1-u^2)^{(s-m+\frac{mk_F}{2})n} \\ & \times \prod_{i=1}^{n} \det(I_m - (\sum_{\gamma\in\Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_m)u + (Q_G + k_F I_m)u^2). \end{aligned}$$

Since the cartesian product $G \times F$ of two graphs $G$ and $F$ is the $F$-bundle over $G$ associated with the trivial voltage assignment $\phi$, i.e., $\phi(e) = 1$ for all $e \in E(\vec{G})$ and $A(G) = A(\vec{G})$, we get the following corollary.

COROLLARY 8. *For any connected graph $G$ and a connected $k_F$-regular graph $F$, the reciprocal of the zeta function of the cartesian product $G \times F$ is*

$$\begin{aligned} Z_{G\times F}(u)^{-1} = \; & (1-u^2)^{\left(s-m+\frac{mk_F}{2}\right)n} \\ & \times \prod_{i=1}^{n} \det\left(I_m - (A(G) + \lambda_{(F,i)} I_m)u + (Q_G + k_F I_m)u^2\right), \end{aligned}$$

*where $|V(G)| = m$, $|E(G)| = s$ and $|V(F)| = n$.*

*In particular, if $G$ is regular of degree $k_G$, then the reciprocal of the zeta function of the cartesian product $G \times F$ is*

$$\begin{aligned} Z_{G\times F}(u)^{-1} = \; & (1-u^2)^{\left(\frac{k_G+k_F}{2}-1\right)mn} \\ & \times \prod_{i=1}^{n}\prod_{j=1}^{m} \left(1 - (\lambda_{(G,j)} + \lambda_{(F,i)})u + (k_G + k_F - 1)u^2\right), \end{aligned}$$

*where $\lambda_{(G,j)}$, $1 \le j \le m$, are the eigenvalues of the graph $G$.*

EXAMPLE 2. Let $G = F = K_3$, the complete graph with 3 vertices. Then $Z(K_3, u)^{-1} = (1-u^3)^2$. The eigenvalues of $K_3$ are 2, $-1$ and $-1$. By Corollary 8, we get

$$Z_{K_3\times K_3}(u)^{-1} = (1-u^2)^9(1-4u+3u^2)(1-u+3u^2)^4(1+2u+3u^2)^4.$$

It is clear that $Z(K_3, u)^{-1}$ is not a divisor of $Z(K_3 \times K_3, u)^{-1}$. This example says that in general, $Z_G(u)^{-1}$ or $Z_F(u)^{-1}$ is not necessarily to be a divisor of $Z_{G\times F}(u)^{-1}$.

In general, for the complete graph $K_m$, its eigenvalues are $m-1$ with multiplicity 1 and $-1$ with multiplicity $m-1$. Therefore

$$\begin{aligned} & Z_{K_m \times K_n}(u)^{-1} \\ = \; & (1-u^2)^{(\frac{m+n}{2}-2)mn}(1-(m+n-2)u+(m+n-3)u^2) \\ & \times(1-(m-2)u+(m+n-3)u^2)^{n-1} \\ & \times(1-(n-2)u+(m+n-3)u^2)^{m-1} \\ & \times(1+2u+(m+n-3)u^2)^{(m-1)(n-1)}. \end{aligned}$$

EXAMPLE 3. Let $C_n$ be the $n$-cycle. It is known [15] that the eigenvalues of $C_n$ are $2\cos(2\pi k/n)$, $0 \le k \le n-1$. By Corollary 8, we get

$$\begin{aligned} & Z_{C_m \times C_n}(u)^{-1} \\ = \; & (1-u^2)^{mn} \prod_{k_2=0}^{n-1} \prod_{k_1=0}^{m-1} (1-2(\cos(2\pi k_1/m)+\cos(2\pi k_2/n))u+3u^2). \end{aligned}$$

## 4. Graph bundles having voltages in a dihedral group

In this section, we compute the zeta function of a graph bundle having voltages in a dihedral group. Assume that $F$ is $k_F$-regular with $n$ vertices again and let $\mathrm{Aut}(F)$ contains the dihedral group $D_n$ of order $2n$, which is described below, as a subgroup in addition.

Let $S_n$ denote the symmetric group on $V(F)$. Set $V(F) = \{1, 2, \ldots, n\}$ for a notational convenience. Let $a = (1\ 2\ \cdots\ n-1\ n)$ be an $n$-cycle and let

$$b = \begin{cases} (1\ n)(2\ n-1)\cdots(\frac{n-1}{2}\ \frac{n+3}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd,} \\ (1\ n)(2\ n-1)\cdots(\frac{n}{2}\ \frac{n+2}{2}) & \text{if } n \text{ is even} \end{cases}$$

be a permutation in $S_n$. Note that the permutations $a$ and $b$ generate the dihedral subgroup $D_n$ of $S_n$, where

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ bab = a^{-1} \rangle = \{1, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}.$$

Let $\mu = \exp(2\pi i/n)$ and let $\mathbf{x}_k \doteq (1, \mu^k, \mu^{2k}, \ldots, \mu^{(n-1)k})^T$ be a column vector in the complex $n$-space $\mathbb{C}^n$. Then $1, \mu, \ldots, \mu^{n-1}$ are distinct eigenvalues of the permutation matrix $P(a)$ and $\mathbf{x}_k$ is an eigenvector of $P(a)$ belonging to the eigenvalue $\mu^k$ for $0 \le k \le n-1$. Let $P(b)$ be the permutation matrix of $b$ and let

$$M = \begin{cases} [\mathbf{x}_0\ \mathbf{x}_1\ P(b)\mathbf{x}_1\ \mathbf{x}_2\ P(b)\mathbf{x}_2\ \cdots\ \mathbf{x}_{(n-1)/2}\ P(b)\mathbf{x}_{(n-1)/2}] & \text{if } n \text{ is odd,} \\ [\mathbf{x}_0\ \mathbf{x}_1\ P(b)\mathbf{x}_1\ \mathbf{x}_2\ P(b)\mathbf{x}_2\ \cdots\ \mathbf{x}_{(n-2)/2}\ P(b)\mathbf{x}_{(n-2)/2}\ \mathbf{x}_{n/2}] & \text{if } n \text{ is even.} \end{cases}$$

From $P(a)P(b)\mathbf{x}_k=P(ab)\mathbf{x}_k=P(ba^{-1})\mathbf{x}_k=P(b)P(a^{-1})\mathbf{x}_k = \mu^{n-k}P(b)\mathbf{x}_k$, we know that for any $0 \le k \le n-1$, $P(b)\mathbf{x}_k$ is an eigenvector of $P(a)$ associated with the eigenvalue $\mu^{n-k}$. Thus the matrix $M$ is invertible because its column vectors are eigenvectors of $P(a)$ associated with distinct eigenvalues. Moreover, $P(a)$ and $A(F)$ are commutative since $a \in \mathrm{Aut}(F)$ from the assumption. Thus $P(a)$ and $A(F)$ are simultaneously diagonalizable. Note that the columns of $M$ are eigenvectors of $P(a)$ associated with distinct eigenvalues, so each column of $M$ is again an eigenvector of $A(F)$. Since $P(b)A(F) = A(F)P(b)$ for $1 \le k \le (n-1)/2$ when $n$ is odd and for $1 \le k \le (n-2)/2$ when $n$ is even, $\mathbf{x}_k$ and $P(b)\mathbf{x}_k$ are eigenvectors of $A(F)$ belonging to the same eigenvalue, say it $\lambda_{(F,k)}$. The all-one vector $\mathbf{x}_0$ is an eigenvector of $A(F)$ belonging to the eigenvalue $k_F$, the degree of the graph $F$. When $n$ is even, denote by $\lambda_{(F,n/2)}$ the eigenvalue of $A(F)$ associated with the eigenvector $\mathbf{x}_{n/2}$. From [8], we know that

$$(I_m \otimes M)^{-1}\left(\sum_{\gamma\in D_n} A(\vec{G}_{(\phi,\gamma)})\otimes P(\gamma) + I_m \otimes A(F)\right)(I_m\otimes M)$$

$$= \begin{cases} (A(G)+k_F I_m) \oplus \left(\displaystyle\bigoplus_{t=1}^{(n-1)/2} (A_t + \lambda_{(F,t)} I_{2m})\right) & \text{if } n \text{ is odd,} \\ (A(G)+k_F I_m) \oplus \left(\displaystyle\bigoplus_{t=1}^{(n-2)/2} (A_t + \lambda_{(F,t)} I_{2m})\right) & \\ \qquad\qquad \oplus\ (B + \lambda_{(F,n/2)} I_m) & \text{if } n \text{ is even,} \end{cases}$$

where

$$A_t = \sum_{k=0}^{n-1} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,a^k b)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k b)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix} \tag{2}$$

is a $2m \times 2m$ matrix and

$$B = \sum_{k=0}^{n-1} \Big((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,a^k b)})\Big) \tag{3}$$

is an $m \times m$ matrix. Again we have

$$(I_m \otimes M)^{-1}(Q_G \otimes I_n + I_m \otimes D_F)(I_m \otimes M) = \bigoplus_{i=1}^{n} (Q_G + k_F I_m).$$

We denote the $2m \times 2m$ matrix $(Q_G + k_F I_m) \otimes I_2$ by $L_G$. Thus when $n$ is odd,

$$\begin{aligned}
& \det(I_{mn} - (\sum_{\gamma \in D_n} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) + I_m \otimes A(F))u \\
& +(Q_G \otimes I_n + I_m \otimes D_F)u^2) \\
= \; & \det(I_m - (A(G) + k_F I_m)u + (Q_G + k_F I_m)u^2) \\
& \times \prod_{t=1}^{(n-1)/2} \det(I_{2m} - (A_t + \lambda_{(F,t)} I_{2m})u + L_G u^2),
\end{aligned}$$

and when $n$ is even,

$$\begin{aligned}
& \det(I_{mn} - (\sum_{\gamma \in D_n} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) + I_m \otimes A(F))u \\
& +(Q_G \otimes I_n + I_m \otimes D_F)u^2) \\
= \; & \det(I_m - (A(G) + k_F I_m)u + (Q_G + k_F I_m)u^2) \\
& \times \det(I_m - (B + \lambda_{(F,n/2)} I_m)u + (Q_G + k_F I_m)u^2) \\
& \times \prod_{t=1}^{(n-2)/2} \det(I_{2m} - (A_t + \lambda_{(F,t)} I_{2m})u + L_G u^2).
\end{aligned}$$

The following theorem comes from Theorem 3.

THEOREM 9. *Let $G$ be a connected graph with $m$ vertices and $s$ edges, and let $F$ be a connected $k_F$-regular graph with $n$ vertices such that* $\mathrm{Aut}(F)$ *contains the dihedral group $D_n$. Then for any $D_n$-voltage assignment $\phi$ on $G$, the reciprocal of the zeta function of the graph bundle $G \times^\phi F$ is*

$$Z_{G \times^\phi F}(u)^{-1} = (1-u^2)^{\left(s-m+\frac{mk_F}{2}\right)n} f_{G,F}(u) \prod_{t=1}^{(n-1)/2} g_{G,F,t}(u)$$

*when $n$ is odd, and*

$$Z_{G \times^\phi F}(u)^{-1} = (1-u^2)^{\left(s-m+\frac{mk_F}{2}\right)n} f_{G,F}(u) h_{G,F}(u) \prod_{t=1}^{(n-2)/2} g_{G,F,t}(u)$$

*when $n$ is even, where the polynomials $f_{G,F}(u)$, $g_{G,F,t}(u)$ and $h_{G,F}(u)$ are*

$$\begin{aligned} f_{G,F}(u) &= \det(I_m - (A(G) + k_F I_m)u + (Q_G + k_F I_m)u^2), \\ g_{G,F,t}(u) &= \det(I_{2m} - (A_t + \lambda_{(F,t)} I_{2m})u + L_G u^2), \\ h_{G,F}(u) &= \det(I_m - (B + \lambda_{(F,n/2)} I_m)u + (Q_G + k_F I_m)u^2). \end{aligned}$$

## 5. Applications

For a graph $G$ with $m$ vertices, let $\omega : E(\vec{G}) \to \mathbb{C}$ be a function on the set of directed edges of $G$ such that $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$ for each $e \in E(\vec{G})$. Such a function $\omega$ is called a *symmetric weight function* on the graph $G$. Define an $m \times m$ matrix $A(G_\omega) = (a_{ij})$ as

$$a_{ij} = \begin{cases} \omega(u_i u_j) & \text{if } u_i u_j \in E(\vec{G}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A(G_\omega)$ is a Hermitian matrix and $A(G_\omega) = A(G)$ when $\omega(e) = 1$ for all $e \in E(\vec{G})$.

For any $D_n$-voltage assignment $\phi$ on $G$, define a new $\mathbb{Z}_2$-voltage assignment $\psi_\phi$ on $G$ by

$$\psi_\phi(e) = \begin{cases} 1 & \text{if } \phi(e) = a^k \text{ for some } 0 \le k \le n-1, \\ -1 & \text{if } \phi(e) = a^k b \text{ for some } 0 \le k \le n-1, \end{cases}$$

for $e = u_i u_j \in E(\vec{G})$. The derived double covering of $G$ by $\psi_\phi$ is denoted by $G^{\psi_\phi}$. For any $t$, $1 \le t \le \lfloor (n-1)/2 \rfloor$, define a function $\omega_t(\phi) : E(\vec{G}^{\psi_\phi}) \to \mathbb{C}$ as follows: for any $e = (u_i, g)(u_j, \psi_\phi(u_i u_j) g) \in E(\vec{G}^{\psi_\phi})$,

$$\omega_t(\phi)(e) = \begin{cases} \mu^{tk} & \text{if } g = 1 \text{ and } (\phi(u_i u_j) = a^k \text{ or } a^k b), \\ \mu^{(n-t)k} & \text{if } g = -1 \text{ and } (\phi(u_i u_j) = a^k \text{ or } a^k b), \end{cases}$$

where $\mu = \exp(2\pi i/n)$. Then $\omega_t(\phi)$ is a symmetric weight function on the graph $G^{\psi_\phi}$.

Define another function $\omega_{-1}(\phi) : E(\vec{G}) \to \mathbb{C}$ on the graph $G$ by

$$\omega_{-1}(\phi)(u_i u_j) = \begin{cases} (-1)^k & \text{if } \phi(u_i u_j) = a^k, \\ (-1)^{k+1} & \text{if } \phi(u_i u_j) = a^k b \end{cases}$$

for $u_i u_j \in E(\vec{G})$. Then $\omega_{-1}(\phi)$ is a symmetric weight function on $G$. The following lemma was obtained by Kwak and Kwon.

LEMMA 10. ([8]) *Let $A_t$ and $B$ be the matrices in Eq (2) and (3), respectively. Then*

(1) *for any $t = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$, $A(G^{\psi_\phi}_{\omega_t(\phi)}) = A_t$ as $2m \times 2m$ matrices under vertex order $(u_1, 1), (u_2, 1), \ldots, (u_m, 1), (u_1, -1), (u_2, -1), \ldots, (u_m, -1)$, and*
(2) *if $n$ is even, $A(G_{\omega_{-1}(\phi)}) = B$ as $m \times m$ matrices.*

Let $G = C_m$ be the $m$-cycle with consecutive vertices $u_1, u_2, \ldots, u_m$. The digraph $\vec{C_m}$ is an edge-disjoint union of two directed cycles $C_m^+ = (u_1, u_2, \ldots, u_m, u_1)$ and $C_m^- = (u_1, u_m, \ldots, u_2, u_1)$. Now, let $F = C_n$ be another cycle with vertices $v_1, v_2, \ldots, v_n$ so that $\mathrm{Aut}(F) = \mathrm{Aut}(C_n) = D_n$. Let $\phi \in C^1(C_m, D_n)$ be a $D_n$-voltage assignment on $C_m$. Define the net voltage on $C_m^+$ by $\phi(C_m^+) = \phi(u_1u_2) \cdots \phi(u_{m-1}u_m)\phi(u_mu_1)$. The graph bundle $C_m \times^\phi C_n$ is called a *discrete torus* if $\phi(C_m^+) = a^k$ for some $0 \le k \le n-1$ and a *discrete Klein bottle* if $\phi(C_m^+) = a^k b$ for some $0 \le k \le n-1$. As an application of our formula, we compute the zeta functions of a discrete torus and of a discrete Klein bottle in this section.

Let $H_1(x) = x$ and $H_2(x) = x^2 - 1$ be polynomials in $x$ and let $H_j(x)$ be a sequence of polynomials satisfying the recurrence relation

$$H_{j+2}(x) = xH_{j+1}(x) - H_j(x).$$

Set

$$P_j(x) = H_j(x) - H_{j-2}(x). \tag{4}$$

Then a straightforward calculation gives the following lemma.

LEMMA 11. *Let $\omega$ be any symmetric weight function on $C_m$. Then*

$$\begin{aligned} &\det\left(I_m - (A((C_m)_\omega) + \lambda I_m)u + (\delta I_m)u^2\right) \\ &= u^m \left( P_m\left(\frac{1}{u} - \lambda + \delta u\right) - \left(\omega(C_m^+) + \overline{\omega(C_m^+)}\right)\right), \end{aligned}$$

*where $\omega(C_m^+) = \omega(u_1u_2) \cdots \omega(u_{m-1}u_m)\omega(u_mu_1)$.*

Now, we are ready to compute the zeta functions of a discrete torus and of a discrete Klein bottle. To do this, one needs to compute three polynomials $f_{C_m,C_n}(u)$, $h_{C_m,C_n}(u)$ and $g_{C_m,C_n,t}(u)$ defined in Theorem 9. For $G = C_m$ and $F = C_n$, we have $k_F = 2$ and $Q_G + k_F I_m = 3I_m$. From Lemma 11, it is easy to show that

$$f_{C_m,C_n}(u) = u^m \left( P_m\left(\frac{1}{u} - 2 + 3u\right) - 2\right).$$

For any even $n$, since $\lambda_{C_n,n/2} = -2$, we have by Lemmas 10 and 11

$$h_{C_m,C_n}(u) = u^m \left(P_m\left(\frac{1}{u}+2+3u\right) - 2\omega_{-1}(\phi)(C_m^+)\right),$$

where

$$\omega_{-1}(\phi)(C_m^+) = \begin{cases} (-1)^k & \text{if } \phi(C_m^+) = a^k, \\ (-1)^{k+1} & \text{if } \phi(C_m^+) = a^k b. \end{cases}$$

For the polynomial $g_{C_m,C_n,t}(u)$, first note that the eigenvalues of $C_n$ are $\lambda_{(C_n,t)} = 2\cos\frac{2\pi t}{n}$ for $1 \le t \le \lfloor\frac{n-1}{2}\rfloor$. When $\phi(C_m^+) = a^k$ for some $0 \le k \le n-1$, $C_m^{\psi_\phi}$ is a disjoint union of two copies of $C_m$ but when $\phi(C_m^+) = a^k b$ for some $0 \le k \le n-1$, $C_m^{\psi_\phi}$ is the cycle $C_{2m}$. By a method similar as in [8] with Lemmas 10 and 11, one can get

$$g_{C_m,C_n,t}(u) = \begin{cases} u^{2m}\left(P_m(\frac{1}{u} - 2\cos\frac{2\pi t}{n} + 3u) - 2\cos\frac{2\pi tk}{n}\right)^2 & \text{if } \phi(C_m^+) = a^k, \\ u^{2m}\left(P_{2m}(\frac{1}{u} - 2\cos\frac{2\pi t}{n} + 3u) - 2\right) & \text{if } \phi(C_m^+) = a^k b. \end{cases}$$

Summarizing our discussions, we have the following theorems.

THEOREM 12. *Let a discrete torus $C_m \times^\phi C_n$ have the net voltage $\phi(C_m^+) = a^k$ for some $0 \le k \le n-1$. Then the reciprocal of its zeta function is*

$$Z_{C_m\times^\phi C_n}(u)^{-1} = \begin{cases} (1-u^2)^{mn}u^{mn}\left(P_m(\frac{1}{u}-2+3u)-2\right) \\ \quad\times \prod_{t=1}^{(n-1)/2}\left(P_m(\frac{1}{u} - 2\cos\frac{2\pi t}{n} + 3u) - 2\cos\frac{2\pi tk}{n}\right)^2 & \text{if } n \text{ is odd,} \\ (1-u^2)^{mn}u^{mn}\left(P_m(\frac{1}{u}-2+3u)-2\right) \\ \quad\times \left(P_m(\frac{1}{u}+2+3u)-2(-1)^k\right) \\ \quad\times \prod_{t=1}^{(n-2)/2}\left(P_m(\frac{1}{u} - 2\cos\frac{2\pi t}{n} + 3u) - 2\cos\frac{2\pi tk}{n}\right)^2 & \text{if } n \text{ is even,} \end{cases}$$

*where $P_m(x)$ is defined in Eq (4).*

THEOREM 13. *Let a discrete Klein bottle $C_m \times^\phi C_n$ have the net voltage $\phi(C_m^+) = a^k b$ for some $0 \le k \le n-1$. Then the reciprocal of its* *zeta function is*

$$Z_{C_m \times^{\phi} C_n}(u)^{-1} = \begin{cases} (1-u^2)^{mn}u^{mn}\left(P_m(\frac{1}{u}-2+3u)-2\right) & \\ \times \displaystyle\prod_{t=1}^{(n-1)/2}\left(P_{2m}(\frac{1}{u}-2\cos\frac{2\pi t}{n}+3u)-2\right) & \text{if } n \text{ is odd,} \\ & \\ (1-u^2)^{mn}u^{mn}\left(P_m(\frac{1}{u}-2+3u)-2\right) & \\ \times\left(P_m(\frac{1}{u}+2+3u)-2(-1)^{k+1}\right) & \\ \times \displaystyle\prod_{t=1}^{(n-2)/2}\left(P_{2m}(\frac{1}{u}-2\cos\frac{2\pi t}{n}+3u)-2\right) & \text{if } n \text{ is even,} \end{cases}$$

*where $P_m(x)$ is defined in Eq (4).*

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