UPPER AND LOWER SOLUTIONS METHOD FOR SECOND ORDER NONLINEAR FOUR POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We develop the upper and lower solutions method for the four point problem relative to second order differential equations in order to obtain the existence of solution.

1. Introduction

We consider the following second order nonlinear differential equation with four point boundary conditions

$$(1) \qquad \begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in I = [a, b], \\ x(a) = x(c), \ x(b) = x(d), \end{cases}$$

where $a < c \le d < b$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

The method of upper and lower solutions is used to prove the existence of solution to (1). To obtain a sequence which converges to a solution for this equation, approximated problems are considered. We extend some results of [5].

Note that (1) is a problem at resonance since any constant function is a solution for the linear equation $x'' = 0$ with the four-point boundary condition.

In this paper, the type of upper and lower solutions of (1) admissible is more general, whereas in [5], upper and lower solutions were considered constant functions.

In Section 2, the terms of upper and lower solutions of (1) are defined and an existence result for problem (1) is given, considering the continuity of f and the existence of well-ordered upper and lower solutions.

In [1], it is considered different four-point boundary condition so that the linear part is invertible (see Lemma 2.2 for the expression of the corresponding Green's function).

Also, in [2], it is studied a four-point boundary value problem for second-order nonlinear ordinary differential equations with the nonlinearity independent of the derivative. Positive solutions of nonlinear four-point boundary value problems have been studied recently [3, 4].

2. Main results

We consider the spaces of functions $C(I)$ and $C^1(I)$ furnished, respectively, with the norms

$$\|x\| = \sup\{|x(t)| : t \in I\}, \quad \|x\|_1 = \|x\| + \|x'\|.$$

DEFINITION 1. A function $\alpha \in C^2[a, b]$ is said to be a lower solution to problem (1) if

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \ t \in [a, b],$$

$$\alpha(a) \leq \alpha(c), \ \alpha(d) \geq \alpha(b).$$

Similarly, $\beta \in C^2[a, b]$ is an upper solution to problem (1) if

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)), \ t \in [a, b],$$

$$\beta(a) \geq \beta(c), \ \beta(d) \leq \beta(b).$$

This formulation includes the constant case, considered in [5], since functions

$$\alpha(t) = r_1, \ \beta(t) = r_2, \ t \in [a, b],$$

where r_1, $r_2 \in \mathbb{R}$, verify the boundary conditions in the previous definition. So that the existence result we present is a generalization of Lemma 2 in [5].

THEOREM 2.1. *Suppose that α, $\beta \in C^2[a,b]$ are, respectively, lower and upper solutions to problem (1), with $\alpha \leq \beta$ on $I = [a,b]$ and that there exists $K \in \mathbb{R}$, $K > 0$ such that*

$$\int_a^b |f(t,x,y)|\,dt \leq K, \ \forall x \in [\alpha(t), \beta(t)], \ y \in \mathbb{R}.$$

Then problem (1) has at least one solution u verifying that

$$\alpha(t) \leq u(t) \leq \beta(t), \ t \in [a,b].$$

Proof. Consider $m \in \mathbb{N}$, $m > 1$ fixed and define

$f_m(t,x,y)$

$$= \begin{cases} f(t,\beta(t),\beta'(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } \beta(t) + \frac{1}{m} \leq x, \\[2mm] f(t,\beta(t),y) + \left[f(t,\beta(t),\beta'(t)) - f(t,\beta(t),y) + \frac{x-\beta(t)}{1+|x-\beta(t)|} \right] \\[1mm] \quad \times m(x-\beta(t)), & \text{if } \beta(t) < x < \beta(t) + \frac{1}{m}, \\[2mm] f(t,x,y), \ \text{if } \alpha(t) \leq x \leq \beta(t), \\[2mm] f(t,\alpha(t),y) - \left[f(t,\alpha(t),\alpha'(t)) - f(t,\alpha(t),y) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|} \right] \\[1mm] \quad \times m(x-\alpha(t)), & \text{if } \alpha(t) - \frac{1}{m} < x < \alpha(t), \\[2mm] f(t,\alpha(t),\alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, \ \text{if } x \leq \alpha(t) - \frac{1}{m}. \end{cases}$$

Note that f_m is continuous on $I \times \mathbb{R}^2$. We consider the problem

(2) $\quad \begin{cases} x''(t) = f_m(t,x(t),x'(t)), & t \in I = [a,b], \\ x(a) = x(c), \ x(b) = x(d). \end{cases}$

If x is a solution to (2) such that $\alpha \leq x \leq \beta$, then x is a solution to (1).

Take $r = \max_I \beta - \min_I \alpha$. If $\min_I \alpha = \max_I \beta$, then $\alpha = \beta$ is a constant solution to (1). We can assume, therefore, that $r > 0$, and define $\sigma = \frac{r}{1+r}$, then $\sigma > 0$. Choose $R \geq 0$ such that

$$|f(t,\alpha(t),\alpha'(t))|, \ |f(t,\beta(t),\beta'(t))| \leq R, \ t \in I.$$

Take $M_1 > 0$ large enough such that

$$0 \leq R \leq M_1 \sigma.$$

Therefore,

$$-M_1 \sigma \leq -R \leq f(t,\alpha(t),\alpha'(t)), \ f(t,\beta(t),\beta'(t)) \leq R \leq M_1 \sigma, \ t \in I.$$

Define bounded and continuous functions

$$K_1(t, x) = \begin{cases} \frac{R}{M_1\sigma}, & \text{for } x \geq \alpha(t) + \frac{R}{mM_1\sigma}, \\ m(x - \alpha(t)), & \text{for } \alpha(t) \leq x \leq \alpha(t) + \frac{R}{mM_1\sigma}, \\ 0, & \text{for } x \leq \alpha(t), \end{cases}$$

$$K_2(t, x) = \begin{cases} 0, & \text{for } x \geq \beta(t), \\ m(x - \beta(t)), & \text{for } \beta(t) - \frac{R}{mM_1\sigma} \leq x \leq \beta(t), \\ -\frac{R}{M_1\sigma}, & \text{for } x \leq \beta(t) - \frac{R}{mM_1\sigma}. \end{cases}$$

Taking

$$\phi(t, x) = [K_1(t, x) + K_2(t, x)]M_1\sigma, \quad (t, x) \in I \times \mathbb{R},$$

then ϕ is continuous and bounded on $I \times \mathbb{R}$ and satisfies $-R \leq \phi \leq R$ on $I \times \mathbb{R}$.

Denote

$$p(t, x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}.$$

To prove existence of solution to (2), consider problem

$$(3) \qquad \begin{cases} x''(t) = \bar{f}_m(t, x(t), x'(t), \lambda), & t \in I = [a, b], \\ x(a) = x(c), \ x(b) = x(d), \end{cases}$$

where

$$\bar{f}_m(t, x, y, \lambda) = \lambda f_m(t, x, y) + (1 - \lambda)\left\{\phi(t, x) - p(t, x) + x\right\},$$

and $\lambda \in [0, 1]$.

Note that if x is a solution to (3) for $\lambda = 1$ and $x(t) \in [\alpha(t), \beta(t)]$, $\forall t \in I$, then

$$x''(t) = f_m(t, x(t), x'(t)) = f(t, x(t), x'(t))$$

and x is a solution to (1).

If $\lambda = 0$, problem (3) is reduced to

$$(4) \qquad \begin{cases} x''(t) = \phi(t, x(t)) - p(t, x(t)) + x(t), & t \in I, \\ x(a) = x(c), \ x(b) = x(d), \end{cases}$$

and, for $\lambda = 1$, it is (2). Since $\phi(t, x) - p(t, x)$ is continuous and bounded and the linear problem

$$\begin{cases} x''(t) = x(t), & t \in I, \\ x(a) = x(c), \ x(b) = x(d), \end{cases}$$

has only the trivial solution, then problem (4) has a solution, i.e., problem (3) has a solution for $\lambda = 0$.

For $\lambda \in [0, 1]$, any solution u to problem (3) satisfies that

$$\alpha(t) - \frac{1}{m} \leq u(t) \leq \beta(t) + \frac{1}{m}, \ t \in I.$$

Indeed, let $\lambda \in [0,1]$ and u a solution to (3). Consider the function $v(t) = u(t) - \beta(t) - \frac{1}{m}$ and suppose that $\max_{t \in I} v(t) = v(t_0) > 0$. Using the boundary conditions, we obtain that

$$v(a) = u(a) - \beta(a) - \frac{1}{m} \leq u(c) - \beta(c) - \frac{1}{m} = v(c), \ v(d) \geq v(b),$$

so that we can assume that $t_0 \in (a,b)$. Then

$$v(t_0) > 0, \ v'(t_0) = 0, \ v''(t_0) \leq 0,$$

that is,

$$u(t_0) > \beta(t_0) + \frac{1}{m}, \ u'(t_0) = \beta'(t_0).$$

Then

$$u(t_0) > \beta(t_0) + \frac{1}{m} \geq \alpha(t_0) + \frac{1}{m} \geq \alpha(t_0) + \frac{R}{mM_1\sigma},$$

so that

$$K_1(t_0, u(t_0)) = \frac{R}{M_1\sigma}, \ K_2(t_0, u(t_0)) = 0,$$

and, using the definition of ϕ and the hypotheses on β,

$$
\begin{aligned}
v''(t_0) &= u''(t_0) - \beta''(t_0) \\
&= \lambda f_m(t_0, u(t_0), u'(t_0)) \\
&\quad + (1-\lambda)\left\{\phi(t_0, u(t_0)) - p(t_0, u(t_0)) + u(t_0)\right\} - \beta''(t_0) \\
&\geq \lambda f_m(t_0, u(t_0), u'(t_0)) \\
&\quad + (1-\lambda)\left\{\phi(t_0, u(t_0)) - p(t_0, u(t_0)) + u(t_0)\right\} \\
&\quad - f(t_0, \beta(t_0), \beta'(t_0)) \\
&= \lambda\left[f(t_0, \beta(t_0), \beta'(t_0)) + \frac{u(t_0) - \beta(t_0)}{1 + |u(t_0) - \beta(t_0)|}\right] \\
&\quad + (1-\lambda)\left[R - \beta(t_0) + u(t_0)\right] - f(t_0, \beta(t_0), \beta'(t_0)) \\
&= (1-\lambda)\left[R - f(t_0, \beta(t_0), \beta'(t_0))\right] + (u(t_0) - \beta(t_0)) \\
&\quad \times \left[\frac{\lambda}{1 + |u(t_0) - \beta(t_0)|} + (1-\lambda)\right] \\
&> (1-\lambda)\left[R - f(t_0, \beta(t_0), \beta'(t_0))\right] \geq 0,
\end{aligned}
$$

which is a contradiction. This proves that $v \leq 0$ on I, so that $u(t) \leq \beta(t) + \frac{1}{m}$, $t \in I$.

Also, we can prove that $u(t) \geq \alpha(t) - \frac{1}{m}$, for all $t \in I$. Defining $v(t) = u(t) - \alpha(t) + \frac{1}{m}$, if $\min_{t \in I} v(t) = v(t_1) < 0$, then

$$v(a) \geq v(c), \ v(d) \leq v(b),$$

so that we can assume that $t_1 \in (a, b)$. Then

$$v(t_1) < 0, \ v'(t_1) = 0, \ v''(t_1) \geq 0,$$

which imply

$$u(t_1) < \alpha(t_1) - \frac{1}{m}, \ u'(t_1) = \alpha'(t_1).$$

Hence

$$u(t_1) < \alpha(t_1) - \frac{1}{m} \leq \beta(t_1) - \frac{1}{m} \leq \beta(t_1) - \frac{R}{mM_1\sigma},$$

and, therefore,

$$K_1(t_1, u(t_1)) = 0, \ K_2(t_1, u(t_1)) = -\frac{R}{M_1\sigma}.$$

Using the definition of ϕ and the hypotheses satisfied by α,

$$
\begin{aligned}
v''(t_1) &= u''(t_1) - \alpha''(t_1) \\
&\leq \lambda f_m(t_1, u(t_1), u'(t_1)) \\
&\quad + (1 - \lambda)\{\phi(t_1, u(t_1)) - p(t_1, u(t_1)) + u(t_1)\} \\
&\quad - f(t_1, \alpha(t_1), \alpha'(t_1)) \\
&= \lambda \left[f(t_1, \alpha(t_1), \alpha'(t_1)) + \frac{u(t_1) - \alpha(t_1)}{1 + |u(t_1) - \alpha(t_1)|} \right] \\
&\quad + (1 - \lambda)\left[-R - \alpha(t_1) + u(t_1) \right] - f(t_1, \alpha(t_1), \alpha'(t_1)) \\
&= -(1 - \lambda)\left[R + f(t_1, \alpha(t_1), \alpha'(t_1)) \right] + (u(t_1) - \alpha(t_1)) \\
&\quad \times \left[\frac{\lambda}{1 + |u(t_1) - \alpha(t_1)|} + (1 - \lambda) \right] \\
&< -(1 - \lambda)\left[R + f(t_1, \alpha(t_1), \alpha'(t_1)) \right] \leq 0,
\end{aligned}
$$

obtaining a contradiction again. This proves that $u(t) \geq \alpha(t) - \frac{1}{m}$, for all $t \in I$.

The boundary conditions on u imply that there exists $\tilde{t} \in (a, b)$ with $u'(\tilde{t}) = 0$, so that, integrating the equation in (3), we obtain, for $t > \tilde{t}$, that

$$u'(t) = u'(t) - u'(\tilde{t}) = \int_{\tilde{t}}^{t} u''(s) \, ds$$

$$= \int_{\tilde{t}}^{t} \left[\lambda f_m(s, u(s), u'(s)) + (1 - \lambda)\{\phi(s, u(s)) - p(s, u(s)) + u(s)\} \right] \, ds,$$

so that

$$|u'(t)| \leq \int_{\tilde{t}}^{t} |\lambda f_m(s, u(s), u'(s)) + (1 - \lambda)\{\phi(s, u(s)) - p(s, u(s)) + u(s)\}| \, ds$$

$$\leq \int_{a}^{b} |\lambda f_m(s, u(s), u'(s)) + (1 - \lambda)\{\phi(s, u(s)) - p(s, u(s)) + u(s)\}| \, ds.$$

A similar estimate can be obtained for $t < \tilde{t}$, so that, for all $t \in I$, we get

$$|u'(t)| \leq \int_{a}^{b} |f_m(s, u(s), u'(s))| \, ds + \int_{a}^{b} |\phi(s, u(s)) - p(s, u(s)) + u(s)| \, ds.$$

Since $\alpha(t) - \dfrac{1}{m} \leq u(t) \leq \beta(t) + \dfrac{1}{m}$, for $t \in I$, and $-R \leq \phi(s, u(s)) \leq R$, $s \in I$, then

$$-R - \beta(s) + \alpha(s) - \frac{1}{m} \leq \phi(s, u(s)) - p(s, u(s)) + u(s)$$

$$\leq R - \alpha(s) + \beta(s) + \frac{1}{m}, \ s \in I,$$

and

$$|\phi(s, u(s)) - p(s, u(s)) + u(s)| \leq R + \beta(s) - \alpha(s) + \frac{1}{m}, \ s \in I,$$

which implies that

$$|u'(t)| \leq \int_{a}^{b} |f_m(s, u(s), u'(s))| \, ds + \int_{a}^{b} \left[R + \beta(s) - \alpha(s) + \frac{1}{m} \right] \, ds, t \in I.$$

The expression $\alpha(t) - \dfrac{1}{m} \leq u(t) \leq \beta(t) + \dfrac{1}{m}$, for $t \in I$, also provides that $f_m(s, u(s), u'(s))$ can be equal to

$$f(s, \beta(s), u'(s))$$
$$+ \left[f(s, \beta(s), \beta'(s)) - f(s, \beta(s), u'(s)) + \frac{u(s) - \beta(s)}{1 + |u(s) - \beta(s)|} \right]$$
$$\times m(u(s) - \beta(s)), \qquad \qquad \text{if } \beta(s) < u(s) < \beta(s) + \frac{1}{m},$$

$$f(s, u(s), u'(s)), \qquad \qquad \text{if } \alpha(s) \leq u(s) \leq \beta(s),$$

$$f(s, \alpha(s), u'(s))$$
$$- \left[f(s, \alpha(s), \alpha'(s)) - f(s, \alpha(s), u'(s)) + \frac{u(s) - \alpha(s)}{1 + |u(s) - \alpha(s)|} \right]$$
$$\times m(u(s) - \alpha(s)), \qquad \qquad \text{if } \alpha(s) - \frac{1}{m} < u(s) < \alpha(s).$$

For $\beta(s) < u(s) < \beta(s) + \dfrac{1}{m}$,

$$|f_m(s, u(s), u'(s))|$$
$$\leq |f(s, \beta(s), u'(s))|$$
$$+ \left| f(s, \beta(s), \beta'(s)) - f(s, \beta(s), u'(s)) + \frac{u(s) - \beta(s)}{1 + |u(s) - \beta(s)|} \right|$$
$$\times m\,(u(s) - \beta(s))$$
$$\leq |f(s, \beta(s), u'(s))|$$
$$+ \left| f(s, \beta(s), \beta'(s)) - f(s, \beta(s), u'(s)) + \frac{u(s) - \beta(s)}{1 + |u(s) - \beta(s)|} \right| m\frac{1}{m}$$
$$\leq |f(s, \beta(s), u'(s))| + |f(s, \beta(s), \beta'(s))| + |f(s, \beta(s), u'(s))|$$
$$+ \frac{u(s) - \beta(s)}{1 + u(s) - \beta(s)}$$
$$< |f(s, \beta(s), u'(s))| + |f(s, \beta(s), \beta'(s))| + |f(s, \beta(s), u'(s))| + \frac{1}{m},$$

and, for $\alpha(s) - \dfrac{1}{m} < u(s) < \alpha(s)$,

$$|f_m(s, u(s), u'(s))|$$
$$\leq |f(s, \alpha(s), u'(s))|$$
$$+ \left| f(s, \alpha(s), \alpha'(s)) - f(s, \alpha(s), u'(s)) + \frac{u(s) - \alpha(s)}{1 + |u(s) - \alpha(s)|} \right|$$
$$\times m\,|u(s) - \alpha(s)|$$
$$\leq |f(s, \alpha(s), u'(s))|$$
$$+ \left| f(s, \alpha(s), \alpha'(s)) - f(s, \alpha(s), u'(s)) + \frac{u(s) - \alpha(s)}{1 + |u(s) - \alpha(s)|} \right| m\frac{1}{m}$$
$$\leq |f(s, \alpha(s), u'(s))| + |f(s, \alpha(s), \alpha'(s))| + |f(s, \alpha(s), u'(s))|$$
$$+ \left| \frac{u(s) - \alpha(s)}{1 + |u(s) - \alpha(s)|} \right|$$
$$< |f(s, \alpha(s), u'(s))| + |f(s, \alpha(s), \alpha'(s))| + |f(s, \alpha(s), u'(s))| + \frac{1}{m},$$

so that

$$\int_a^b |f_m(s, u(s), u'(s))|\, ds \leq 3\,K + \frac{1}{m}(b - a),$$

which implies

$$\|u'\| \leq 3\,K + \frac{1}{m}(b-a) + \int_a^b \left[R + \beta(s) - \alpha(s) + \frac{1}{m} \right]\, ds,$$

in particular,

$$\|u'\| \leq 3\,K + (b-a) + \int_a^b [R + \beta(s) - \alpha(s) + 1]\, ds = \gamma,$$

with γ independent of m.

Then, for $\lambda = 1$, problem (3) has a solution u_m such that

$$\alpha(t) - \frac{1}{m} \leq u_m(t) \leq \beta(t) + \frac{1}{m},\ t \in I.$$

Repeating this procedure for all $m \in \mathbb{N}$, $m > 1$, we obtain a sequence $\{u_m\}_{m=2}^{+\infty}$, where u_m is a solution to problem

$$\begin{cases} x''(t) = f_m(t, x(t), x'(t)), & t \in I, \\ x(a) = x(c),\ x(b) = x(d), \end{cases}$$

and

$$\|u_m\| \leq \max\{\|\alpha\|, \|\beta\|\} + 1,\ \|u_m'\| \leq \gamma,$$

so that the sequence $\{u_m\}$ is bounded in $C^1(I)$ and

$$\|u_m''(t)\| = \|f_m(t, u_m(t), u_m'(t))\|$$

is bounded, so that $\{u_m\}$ is equicontinuous in $C^1(I)$. Thus, Arzelá-Ascoli Theorem implies the existence of a subsequence of $\{u_m\}$ convergent in $C^1(I)$ to a function u. Since

$$\alpha(t) - \frac{1}{m} \leq u_m(t) \leq \beta(t) + \frac{1}{m},\ \text{for all } t \in I,$$

then

$$\alpha(t) \leq u(t) \leq \beta(t),\ t \in I.$$

Moreover, $\|u'\| \leq \gamma$, or more specifically,

$$\|u'\| \leq 3\,K + \int_a^b [R + \beta(s) - \alpha(s)]\, ds.$$

Function u is a solution to problem (1). $\qquad\qquad\qquad\square$

REMARK 1. Theorem 2.1 extends Lemma 2 in [5], where constant upper and lower solutions for (1) are considered.

THEOREM 2.2. *Suppose that α, β are, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ on I and that there exist real numbers R_1, R_2, R_3, R_4 such that $R_1 \neq R_3$, $R_2 \neq R_4$,*

$$R_1, R_3 \leq - \max\{\|\alpha'\|, \|\beta'\|\},$$

$$R_2, R_4 \geq \max\{\|\alpha'\|, \|\beta'\|\}$$

and

$$f(t, x, R_2) \geq 0, \ f(t, x, R_1) \leq 0, \ \forall t \in [a, b], \ \forall x \in [\alpha(t), \beta(t)],$$

$$f(t, x, R_3) \geq 0, \ f(t, x, R_4) \leq 0, \ for \ t \in [d, b], \ x \in [\alpha(t), \beta(t)].$$

Then problem (1) has at least one solution satisfying, for every $t \in [a, b]$, that

$$\alpha(t) \leq u(t) \leq \beta(t),$$

$$\min\{R_1, R_3\} \leq u'(t) \leq \max\{R_2, R_4\}.$$

Proof. Suppose that $R_3 < R_1$ and $R_4 > R_2$. Following the proof of Theorem 1 in [5], we can choose $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n \geq n_0$, we have that

$$R_2 + \frac{2}{n} < R_4, \ R_1 - \frac{2}{n} > R_3.$$

For $n \geq n_0$, let

$$h_n(t, x, y) = \begin{cases} f(t, x, R_4), & R_4 < y, \\[2mm] f(t, x, y), & R_2 + \frac{2}{n} \leq y \leq R_4, \\[2mm] f(t, x, R_2 + \frac{2}{n}) + w_2, & R_2 + \frac{1}{n} < y < R_2 + \frac{2}{n}, \\[2mm] f(t, x, R_2), & R_2 < y \leq R_2 + \frac{1}{n}, \\[2mm] f(t, x, y), & R_1 \leq y \leq R_2, \\[2mm] f(t, x, R_1), & R_1 - \frac{1}{n} \leq y < R_1, \\[2mm] f(t, x, R_1 - \frac{2}{n}) - w_1, & R_1 - \frac{2}{n} < y < R_1 - \frac{1}{n}, \\[2mm] f(t, x, y), & R_3 \leq y \leq R_1 - \frac{2}{n}, \\[2mm] f(t, x, R_3), & y < R_3, \end{cases}$$

where

$$w_2 = \left[f\left(t, x, R_2 + \frac{2}{n}\right) - f(t, x, R_2) \right] n \left(y - R_2 - \frac{2}{n}\right),$$

$$w_1 = \left[f\left(t, x, R_1 - \frac{2}{n}\right) - f(t, x, R_1) \right] n \left(y - R_1 + \frac{2}{n}\right).$$

We check that problem

$$(5) \qquad \begin{cases} x''(t) = h_n(t, x(t), x'(t)), & t \in I = [a, b], \\ x(a) = x(c), \ x(b) = x(d) \end{cases}$$

is under the hypotheses of Theorem 2.1. Indeed,

$$\int_a^b |h_n(t, x, y)| \, dt \leq K, \quad \text{for } x \in [\alpha(t), \beta(t)], \ y \in \mathbb{R},$$

where

$$K = \int_a^b \sup\{|h_n(t, x, y)| \ : \ x \in [\alpha(t), \beta(t)], \ y \in \mathbb{R}\} \, dt.$$

Moreover, α, β are, respectively, lower and upper solutions to problem (5), since

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) = h_n(t, \alpha(t), \alpha'(t)), \ t \in I,$$

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)) = h_n(t, \beta(t), \beta'(t)), \ t \in I,$$

where we use that $R_1 \leq \alpha'(t), \beta'(t) \leq R_2$. Using Theorem 2.1, we get that problem (5) has a solution u_n satisfying

$$\alpha(t) \leq u_n(t) \leq \beta(t), t \in I.$$

Now, we give estimates for u_n'. Since $u_n(a) = u_n(c), u_n(d) = u_n(b)$, then there exist $a_0^n \in (a, c)$ and $b_0^n \in (d, b)$ with $u_n'(a_0^n) = 0$, $u_n'(b_0^n) = 0$.

Suppose that $\max\{u_n'(t) \ : \ t \in [a, b_0^n]\} = u_n'(z_0^n) > R_2 + \frac{1}{n}$. Then $z_0^n \neq b_0^n$ and there exists $(t_1^n, t_2^n) \subset (a, b_0^n)$, with $u_n'(t_2^n) = R_2$, $u_n'(t_1^n) = R_2 + \frac{1}{n}$ and $R_2 \leq u_n'(t) \leq R_2 + \frac{1}{n}$, for all $t \in (t_1^n, t_2^n)$. Integrating u_n'' in the interval (t_1^n, t_2^n) and using the hypotheses on f, we obtain

$$0 > -\frac{1}{n} = R_2 - \left(R_2 + \frac{1}{n}\right) = u_n'(t_2^n) - u_n'(t_1^n) = \int_{t_1^n}^{t_2^n} u_n''(s) \, ds$$

$$= \int_{t_1^n}^{t_2^n} h_n(s, u_n(s), u_n'(s)) \, ds = \int_{t_1^n}^{t_2^n} f(s, u_n(s), R_2) \, ds \geq 0,$$

which is a contradiction. If $\min\{u_n'(t) \ : \ t \in [a, b_0^n]\} = u_n'(z_1^n) < R_1 - \frac{1}{n}$, then $z_1^n \neq b_0^n$ and there exists $(t_1^n, t_2^n) \subset (a, b_0^n)$, with $u_n'(t_2^n) = R_1$,

$u'_n(t_1^n) = R_1 - \frac{1}{n}$ and $R_1 - \frac{1}{n} \leq u'_n(t) \leq R_1$, for all $t \in (t_1^n, t_2^n)$. Integrating again, we get

$$0 < \frac{1}{n} = R_1 - \left(R_1 - \frac{1}{n} \right) = u'_n(t_2^n) - u'_n(t_1^n) = \int_{t_1^n}^{t_2^n} u''_n(s)\, ds$$

$$= \int_{t_1^n}^{t_2^n} h_n(s, u_n(s), u'_n(s))\, ds = \int_{t_1^n}^{t_2^n} f(s, u_n(s), R_1)\, ds \leq 0,$$

another contradiction. We have proved that

$$R_1 - \frac{1}{n} \leq u'_n(t) \leq R_2 + \frac{1}{n}, \quad \text{for all } t \in [a, b_0^n].$$

Now, suppose that $\max\{u'_n(t) : t \in [b_0^n, b]\} = u'_n(z_0^n) > R_4 + \frac{1}{n}$. Then $z_0^n \in (b_0^n, b]$ and there exists $(t_1^n, t_2^n) \subset (b_0^n, b)$, with $u'_n(t_1^n) = R_4$, $u'_n(t_2^n) = R_4 + \frac{1}{n}$, and $R_4 \leq u'_n(t) \leq R_4 + \frac{1}{n}$, for all $t \in (t_1^n, t_2^n)$, so that

$$0 < \frac{1}{n} = R_4 + \frac{1}{n} - R_4 = u'_n(t_2^n) - u'_n(t_1^n) = \int_{t_1^n}^{t_2^n} u''_n(s)\, ds$$

$$= \int_{t_1^n}^{t_2^n} h_n(s, u_n(s), u'_n(s))\, ds = \int_{t_1^n}^{t_2^n} f(s, u_n(s), R_4)\, ds \leq 0,$$

which is absurd and, similarly, if $\min\{u'_n(t) : t \in [b_0^n, b]\} = u'_n(z_1^n) < R_3 - \frac{1}{n}$, then $z_1^n \in (b_0^n, b]$ and there exists $(t_1^n, t_2^n) \subset (b_0^n, b)$, with $u'_n(t_2^n) = R_3 - \frac{1}{n}$, $u'_n(t_1^n) = R_3$ and $R_3 - \frac{1}{n} \leq u'_n(t) \leq R_3$, for all $t \in (t_1^n, t_2^n)$. Thus

$$0 > -\frac{1}{n} = R_3 - \frac{1}{n} - R_3 = u'_n(t_2^n) - u'_n(t_1^n) = \int_{t_1^n}^{t_2^n} u''_n(s)\, ds$$

$$= \int_{t_1^n}^{t_2^n} h_n(s, u_n(s), u'_n(s))\, ds = \int_{t_1^n}^{t_2^n} f(s, u_n(s), R_3)\, ds \geq 0,$$

which is again a contradiction. Therefore,

$$R_3 - \frac{1}{n} \leq u'_n(t) \leq R_4 + \frac{1}{n}, \; t \in [b_0^n, b],$$

$$R_3 - \frac{1}{n} < R_1 - \frac{1}{n} \leq u'_n(t) \leq R_2 + \frac{1}{n} < R_4 + \frac{1}{n}, \; t \in [a, b_0^n],$$

that is,

$$\min\{R_1, R_3\} - \frac{1}{n} = R_3 - \frac{1}{n} \leq u'_n(t) \leq R_4 + \frac{1}{n} = \max\{R_2, R_4\} + \frac{1}{n}, t \in I.$$

This proves that the sequence $\{u_n\}_{n_0}^{+\infty}$ is bounded and equicontinuous in $C^1(I)$, so that there exists a subsequence convergent in $C^1(I)$ to a

function u satisfying

$$\alpha(t) \leq u(t) \leq \beta(t),$$

$$R_3 \leq u'(t) \leq R_4.$$

In consequence, u is a solution to equation

$$x''(t) = h(t, x(t), x'(t)), \ t \in I,$$

$$x(a) = x(c), \ x(b) = x(d),$$

where

$$h(t, x, y) = \begin{cases} f(t, x, R_4), & y > R_4, \\ f(t, x, y), & R_3 \leq y \leq R_4, \\ f(t, x, R_3), & y < R_3, \end{cases}$$

but $R_3 \leq u'(t) \leq R_4$, $t \in I$, then u is a solution to (1) with

$$\alpha(t) \leq u(t) \leq \beta(t),$$

$$\min\{R_1, R_3\} = R_3 \leq u'(t) \leq R_4 = \max\{R_2, R_4\}.$$

If $R_3 < R_1$, $R_2 < R_4$ is not true, then the proof can be completed following a similar argument. $\qquad\square$

The following result studies the case where $R_1 = R_3$, $R_2 = R_4$.

THEOREM 2.3. *Suppose that α, β are, respectively, lower and upper solutions of problem (1) such that $\alpha \leq \beta$ on I and that there exist real numbers R_1, R_2, such that $R_1 \leq -\max\{\|\alpha'\|, \|\beta'\|\}$, $R_2 \geq \max\{\|\alpha'\|, \|\beta'\|\}$ and*

$$f(t, x, R_2) \geq 0, \ f(t, x, R_1) \leq 0, \ \forall t \in [a, b], \ \forall x \in [\alpha(t), \beta(t)],$$

$$f(t, x, R_1) \geq 0, \ f(t, x, R_2) \leq 0, \ for \ t \in [d, b], \ x \in [\alpha(t), \beta(t)].$$

Then problem (1) has at least one solution satisfying

$$\alpha(t) \leq u(t) \leq \beta(t), \ R_1 \leq u'(t) \leq R_2, \ t \in I.$$

Proof. Let

$$h(t, x, y) = \left.\begin{cases} f(t, x, R_2), & R_2 < y, \\ f(t, x, y), & R_1 \leq y \leq R_2, \\ f(t, x, R_1), & y < R_1, \end{cases}\right\} = f(t, x, q(y)),$$

where

$$q(y) = \max\{R_1, \min\{y, R_2\}\}.$$

In this case, for problem

(6) $\quad \begin{cases} x''(t) = h(t, x(t), x'(t)) = f(t, x(t), q(x'(t))), & t \in I = [a, b], \\ x(a) = x(c), \ x(b) = x(d), \end{cases}$

the hypotheses of Theorem 2.1 hold. Indeed,

$$\int_a^b |h(t,x,y)|\,dt \leq K, \quad \text{for } x \in [\alpha(t),\beta(t)],\ y \in \mathbb{R},$$

where

$$K = \int_a^b \sup\{|f(t,x,y)| \ : \ x \in [\alpha(t),\beta(t)],\ y \in [R_1,R_2]\}\,dt.$$

The conditions on R_1 and R_2 allow to prove that

$$\alpha''(t) \geq f(t,\alpha(t),\alpha'(t)) = h(t,\alpha(t),\alpha'(t)),\ t \in I,$$

$$\beta''(t) \leq f(t,\beta(t),\beta'(t)) = h(t,\beta(t),\beta'(t)),\ t \in I,$$

so that functions α, β are, respectively, lower and upper solutions to problem (6).

Applying Theorem 2.1, we get that problem (6) has a solution u satisfying

$$\alpha(t) \leq u(t) \leq \beta(t), t \in I.$$

We prove that $R_1 \leq u'(t) \leq R_2,\ t \in I$. The boundary conditions imply that $u(a) = u(c)$, $u(d) = u(b)$, then there exist $a_0 \in (a,c)$ and $b_0 \in (d,b)$ with $u'(a_0) = 0$, $u'(b_0) = 0$.

Suppose that $\max\{u'(t) \ : \ t \in [a,b_0]\} = u'(z_0) > R_2$. Then $z_0 \neq b_0$ and there exists $(t_1,t_2) \subset (a,b_0)$, with $u'(t_2) = R_2$, $u'(t_1) > R_2$ and $R_2 \leq u'(t) \leq u'(t_1)$, for all $t \in (t_1,t_2)$. Integrating u'' in the interval (t_1,t_2) and using the hypotheses, we get

$$0 > R_2 - u'(t_1) = u'(t_2) - u'(t_1) = \int_{t_1}^{t_2} u''(s)\,ds$$

$$= \int_{t_1}^{t_2} h(s,u(s),u'(s))\,ds = \int_{t_1}^{t_2} f(s,u(s),R_2)\,ds \geq 0,$$

getting a contradiction. If $\min\{u'(t) \ : \ t \in [a,b_0]\} = u'(z_1) < R_1$, then $z_1 \neq b_0$ and there exists $(t_1,t_2) \subset (a,b_0)$, with $u'(t_2) = R_1$, $u'(t_1) < R_1$ and $u'(t_1) \leq u'(t) \leq R_1$, for all $t \in (t_1,t_2)$. Integrating again, we get

$$0 < R_1 - u'(t_1) = u'(t_2) - u'(t_1) = \int_{t_1}^{t_2} u''(s)\,ds$$

$$= \int_{t_1}^{t_2} h(s,u(s),u'(s))\,ds = \int_{t_1}^{t_2} f(s,u(s),R_1)\,ds \leq 0,$$

again a contradiction. This provides that

$$R_1 \leq u'(t) \leq R_2, \quad \text{for all } t \in [a,b_0].$$

Now, suppose that $\max\{u'(t) : t \in [b_0, b]\} = u'(z_0) > R_2$. Then $z_0 \in (b_0, b]$ and there exists $(t_1, t_2) \subset (b_0, b)$, with $u'(t_1) = R_2$, $u'(t_2) > R_2$, and $R_2 \leq u'(t) \leq u'(t_2)$, for all $t \in (t_1, t_2)$, so that

$$0 < u'(t_2) - R_2 = u'(t_2) - u'(t_1) = \int_{t_1}^{t_2} u''(s)\,ds$$

$$= \int_{t_1}^{t_2} h(s, u(s), u'(s))\,ds = \int_{t_1}^{t_2} f(s, u(s), R_2)\,ds \leq 0,$$

which is absurd and, similarly, if $\min\{u'(t) : t \in [b_0, b]\} = u'(z_1) < R_1$, then $z_1 \in (b_0, b]$ and there exists $(t_1, t_2) \subset (b_0, b)$, with $u'(t_2) < R_1$, $u'(t_1) = R_1$, and $u'(t_2) \leq u'(t) \leq R_1$, for all $t \in (t_1, t_2)$, thus

$$0 > u'(t_2) - R_1 = u'(t_2) - u'(t_1) = \int_{t_1}^{t_2} u''(s)\,ds$$

$$= \int_{t_1}^{t_2} h(s, u(s), u'(s))\,ds = \int_{t_1}^{t_2} f(s, u(s), R_1)\,ds \geq 0,$$

which is a contradiction.

In consequence,

$$R_1 \leq u'(t) \leq R_2, \ t \in I,$$

and $q(u'(t)) = u'(t)$, $t \in I$, hence u is a solution to (1) with the properties

$$\alpha(t) \leq u(t) \leq \beta(t), \ t \in I,$$

$$R_1 \leq u'(t) \leq R_2, \ t \in I.$$

The proof is complete. $\square$

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