J. Korean Math. Soc. **43** (2006), No. 6, pp. 1231–1252

# EMBEDDING OF THE TEICHMÜLLER SPACE INTO THE GOLDMAN SPACE

HONG CHAN KIM

ABSTRACT. In this paper we shall explicitly calculate the formula of the algebraic presentation of an embedding of the Teichmüller space $\mathfrak{T}(M)$ into the Goldman space $\mathcal{G}(M)$. From this algebraic presentation, we shall show that the Goldman's length parameter on $\mathcal{G}(M)$ is an isometric extension of the Fenchel-Nielsen's length parameter on $\mathfrak{T}(M)$.

## 1. Introduction

A *convex* real projective structure on a smooth surface $M$ is a representation of $M$ as a quotient space $\Omega/\Gamma$ of a convex domain $\Omega \subset \mathbb{RP}^2$ by a discrete group $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ acting properly and freely. If $\chi(M) < 0$, then the equivalence classes of convex real projective structures on $M$ form a deformation space $\mathcal{G}(M)$ called the *Goldman space.*

The study of $\mathbb{RP}^2$ structures has been quite active. Ehresmann, Kuiper, Benzécri, Kobayashi, and Thurston have done important work. Recently Goldman and S. Choi lead this field.

The deformation space of *hyperbolic* structures on $M$ is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$. Choi and Goldman [2] proved $\mathcal{G}(M)$ is a component of the deformation space $\mathbb{RP}^2(M)$ of real projective structure on $M$ and $\mathcal{G}(M)$ contains the Teichmüller space $\mathfrak{T}(M)$.

Goldman [5] defined the length parameters $\ell, m$ on $\mathcal{G}(M)$. They seems to be an extension of Fenchel-Nielsen's length parameter $\ell$ on $\mathfrak{T}(M)$. But during the calculation of the translation length of Goldman and Fenchel-Nielsen's parameters, author realized that they do not fit. Thus we define the *modified* Goldman's length parameters on $\mathcal{G}(M)$.

Received June 21, 2005. Revised November 10, 2005.

2000 Mathematics Subject Classification: 58D27, 53D30.

Key words and phrases: convex real projective structure, Hilbert metric, Teichmüller space, Goldman space, length parameter.

The author gratefully acknowledges the support from a Korea University Grant.

The purpose of this paper is to formulate the explicit algebraic presentation of an embedding of $\mathfrak{T}(M)$ into $\mathcal{G}(M)$ which isometrically extends Fenchel-Nielsen's length parameter on $\mathfrak{T}(M)$ to the modified Goldman's length parameters on $\mathcal{G}(M)$.

In Section 2, we recall some preliminary definitions about $(G, X)$-structures on a smooth manifold $M$. In Section 3, we describe the relation between the deformation space $\mathfrak{D}(M)$ of $(G, X)$-structures on $M$ and the orbit space $\mathrm{Hom}(\pi, G)^{--}/G$. In Section 4, we recall the positive hyperbolic elements of $\mathbf{SL}(3, \mathbb{R})$. In Section 5, we give some knowledge about the Hilbert metric. In Section 6, we compare the relations between the Poincaré metric and the Hilbert metric on the unit disc. In Section 7, we define an embedding formula of hyperbolic structures into convex real projective structures. To realize a hyperbolic structure on $M$ as a convex real projective structure, we define an isometry $\mathbb{H}^2 \to \Omega$ and an embedding $\mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$, where $\Omega$ is a strictly convex subset of $\mathbb{RP}^2$ with the conic boundary. In Section 8, we show the modified Goldman's length parameters $\ell, m$ on $\mathcal{G}(M)$ isometrically extend Fenchel-Nielsen's length parameter $\ell$.

## 2. Preliminaries

The followings are from Kim's paper [7]. For more detail see Kim [7].

Let $X$ be a smooth manifold and $G$ a connected algebraic Lie group. We assume that $G$ acts on $X$ *strongly effectively*; that is, if $g_1, g_2 \in G$ agree on a nonempty open set of $X$, then $g_1 = g_2$. We start this section with examples of strongly effective action.

EXAMPLE 2.1. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane. Then $\mathbf{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}. \tag{2.1}$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$.

EXAMPLE 2.2. Let $\mathbb{RP}^2$ be the space of all lines through the origin in $\mathbb{R}^3$. For a nonzero vector $v$ in $\mathbb{R}^3$, the corresponding point in $\mathbb{RP}^2$ will be denoted by $[v]$. Then $\mathbf{GL}(3, \mathbb{R})$ acts on $\mathbb{RP}^2$ by

$$B \cdot [v] = [Bv]. \tag{2.2}$$

Since the scalar matrices $\mathbb{R}^* \subset \mathbf{GL}(3, \mathbb{R})$ acts trivially on $\mathbb{RP}^2$, the Lie group $\mathbf{PGL}(3, \mathbb{R}) = \mathbf{GL}(3, \mathbb{R})/\mathbb{R}^*$ acts strongly effectively on $\mathbb{RP}^2$.

Let $\Omega$ be an open subset of $X$. A map $\phi : \Omega \to X$ is called *locally-*$(G, X)$ if for each component $W \subset \Omega$, there exists $g \in G$ such that $\phi|_W = g|_W$. Since $G$ acts strongly effectively on $X$, above element $g$ is unique for each component. Clearly a locally-$(G, X)$ map is a local diffeomorphism. A $(G, X)$*-structure* on a connected smooth $n$-manifold $M$ is a maximal collection of coordinate charts $\{(U_\alpha, \psi_\alpha)\}$ such that

1. $G$ acts strongly effectively on $X$.
2. $\{U_\alpha\}$ is an open covering of $M$.
3. For each $\alpha$, $\psi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image.
4. If two coordinate charts $U_\alpha \cap U_\beta \neq \emptyset$, then the transition function $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is locally-$(G, X)$.

The $(\mathbf{PSL}(2, \mathbb{R}), \mathbb{H}^2)$-structures and $(\mathbf{PGL}(3, \mathbb{R}), \mathbb{RP}^2)$-structures on a smooth surface $M$ are called the *hyperbolic* structures and *real projective* structures on $M$ respectively.

A smooth manifold equipped with a $(G, X)$-structure is called a $(G, X)$-manifold. If $N$ is a $(G, X)$-manifold and $f : M \to N$ is a local diffeomorphism, then we can give the induced $(G, X)$-structure on $M$ via $f$. In particular every covering space of a $(G, X)$-manifold has the canonically induced $(G, X)$-structure.

A smooth map $f : M \to N$ of $(G, X)$-manifolds is called a $(G, X)$-*map* if for each coordinate chart $(U, \psi_U)$ on $M$ and $(V, \psi_V)$ on $N$, the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V) \cap U) \to \psi_V(f(U) \cap V)$ is locally-$(G, X)$. If $f : M \to N$ is a diffeomorphism such that $f$ and $f^{-1}$ are $(G, X)$-maps, then $f$ is called a $(G, X)$-*diffeomorphism.*

The following *Development Theorem* is the fundamental fact about $(G, X)$-structures. See Thurston's book [13] for details.

THEOREM 2.3. *Let $p : \tilde{M} \to M$ denote a universal covering map of a $(G, X)$-manifold $M$, and $\pi$ the corresponding group of covering transformations. Then*

1. *There exist a $(G, X)$-map* $\mathbf{dev} : \tilde{M} \to X$ (called the *developing map*) *and homomorphism $h : \pi \to G$* (called the *holonomy homomorphism*) *such that for each $\gamma \in \pi$ the following diagram commutes:*

$$\begin{tikzcd} \tilde{M} \arrow[r, "\mathbf{dev}"] \arrow[d, "\gamma"'] & X \arrow[d, "h(\gamma)"] \\ \tilde{M} \arrow[r, "\mathbf{dev}"'] & X \end{tikzcd}$$

2. *Suppose* $(\mathbf{dev}', h')$ *is another pair satisfying above conditions. Then there exists* $g \in G$ *such that* $\mathbf{dev}' = g \circ \mathbf{dev}$ *and* $h' = \iota_g \circ h$, *where* $\iota_g : G \to G$ *denotes the inner automorphism defined by* $g$.

By Theorem 2.3, the *developing pair* $(\mathbf{dev}, h)$ is unique up to the $G$-action by composition and conjugation respectively.

## 3. Deformation space of $(G, X)$-structures

For a smooth manifold $M$, consider a pair $(f, N)$ where $N$ is a $(G, X)$-manifold and $f : M \to N$ is a diffeomorphism. Then $M$ admits the induced $(G, X)$-structure via $f$. The set of all such pairs $(f, N)$ is denoted by $\mathcal{A}(M)$. Then $\mathcal{A}(M)$ is the space of all $(G, X)$-structures on $M$. We say two pairs $(f, N)$ and $(f', N')$ in $\mathcal{A}(M)$ are *equivalent* if there exists a $(G, X)$-diffeomorphism $g : N \to N'$ such that $g \circ f$ is isotopic to $f'$. The set of equivalence classes $\mathcal{A}(M)/\sim$ will be denoted by $\mathfrak{D}(M)$ and called the *deformation space* of $(G, X)$-structures on $M$.

The deformation space $\mathfrak{D}(M)$ has the natural topology. Let $\mathrm{Diff}(M)^0$ be the space of all diffeomorphisms of $M$ which are isotopic to the identity map $I_M$. Then we may think the deformation space $\mathfrak{D}(M)$ consists of diffeomorphisms $f : M \to N$ to a $(G, X)$-manifold $N$ modulo the action of $\mathrm{Diff}(M)^0$ given by $g : f \mapsto f \circ g$ where $g \in \mathrm{Diff}(M)^0$. Give $\mathfrak{D}(M)$ the quotient topology induced from the $C^\infty$-topology on the space of diffeomorphisms $f : M \to N$.

DEFINITION 3.1. Let $M$ be a connected smooth surface. The deformation space of hyperbolic structures on $M$ is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$. The deformation space of real projective structures on $M$ is denoted by $\mathbb{RP}^2(M)$.

The deformation space $\mathfrak{D}(M)$ is closely related to $\mathrm{Hom}(\pi, G)/G$ the orbit space of homomorphisms $\phi : \pi \to G$. The group $G$ acts on $\mathrm{Hom}(\pi, G)$ by conjugation; that is, for $g \in G$ and $\phi \in \mathrm{Hom}(\pi, G)$, the action $g \cdot \phi$ is defined by $(g \cdot \phi)(\gamma) = g \circ \phi(\gamma) \circ g^{-1}$ where $\gamma \in \pi$.

If $G$ is an algebraic Lie group, then $\mathrm{Hom}(\pi, G)$ is an *algebraic variety.* But generally $\mathrm{Hom}(\pi, G)$ is not smooth. Suppose $\phi \in \mathrm{Hom}(\pi, G)$ and $Z(\phi)$ is the centralizer of $\phi(\pi)$ in $G$. Goldman [3] showed $\phi$ is a nonsingular point of $\mathrm{Hom}(\pi, G)$ if and only if $\dim Z(\phi)/Z(G) = 0$, where $Z(G)$ denotes the center of $G$. Let $\mathrm{Hom}(\pi, G)^-$ be the set of nonsingular points of $\mathrm{Hom}(\pi, G)$. Then $G$ acts freely on the smooth Zariski open subset $\mathrm{Hom}(\pi, G)^-$. But unfortunately $\mathrm{Hom}(\pi, G)^-/G$ is generally not Hausdorff. Let $\mathrm{Hom}(\pi, G)^{--}$ be the subset of $\mathrm{Hom}(\pi, G)^-$ consisting

of homomorphisms whose image does not lie in a parabolic subgroup of $G$. Then $\mathrm{Hom}(\pi, G)^{--}$ is a Zariski open subset of $\mathrm{Hom}(\pi, G)^{-}$, and $\mathrm{Hom}(\pi, G)^{--}/G$ is a Hausdorff smooth manifold of dimension $-\dim G \cdot \chi(M)$. For more detail see Goldman's paper [3].

Suppose $M$ is a closed surface. Taking the holonomy homomorphism of a $(G, X)$-structure defines a map

$$\mathbf{hol} : \mathfrak{D}(M) \longrightarrow \mathrm{Hom}(\pi, G)^{--}/G$$

which is a local homeomorphism. See Goldman [4] and Johnson-Millson [6] for details. For a hyperbolic structures on $M$,

$$\mathbf{hol} : \mathfrak{T}(M) \longrightarrow \mathrm{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))^{--}/\mathbf{PSL}(2, \mathbb{R})$$

is an embedding onto a real analytic manifold of dimension $-3 \cdot \chi(M) = 6g - 6$. Thus the Teichmüller space $\mathfrak{T}(M)$ is diffeomorphic to $\mathbb{R}^{6g-6}$ and an element of $\mathfrak{T}(M)$ will be identified with a conjugacy class of $\mathrm{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))^{--}$. Furthermore the developing map $\mathbf{dev}$ is a diffeomorphism from $\tilde{M}$ onto a convex domain $\Omega = \mathbf{dev}(\tilde{M}) \subset \mathbb{H}^2$. In this case the holonomy homomorphism $h$ is an isomorphism from $\pi$ onto a discrete subgroup $\Gamma = h(\pi) \subset \mathbf{PSL}(2, \mathbb{R})$ which acts properly and freely on $\Omega$. Thus $M$ is diffeomorphic to the quotient space $\Omega/\Gamma$.

But for a real projective structures on a closed surface $M$,

$$\mathbf{hol} : \mathbb{RP}^2(M) \longrightarrow \mathrm{Hom}(\pi, \mathbf{PGL}(3, \mathbb{R}))^{--}/\mathbf{PGL}(3, \mathbb{R})$$

is just a local homeomorphism. And the developing map is just a local diffeomorphism and the developing image may be not convex. We can find such examples in Sullivan and Thurston's paper [12].

We consider the convex real projective structures on $M$. A domain $\Omega \subset \mathbb{RP}^2$ is called convex if there exist a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - l)$ and $\Omega$ is a convex subset of the affine plane $\mathbb{RP}^2 - l$. A real projective structure on $M$ is called convex if the developing map $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$. The following fundamental theorem is due to Goldman [5].

THEOREM 3.2. *Let $M$ be a closed real projective surface. Then the following statements are equivalent.*

1. *$M$ has a convex real projective structure.*
2. *$M$ is projectively diffeomorphic to a quotient space $\Omega/\Gamma$, where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group acting properly and freely on $\Omega$.*

DEFINITION 3.3. The *Goldman space* $\mathcal{G}(M)$ is the subset of $\mathbb{RP}^2(M)$ consisting of the equivalence classes of convex real projective structures.

Choi and Goldman ([2], [5]) proved that if $M$ is a closed real projective surface, then $\mathcal{G}(M)$ is a component of $\mathbb{RP}^2(M)$ and the restriction

$$\mathbf{hol} : \mathcal{G}(M) \to \mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))^{--}/\mathbf{PGL}(3,\mathbb{R})$$

is an embedding onto a real analytic manifold of dimension $-8 \cdot \chi(M) = 16g - 16$. Thus the Goldman space $\mathcal{G}(M)$ is diffeomorphic to $\mathbb{R}^{16g-16}$. The Goldman space $\mathcal{G}(M)$ is an analogue of the Teichmüller space $\mathfrak{T}(M)$. Classically known that $\mathfrak{T}(M)$ embeds in $\mathcal{G}(M)$. That means every hyperbolic structure on $M$ defines a convex real projective structure on $M$. We shall explicitly calculate the formula of the algebraic presentation of an embedding of $\mathfrak{T}(M)$ into $\mathcal{G}(M)$.

## 4. Positive hyperbolic elements

An element $A$ of $\mathbf{SL}(2,\mathbb{R})$ is said to be *hyperbolic* if $A$ has two distinct real eigenvalues. Since the characteristic polynomial of $A$ is $f(\lambda) = \lambda^2 - t\lambda + 1$, where $t = \mathrm{tr}(A)$, $A$ is hyperbolic if and only if $\mathrm{tr}(A)^2 > 4$.

Let $A$ be an element of $\mathbf{PSL}(2,\mathbb{R})$. Since the absolute value of trace is still defined, $A \in \mathbf{PSL}(2,\mathbb{R})$ is said to be *hyperbolic* if $|\mathrm{tr}(A)| > 2$. A hyperbolic element $A$ in $\mathbf{PSL}(2,\mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \overset{\text{let}}{=} \pm \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \tag{4.1}$$

via an $\mathbf{SL}(2,\mathbb{R})$-conjugation where $\alpha > 1$.

The homomorphism $\mathbf{GL}(3,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ defined by

$$B \mapsto (\det B)^{-1/3} B$$

induces an isomorphism $\mathbf{PGL}(3,\mathbb{R}) = \mathbf{GL}(3,\mathbb{R})/\mathbb{R}^* \to \mathbf{SL}(3,\mathbb{R})$. Thus from now on we shall identify the groups $\mathbf{PGL}(3,\mathbb{R})$ and $\mathbf{SL}(3,\mathbb{R})$.

An element $B \in \mathbf{SL}(3,\mathbb{R})$ is called *positive hyperbolic* if it has three distinct positive real eigenvalues. If $B$ is positive hyperbolic, then it can be represented by the diagonal matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \tag{4.2}$$

via an $\mathbf{SL}(3,\mathbb{R})$-conjugation where $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$.

The following theorem is one of the analogues between hyperbolic structures and convex real projective structures proved by Kuiper [9].

THEOREM 4.1. *Let $M$ be a closed oriented surface with $\chi(M) < 0$.*

1. *If $M$ is a hyperbolic surface, then every nontrivial element of holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ is hyperbolic.*
2. *If $M$ is a convex real projective surface, then every nontrivial element of holonomy group $\Gamma \subset \mathbf{SL}(3,\mathbb{R})$ is positive hyperbolic.*
3. *Either the boundary of the developing image is a conic in $\mathbb{RP}^2$ or is not $C^{1+\varepsilon}$ for any $\varepsilon > 0$.*

It is known that the boundary $\partial\Omega$ is a conic if and only if the convex real projective structure on $M$ arises from a hyperbolic structure on $M$. Let $\Omega$ be the domain in $\mathbb{RP}^2$ defined by

$$\Omega = \{[x_1, x_2, x_3] \in \mathbb{RP}^2 \mid x_1^2 + x_2^2 - x_3^2 < 0\}. \tag{4.3}$$

Then $\Omega$ has a conic boundary $\partial\Omega$. Let $M$ be a surface with a hyperbolic structure. Composing the developing map $\tilde{M} \to \mathbb{H}^2$ with an isometry $\mathbb{H}^2 \to \Omega \subset \mathbb{RP}^2$ and the holonomy homomorphism $\pi \to \mathbf{PSL}(2,\mathbb{R})$ with an embedding $\mathbf{PSL}(2,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ realizes $M$ as a convex real projective surface. Thus we shall define an isometry $\mathbb{H}^2 \to \Omega$ and an embedding $\mathbf{PSL}(2,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$.

## 5. The Hilbert metric

To define an isometry $\mathbb{H}^2 \to \Omega$, we need a knowledge about the Hilbert metric. Hilbert discovered a metric on a bounded convex domain $\Omega$ in $\mathbb{R}^2$ (or equivalently in $\mathbb{C}$). This metric is related to the hyperbolic metric but geodesics are still Euclidean segments. We recall some basic definitions and properties. For more detail, see Kobayashi's paper [8].

DEFINITION 5.1. Let $z_1, z_2, z_3, z_4$ be four points in the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that at least three points are distinct. The *cross-ratio* of $z_1, z_2, z_3, z_4$ is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}. \tag{5.1}$$

There are six different methods to define the cross-ratio on $\hat{\mathbb{C}}$. We adopt this presentation since it is an easy way to understand the Hilbert metric. The properties of this presentation are:

1. $[z_1, z_2, z_3, z_4] = [z_4, z_3, z_2, z_1]$.
2. $[z_1, z_2, z_3, z_4] > 1$ if distinct four points lie in a line segment such that $z_2$ is between $z_1$ and $z_3$ and $z_3$ is between $z_2$ and $z_4$.

DEFINITION 5.2. Let $\Omega$ be a bounded convex domain in $\mathbb{C}$. For a distinct points $z, w$ in $\Omega$, the *Hilbert distance* between $z$ and $w$ is defined by

$$d_H(z,w) = \log\,[z^*, z, w, w^*], \tag{5.2}$$

where $z^*, w^*$ are the boundary points in $\partial\Omega$ which lie on the straight line joining $z$ and $w$ such that $z$ lies between $z^*$ and $w$.

If we add $d_H(z,z) = 0$, then the Hilbert distance $d_H$ defines a complete metric on $\Omega$. A *strictly* convex domain is a convex domain whose boundary $\partial\Omega$ does not contain any line segment. Since a strictly convex domain $\Omega$ does not contain any full straight line, the Hilbert distance $d_H$ still defines a complete metric on $\Omega$ and has the following properties:

1. There is a unique geodesic between two points in $\Omega$.
2. The geodesics are straight lines in Euclidean sense.

Kuiper [9] showed that the developing images $\Omega \subset \mathbb{RP}^2$ of convex real projective structures are strictly convex domains. This yields that every developing image of a convex real projective structure of a closed surface has the complete Hilbert metric.

Let $a_1, a_2, a_3, a_4$ be *collinear* distinct four points in $\mathbb{RP}^2$. Then there exist corresponding four nonzero vectors $v_1, v_2, v_3, v_4$ of $\mathbb{R}^3$ which are contained in a plane $\mathcal{P} \subset \mathbb{R}^3$; that is, $a_k = [v_k] = [x_k, y_k, s_k]$ for each $k$.

REMARK 5.3. Since we use the $z$ for the complex variable $z = x + iy$, the $s$ will be used for the third Euclidean coordinate in $\mathbb{R}^3$.

If $\mathcal{P}$ is the $xy$-plane, then the cross-ratio is defined by

$$[a_1, a_2, a_3, a_4] := \left[\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}\right]. \tag{5.3}$$

If $\mathcal{P}$ is not the $xy$-plane, then the cross-ratio is defined by

$$[a_1, a_2, a_3, a_4] := \left[\frac{x_1 + i\,y_1}{s_1}, \frac{x_2 + i\,y_2}{s_2}, \frac{x_3 + i\,y_3}{s_3}, \frac{x_4 + i\,y_4}{s_4}\right]. \tag{5.4}$$

PROPOSITION 5.4. *Suppose $z_1, z_2, z_3, z_4$ are distinct four points which are contained in a straight line in $\mathbb{C}$. Let $z_k = x_k + i\,y_k$ for each $k$. Then*

$$[z_1, z_2, z_3, z_4] = \begin{cases} [x_1, x_2, x_3, x_4] & \text{if } x_1, x_2, x_3, x_4 \text{ are distinct} \\ [y_1, y_2, y_3, y_4] & \text{if } y_1, y_2, y_3, y_4 \text{ are distinct.} \end{cases} \tag{5.5}$$

By virtue of Proposition 5.4, the cross ratio of collinear distinct four points $a_1, a_2, a_3, a_4$ of $\mathbb{RP}^2$ should be one of the followings:

$$[a_1, a_2, a_3, a_4] = \left[\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \frac{x_4}{y_4}\right] \text{ or } \left[\frac{x_1}{s_1}, \frac{x_2}{s_2}, \frac{x_3}{s_3}, \frac{x_4}{s_4}\right] \text{ or } \left[\frac{y_1}{s_1}, \frac{y_2}{s_2}, \frac{y_3}{s_3}, \frac{y_4}{s_4}\right]$$

when the cross ratios of the right hand side are well-defined.

## 6. Relation between the Poincaré and the Hilbert metric

Let $\mathcal{H}_P = (\mathbb{H}^2, d_P)$ be the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the Poincaré metric $d_P$; i.e., the lines in $\mathcal{H}_P$ are the semi-circles centered at the $x$-axis and the rays orthogonal to the $x$-axis. The Poincaré metric on $\mathcal{H}_P$ is defined by

$$d_P(z, w) = \log [z', z, w, w'], \tag{6.1}$$

where $z', w'$ are the boundary points in the extended $x$-axis $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which lie on the line joining $z$ and $w$ such that $z$ is between $z'$ and $w$.

The elements of $\mathbf{PSL}(2, \mathbb{R})$ act on $\mathcal{H}_P$ as the linear fractional transformations in (2.1). Since the linear fractional transformations on $\hat{\mathbb{C}}$ preserve the cross-ratio, we have the following theorem.

THEOREM 6.1. *The group $\mathcal{I}_1$ of orientation preserving isometries of the upper half plane $\mathcal{H}_P$ is*

$$\mathcal{I}_1 = \mathbf{PSL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{PGL}(2, \mathbb{R}) \,\middle|\, ad - bc = 1 \right\}. \tag{6.2}$$

Let $\mathcal{D}_P = (\mathbb{D}^2, d_P)$ be the unit disc $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ with the Poincaré metric $d_P$; i.e., the lines in $\mathcal{D}_P$ are the arcs of circles which are orthogonal the boundary of $\mathbb{D}^2$ and the segments through the origin. The Poincaré metric on $\mathcal{D}_P$ is defined similarly as in (6.1).

The linear fractional transformation $G_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$w = G_1(z) = \frac{z - i}{-iz + 1} \tag{6.3}$$

maps $\{-1, 0, 1, \infty, i\}$ to $\{-1, -i, 1, i, 0\}$ respectively. Thus the restriction of $G_1$ to $\mathcal{H}_P$ is an orientation preserving isometry onto $\mathcal{D}_P$ with the inverse $G_1^{-1} \overset{\text{let}}{=} F_1$ such that

$$z = F_1(w) = \frac{w + i}{iw + 1}. \tag{6.4}$$

The following Theorem 6.2 is well-known fact. We can find a similar result in Matsuzaki and Taniguchi's book [10] or Ratcliffe's book [11].

THEOREM 6.2. *The group $\mathcal{I}_2$ of orientation preserving isometries of the Poincaré disk $\mathcal{D}_P$ is*

$$\mathcal{I}_2 = \left\{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in \mathbf{PGL}(2, \mathbb{C}) \,\middle|\, |\alpha|^2 - |\beta|^2 = 1 \right\}. \tag{6.5}$$

*Proof.* The linear fractional transformations $G_1$ and $F_1$ correspond to the matrices $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ in $\mathbf{SL}(2,\mathbb{C})$ respectively. Thus the isometry of $\mathcal{D}_P$ is the compositions of $G_1 \circ f_1 \circ F_1$, where $f_1$ is an isometry of $\mathcal{H}_P$ in (6.2). In the matrix representation,

$$\begin{aligned} G_1 \circ f_1 \circ F_1 &= \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} (\frac{a+d}{2}) + i(\frac{b-c}{2}) & (\frac{b+c}{2}) + i(\frac{a-d}{2}) \\ (\frac{b+c}{2}) - i(\frac{a-d}{2}) & (\frac{a+d}{2}) - i(\frac{b-c}{2}) \end{bmatrix} \overset{\text{let}}{=} \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \end{aligned}$$

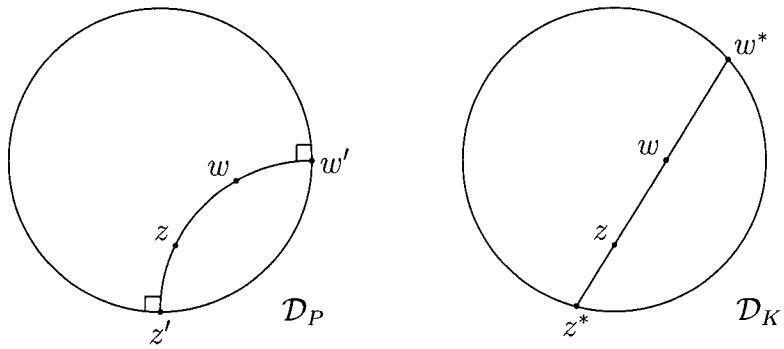
and $|\alpha|^2 - |\beta|^2 = ad - bc = 1$.

Conversely, if $\alpha = \alpha_1 + i\,\alpha_2$, $\beta = \beta_1 + i\,\beta_2$ (for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$) are the complex numbers in the group $\mathcal{I}_2$, then the corresponding element in $\mathcal{I}_1$ is

$$\begin{bmatrix} \alpha_1 + \beta_2 & \beta_1 + \alpha_2 \\ \beta_1 - \alpha_2 & \alpha_1 - \beta_2 \end{bmatrix} \tag{6.6}$$

and $(\alpha_1 + \beta_2)(\alpha_1 - \beta_2) - (\beta_1 + \alpha_2)(\beta_1 - \alpha_2) = |\alpha|^2 - |\beta|^2 = 1$. This completes the proof. □

Let $\mathcal{D}_H = (\mathbb{D}^2, d_H)$ be the unit disc $\mathbb{D}^2$ with the Hilbert metric $d_H$; i.e., the lines in $\mathcal{D}_H$ are the Euclidean line segments.




$d_P(z,w) = \log[z', z, w, w']$ $\qquad$ $d_H(z,w) = \log[z^*, z, w, w^*]$

FIGURE 1. The Poincaré and the Hilbert metric on $\mathbb{D}^2$

Let $\Sigma = \{(x,y,s) \in \mathbb{R}^3 \mid x^2 + y^2 + s^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ with the north pole $n = (0,0,1)$. Consider the stereographic projection

$P : \Sigma - \{n\} \to \mathbb{R}^2$ defined by

$$P(x, y, s) = \left( \frac{x}{1-s}, \frac{y}{1-s} \right).$$

Then $P$ is a conformal diffeomorphism with the inverse $P^{-1}$ such that

$$P^{-1}(u, v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

Let $\Sigma_- = \{(x, y, s) \in \Sigma \mid s < 0\}$ be the lower hemisphere of $\Sigma$. Then the restriction of $P^{-1} : \mathbb{R}^2 \to \Sigma - \{n\}$ to the unit disk $\mathbb{D}^2$ is diffeomorphic to the lower hemisphere $\Sigma_-$. See Figure 2.

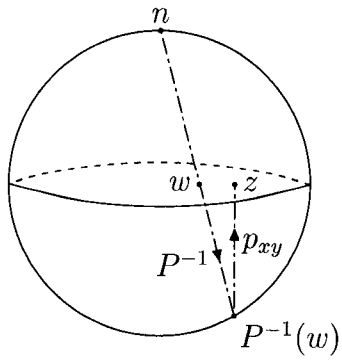



FIGURE 2. Inverse of the stereographic projection $P^{-1}$

We define a mapping $G_2 : \mathcal{D}_P \to \mathcal{D}_H$ by $G_2 = p_{xy} \circ P^{-1}$, where $p_{xy} : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection to the $xy$-plain; i.e.,

$$G_2(u, v) = \left( \frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2} \right). \tag{6.7}$$

Then $G_2$ is a diffeomorphism with the inverse $G_2^{-1} \overset{\text{let}}{=} F_2 : \mathcal{D}_H \to \mathcal{D}_P$

$$F_2(x, y) = \left( \frac{x}{1+\sqrt{1-x^2-y^2}}, \frac{y}{1+\sqrt{1-x^2-y^2}} \right). \tag{6.8}$$

In the complex variables $w \in \mathcal{D}_P$, $z \in \mathcal{D}_H$, the mappings $G_2, F_2$ are represented by

$$z = G_2(w) = \frac{2w}{1+|w|^2}, \qquad w = F_2(z) = \frac{z}{1+\sqrt{1-|z|^2}}. \tag{6.9}$$

The result in Proposition 6.3 is also found in Thurston's book [13].

PROPOSITION 6.3. *The mappings $G_2 : \mathcal{D}_P \to \mathcal{D}_H$ and $F_2 : \mathcal{D}_H \to \mathcal{D}_P$ preserve the lines in $\mathcal{D}_P$ and $\mathcal{D}_H$.*

*Proof.* First we will show that $F_2$ carries the chords in $\mathcal{D}_H$ to the arcs in $\mathcal{D}_P$ which are orthogonal to the boundary $\partial\mathcal{D}_P$. Using the rotations on $\mathcal{D}_H$, we may assume the cord is $\{x = a\}$ for $0 \le a < 1$. If $a = 0$, then the image of $\{x = 0\}$ is itself $\{u = 0\}$. Suppose $0 < a < 1$ and $(u, v) = F_2(a, y) \in \mathcal{D}_P$. Then

$$u = \frac{a}{1 + \sqrt{1 - a^2 - y^2}}, \quad v = \frac{y}{1 + \sqrt{1 - a^2 - y^2}}. \tag{6.10}$$

Since $a \neq 0$, $u$ is also non-vanishing. From (6.10), we get the relation $y = v\frac{a}{u}$. Plug in $y = v\frac{a}{u}$ to the left equation of (6.10), we obtain the following equation:

$$(u - \frac{1}{a})^2 + v^2 = \frac{1 - a^2}{a^2}. \tag{6.11}$$

Therefore the image of the cord $\{x = a\}$ is the part of the circle centered at $C = (\frac{1}{a}, 0)$ with the radius $\frac{\sqrt{1-a^2}}{a}$.

Let $O$ be the origin of $\mathbb{R}^2$ and $A, B$ the points in $\partial\mathcal{D}_P$ which intersect with the circle (6.11). Then $A = (a, \sqrt{1 - a^2})$ and $B = (a, -\sqrt{1 - a^2})$. We can easily compute

$$(\overline{OA})^2 + (\overline{AC})^2 = \left(a^2 + (1 - a^2)\right) + \left(\frac{1 - a^2}{a^2}\right) = \frac{1}{a^2} = (\overline{OC})^2.$$

By the Pythagorean theorem, $\angle OAC = \pi/2$. Similarly we can show $\angle OBC = \pi/2$. Therefore the image of the cords in $\mathcal{D}_H$ are the arcs in $\mathcal{D}_P$ which are orthogonal to the boundary $\partial\mathcal{D}_P$.

Conversely, let $\ell_P$ be an arc in $\mathcal{D}_P$ which is orthogonal to the boundary $\partial\mathcal{D}_P$. Then $\ell_P$ is the $F_2$-image of the chord $\ell_H$ joining the boundary points of the arc $\ell_P$. Since $G_2 : \mathcal{D}_P \to \mathcal{D}_H$ is the inverse mapping of $F_2$, $G_2(\ell_P) = G_2(F_2(\ell_H)) = \ell_H$. Thus $G_2$ maps the lines in $\mathcal{D}_P$ to the lines in $\mathcal{D}_H$. □

Unfortunately the mapping $F_2 : \mathcal{D}_H \to \mathcal{D}_P$ is not an isometry. Through a little modification of the Hilbert metric, we can show that $F_2 : \mathcal{D}_{H'} \to \mathcal{D}_P$ is an isometry. Let $\mathcal{D}_{H'} = (\mathbb{D}^2, d_{H'})$ be the unit disc $\mathbb{D}^2$ with the *modified Hilbert metric* $d_{H'}$ defined by

$$d_{H'}(z, w) = \frac{1}{2}\, d_H(z, w). \tag{6.12}$$

The following Theorem 6.4 is also found in Thurston's book [13]. I give another proof of it.

THEOREM 6.4. *The mapping $F_2 : \mathcal{D}_{H'} \to \mathcal{D}_P$ is an isometry.*

*Proof.* For $z_1, z_2 \in \mathcal{D}_{H'}$, let $w_j = F_2(z_j) \in \mathcal{D}_P$. Since $d_P(w_1, w_2) = \log[w_1', w_1, w_2, w_2']$ and $d_{H'}(z_1, z_2) = \frac{1}{2}\log[z_1^*, z_1, z_2, z_2^*]$, it is equivalent to show that

$$[w_1', w_1, w_2, w_2'] = [z_1^*, z_1, z_2, z_2^*]^{\frac{1}{2}}. \tag{6.13}$$

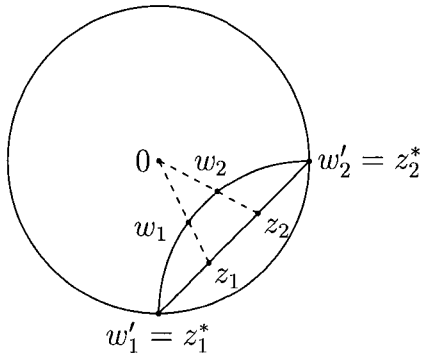



FIGURE 3. The image of lines through $F_2 : \mathcal{D}_{H'} \to \mathcal{D}_P$

Without loss of generality we assume that $z_1, z_2$ lie on the chord $\{x = a\}$ where $0 \le a < 1$. Let $z_1 = a + b\,i$, $z_2 = a + c\,i$ for $b < c$, then the boundary points of line segment in $\mathcal{D}_{H'}$ are $z_1^* = a - \sqrt{1-a^2}\,i$, $z_2^* = a + \sqrt{1+a^2}\,i$. From the proof of Proposition 6.3, we know $w_1' = z_1^*$ and $w_2' = z_2^*$. Therefore the cross-ratio $[w_1', w_1, w_2, w_2']$ is

$$\left[a - \sqrt{1-a^2}\,i, \frac{a+b\,i}{1+\sqrt{1-a^2-b^2}}, \frac{a+c\,i}{1+\sqrt{1-a^2-c^2}}, a + \sqrt{1-a^2}\,i\right]$$

$$= \frac{\left(a - \sqrt{1-a^2}\,i\right) - \left(\frac{a+c\,i}{1+\sqrt{1-a^2-c^2}}\right)}{\left(a - \sqrt{1-a^2}\,i\right) - \left(\frac{a+b\,i}{1+\sqrt{1-a^2-b^2}}\right)} \cdot \frac{\left(\frac{a+b\,i}{1+\sqrt{1-a^2-b^2}}\right) - \left(a + \sqrt{1-a^2}\,i\right)}{\left(\frac{a+c\,i}{1+\sqrt{1-a^2-c^2}}\right) - \left(a + \sqrt{1-a^2}\,i\right)}$$

$$\overset{\text{let}}{=} \frac{B}{C} \cdot \frac{(-A)}{(-D)} = \frac{A}{C}\frac{B}{D}.$$

Since $z_1 = a + b\,i$ is a point in the unit disc $\mathcal{D}_{H'}$, we have $1 - a^2 > b^2$. Thus $\sqrt{1-a^2} - b > 0$, $\sqrt{1-a^2} + b > 0$. It follows

$$\sqrt{1-a^2-b^2} = (\sqrt{1-a^2} - b)^{\frac{1}{2}}(\sqrt{1-a^2} + b)^{\frac{1}{2}}. \tag{6.14}$$

Hence $(1+\sqrt{1-a^2-b^2})\,A = (1+\sqrt{1-a^2-b^2})(a+\sqrt{1-a^2}\,i) - (a+b\,i)$
$= (\sqrt{1-a^2} - b)\,i + \sqrt{1-a^2-b^2}(a + \sqrt{1-a^2}\,i)$
$= (\sqrt{1-a^2} - b)^{\frac{1}{2}}\left((\sqrt{1-a^2} - b)^{\frac{1}{2}}\,i + (\sqrt{1-a^2} + b)^{\frac{1}{2}}(a + \sqrt{1-a^2}\,i)\right)$

$\overset{\text{let}}{=} (\sqrt{1-a^2}-b)^{\frac{1}{2}}\ \alpha$. Through the similar computations, we have the followings:

$$\begin{aligned}
(1+\sqrt{1-a^2-b^2})\,A &= (\sqrt{1-a^2}-b)^{\frac{1}{2}}\ \alpha,\\
(1+\sqrt{1-a^2-c^2})\,B &= (\sqrt{1-a^2}+c)^{\frac{1}{2}}\ \beta,\\
(1+\sqrt{1-a^2-b^2})\,C &= (\sqrt{1-a^2}+b)^{\frac{1}{2}}\ \gamma,\\
(1+\sqrt{1-a^2-c^2})\,D &= (\sqrt{1-a^2}-c)^{\frac{1}{2}}\ \delta,
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= (\sqrt{1-a^2}-b)^{\frac{1}{2}}\,i + (\sqrt{1-a^2}+b)^{\frac{1}{2}}(a+\sqrt{1-a^2}\,i),\\
\beta &= (\sqrt{1-a^2}+c)^{\frac{1}{2}}\,i - (\sqrt{1-a^2}-c)^{\frac{1}{2}}(a+\sqrt{1-a^2}\,i),\\
\gamma &= (\sqrt{1-a^2}+b)^{\frac{1}{2}}\,i - (\sqrt{1-a^2}-b)^{\frac{1}{2}}(a+\sqrt{1-a^2}\,i),\\
\delta &= (\sqrt{1-a^2}-c)^{\frac{1}{2}}\,i + (\sqrt{1-a^2}+c)^{\frac{1}{2}}(a+\sqrt{1-a^2}\,i).
\end{aligned}$$

By simple calculation we can get the relation $\alpha\,\beta = \gamma\,\delta$. Therefore the theorem is proved as follow:

$$\begin{aligned}
[w_1', w_1, w_2, w_2'] &= \frac{A}{C}\cdot\frac{B}{D}\\
&= \frac{(1+\sqrt{1-a^2-b^2})\,A}{(1+\sqrt{1-a^2-b^2})\,C}\cdot\frac{(1+\sqrt{1-a^2-c^2})\,B}{(1+\sqrt{1-a^2-c^2})\,D}\\
&= \frac{(\sqrt{1-a^2}-b)^{\frac{1}{2}}\ \alpha}{(\sqrt{1-a^2}+b)^{\frac{1}{2}}\ \gamma}\cdot\frac{(\sqrt{1-a^2}+c)^{\frac{1}{2}}\ \beta}{(\sqrt{1-a^2}-c)^{\frac{1}{2}}\ \delta}\\
&= \frac{(\sqrt{1-a^2}-b)^{\frac{1}{2}}}{(\sqrt{1-a^2}+b)^{\frac{1}{2}}}\cdot\frac{(\sqrt{1-a^2}+c)^{\frac{1}{2}}}{(\sqrt{1-a^2}-c)^{\frac{1}{2}}}\\
&= \left(\frac{(\sqrt{1-a^2}-b)}{(\sqrt{1-a^2}+b)}\frac{(\sqrt{1-a^2}+c)}{(\sqrt{1-a^2}-c)}\right)^{\frac{1}{2}}\\
&= \left[a-\sqrt{1-a^2}\,i,\ a+b\,i,\ a+c\,i,\ a+\sqrt{1-a^2}\,i\right]^{\frac{1}{2}}\\
&= [z_1^*, z_1, z_2, z_2^*]^{\frac{1}{2}}\,. \qquad \square
\end{aligned}$$

THEOREM 6.5. *The group $\mathcal{I}_3$ of orientation preserving isometries of the Hilbert disk $\mathcal{D}_{H'}$ is*

$$\mathcal{I}_3 = \left\{ \frac{\alpha^2 z + \beta^2 \bar{z} + 2\alpha\beta}{2\,\mathrm{Re}(\alpha\bar{\beta}z) + |\alpha|^2 + |\beta|^2} \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\}. \tag{6.15}$$

*Proof.* Since $G_2 : \mathcal{D}_P \to \mathcal{D}_{H'}$ is an isometry with the inverse $F_2$, the isometry of $\mathcal{D}_{H'}$ has the expression such that $G_2 \circ f_2 \circ F_2$, where $f_2$ is an isometry of $\mathcal{D}_P$ in (6.5). Therefore the isometries of $\mathcal{D}_H$ are

$$\begin{aligned}
&(G_2 \circ f \circ F_2)(z) \\
&= (G_2 \circ f_2)\left(\frac{z}{1+\sqrt{1-|z|^2}}\right) \\
&= G_2\left(\frac{\alpha z + \beta(1+\sqrt{1-|z|^2})}{\bar{\beta} z + \bar{\alpha}(1+\sqrt{1-|z|^2})}\right) \overset{\text{let}}{=} G_2\left(\frac{A}{B}\right) = \frac{2A\bar{B}}{B\bar{B}+A\bar{A}} \\
&= \frac{2(\alpha^2 z + \beta^2 \bar{z} + 2\alpha\beta)(1+\sqrt{1-|z|^2})}{(2\alpha\bar{\beta} z + 2\bar{\alpha}\beta\bar{z} + 2|\alpha|^2 + 2|\beta|^2)(1+\sqrt{1-|z|^2})} \\
&= \frac{\alpha^2 z + \beta^2 \bar{z} + 2\alpha\beta}{2\,\mathrm{Re}(\alpha\bar{\beta} z) + |\alpha|^2 + |\beta|^2},
\end{aligned}$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$. □

Let $\Omega_H = (\Omega, d_H)$ be the convex domain $\Omega \subset \mathbb{RP}^2$ defined by

$$\Omega = \{[x, y, s] \in \mathbb{RP}^2 \mid x^2 + y^2 - s^2 < 0\} \tag{6.16}$$

with the Hilbert metric $d_H$. Then $\Omega_H$ is a strictly convex domain with the conic boundary. Consider the mapping $G_3 : \mathcal{D}_H \to \Omega_H$ defined by

$$G_3(z) = [\,\mathrm{Re}(z), \mathrm{Im}(z), 1\,] = [\,x, y, 1\,] \tag{6.17}$$

for $z = x + i\,y$. Then $G_3$ is a diffeomorphism with the inverse $G_3^{-1} \overset{\text{let}}{=} F_3$ such that

$$F_3\,([x, y, s]) = \frac{x}{s} + i\,\frac{y}{s}. \tag{6.18}$$

THEOREM 6.6. *The mapping $G_3 : \mathcal{D}_H \to \Omega_H$ is an isometry.*

*Proof.* For two distinct points $z_1 = x_1 + i\,y_1$, $z_2 = x_2 + i\,y_2 \in \mathcal{D}_H$ with the boundary points $z_1^*, z_2^*$, we denote $w_j = G_3(z_j)$ and $w_j^* = G_3(z_j^*)$ for each $j$. Then $w_1^*, w_1, w_2, w_2^*$ are four distinct collinear points in $\mathbb{RP}^2$ since the corresponding four lines in $\mathbb{R}^3$ are contained in the plane $\mathcal{P}$ spanned by two linearly independent vectors $(x_1, y_1, 1)$ and $(x_2, y_2, 1)$ in $\mathbb{R}^3$. Since the plane $\mathcal{P}$ is not the $xy$-plane, by Definition (5.4) of the cross-ratio,

$$\begin{aligned}
[w_1^*, w_1, w_2, w_2^*] &= [[x_1^*, y_1^*, 1], [x_1, y_1, 1], [x_2, y_2, 1], [x_2^*, y_2^*, 1]] \\
&= [x_1^* + i\,y_1^*, x_1 + i\,y_1, x_2 + i\,y_2, x_2^* + i\,y_2^*] \\
&= [z_1^*, z_1, z_2, z_2^*].
\end{aligned}$$

This completes the proof because $d_H(w_1, w_2) = \log[w_1^*, w_1, w_2, w_2^*]$ and $d_H(z_1, z_2) = \log[z_1^*, z_1, z_2, z_2^*]$. □

## 7. Embedding of $\mathbf{PSL}(2, \mathbb{R})$ into $\mathbf{SL}(3, \mathbb{R})$

The goal of this section is to define an isometry $\mathbb{H} \to \Omega$ and an embedding of $\mathbf{PSL}(2, \mathbb{R})$ into $\mathbf{SL}(3, \mathbb{R})$. Since $\mathcal{H}_P$ and $\mathcal{D}_{H'}$ are isometric and $\mathcal{D}_H$ and $\Omega_H$ are isometric, to define an isometry between $\mathbb{H}$ and $\Omega$, we need to change the metric on $\Omega$ from $d_H$ to $d_{H'}$ as in (6.12). Let $\Omega_{H'} = (\Omega, d_{H'})$ be the strictly convex domain $\Omega$ in $\mathbb{RP}^2$ with the modified Hilbert metric $d_{H'}$. Then the mapping $G_3 : \mathcal{D}_{H'} \to \Omega_{H'}$ is also an isometry.

THEOREM 7.1. *The mapping $G : \mathcal{H}_P \to \Omega_{H'}$ defined by*

$$G(z) = [\,2x,\ x^2 + y^2 - 1,\ x^2 + y^2 + 1\,] \tag{7.1}$$

*is an isometry with the inverse $G^{-1} \overset{\text{let}}{=} F$ such that*

$$F([x, y, s]) = \left(\frac{x}{s-y}\right) + i\left(\frac{s}{s-y}\right)\sqrt{1 - \frac{x^2}{s^2} - \frac{y^2}{s^2}}. \tag{7.2}$$

*Proof.* Since $G_1 : \mathcal{H}_P \to \mathcal{D}_P$, $G_2 : \mathcal{D}_P \to \mathcal{D}_{H'}$, $G_3 : \mathcal{D}_{H'} \to \Omega_{H'}$ are isometries, clearly their composition $G = G_3 \circ G_2 \circ G_1$ is an isometry from $\mathcal{H}_P$ to $\Omega_{H'}$. With the identification $z = x + i\,y \in \mathcal{H}_P$, the isometry $G$ can be calculated as in (7.1). The expression of the inverse $G^{-1} = F$ is (7.2). Clearly $F : \Omega_{H'} \to \mathcal{H}_P$ is the isometry calculated by the composition $F = F_1 \circ F_2 \circ F_3$. □

To realizes a hyperbolic structure on $M$ as a convex real projective structure, consider the final goal of this section, which is to define an embedding $\mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$.

THEOREM 7.2. *The mapping $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ defined by*

$$\varphi(A) = \begin{pmatrix} ad + bc & ac - bd & ac + bd \\ ab - cd & \frac{a^2 - b^2 - c^2 + d^2}{2} & \frac{a^2 + b^2 - c^2 - d^2}{2} \\ ab + cd & \frac{a^2 - b^2 + c^2 - d^2}{2} & \frac{a^2 + b^2 + c^2 + d^2}{2} \end{pmatrix} \text{ for } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.3}$$

*is an embedding of $\mathbf{PSL}(2, \mathbb{R})$ into $\mathbf{SL}(3, \mathbb{R})$.*

*Proof.* Since the mapping $G : \mathcal{H}_P \to \Omega_{H'}$ is an isometry with the inverse $F$, we can define a mapping $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ such that

the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H}_P & \xrightarrow{G} & \Omega_{H'} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi(A)} \\ \mathcal{H}_P & \xrightarrow[G]{} & \Omega_{H'} \end{array}$$

Let $[x, y, s] = [z, s]$ be a point in $\Omega_{H'} \subset \mathbb{RP}^2$ and $A \in \mathbf{PSL}(2, \mathbb{R})$ the matrix representation of an isometry $f_1$ of $\mathcal{H}_P$. By Theorems 6.1, 6.2, and 6.5,

$$\begin{aligned} (G \circ f_1 \circ F)[x, y, s] &= (G_3 \circ f_3 \circ F_3)[x, y, s] = (G_3 \circ f_3)(\frac{z}{s}) \\ &= G_3\left(\frac{\alpha^2 z + \beta^2 \bar{z} + 2\alpha\beta s}{2\,\mathrm{Re}(\alpha\bar{\beta}z) + |\alpha|^2 s + |\beta|^2 s}\right) \overset{\text{let}}{=} G_3(\frac{w}{t}) = \left[\frac{w}{t}, 1\right] = [w, t]\,. \end{aligned}$$

To describe the matrix representation of $(G \circ f_1 \circ F)$, plug in the standard basis $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ to the linear transformation $(G \circ f_1 \circ F)$. Then we obtain

$$\begin{aligned} (G \circ f_1 \circ F)[1, 0, 0] &= (G \circ f_1 \circ F)[\,1, 0\,] = [\,\alpha^2 + \beta^2,\ 2\,\mathrm{Re}(\alpha\bar{\beta})\,], \\ (G \circ f_1 \circ F)[0, 1, 0] &= (G \circ f_1 \circ F)[\,i\,, 0\,] = [\,\alpha^2 i - \beta^2 i,\ 2\,\mathrm{Re}(\alpha\bar{\beta}i)\,], \\ (G \circ f_1 \circ F)[0, 0, 1] &= (G \circ f_1 \circ F)[\,0, 1\,] = [\,2\,\alpha\beta,\ |\alpha|^2 + |\beta|^2\,]. \end{aligned}$$

Recall $\alpha = (\frac{a+d}{2}) + i(\frac{b-c}{2})$ and $\beta = (\frac{b+c}{2}) + i(\frac{a-d}{2})$ from the proof of Theorem 6.2. After some calculations, we have the following equations.

$$\begin{aligned} \alpha^2 + \beta^2 &= (ad + bc) + i(ab - cd), \\ 2\,\mathrm{Re}(\alpha\bar{\beta}) &= ab + cd, \\ \alpha^2 i - \beta^2 i &= (ac - bd) + i(\frac{a^2 - b^2 - c^2 + d^2}{2}), \\ 2\,\mathrm{Re}(\alpha\bar{\beta}i) &= \frac{a^2 - b^2 + c^2 - d^2}{2}, \\ 2\,\alpha\beta &= (ac + bd) + i(\frac{a^2 + b^2 - c^2 - d^2}{2}), \\ |\alpha|^2 + |\beta|^2 &= \frac{a^2 + b^2 + c^2 + d^2}{2}. \end{aligned}$$

Thus we have the matrix representation $\varphi(A)$ as in (7.3). Suppose $\varphi(A_1) = \varphi(A_2)$ for $A_1, A_2 \in \mathbf{SL}(2, \mathbb{R})$. Then $A_1 = A_2$ or $A_1 = -A_2$. We can also compute $\varphi(A_1 A_2) = \varphi(A_1)\varphi(A_2)$. Thus $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to$

$\mathbf{SL}(3,\mathbb{R})$ is an injective homomorphism. Therefore $\varphi$ is an embedding of $\mathbf{PSL}(2,\mathbb{R})$ into $\mathbf{SL}(3,\mathbb{R})$. □

After some the Mathematica computations, we have the following properties.

PROPOSITION 7.3. *Let $B = \varphi(A) \in \mathbf{SL}(3,\mathbb{R})$ for $A \in \mathbf{PSL}(2,\mathbb{R})$. Then,*

1. $\mathrm{Det}(B) = \mathrm{Det}(A)^3 = (ad - bc)^3 = 1$.
2. $\mathrm{Tr}(B) = \mathrm{Tr}(A)^2 - 1 = (a + d)^2 - 1$.
3. $B \in \mathbf{PSO}(2,1) \subset \mathbf{SL}(3,\mathbb{R})$.
4. *Suppose $\pm\{\alpha, \alpha^{-1}\}$ is the eigenvalues of $A$, respectively. Then $\{\alpha^2, 1, \alpha^{-2}\}$ is the eigenvalues of $B$.*

Therefore if $A \in \mathbf{PSL}(2,\mathbb{R})$ is a hyperbolic element, then $B = \varphi(A) \in \mathbf{SL}(3,\mathbb{R})$ is a positive hyperbolic element. We can derive that the hyperbolic structures embeds into convex real projective structures through the identification of the conjugacy classes of $[\mathbf{PSL}(2,\mathbb{R})] \hookrightarrow [\mathbf{SL}(3,\mathbb{R})]$

$$\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha^{-2}+\alpha^2}{2} & \frac{\alpha^{-2}-\alpha^2}{2} \\ 0 & \frac{\alpha^{-2}-\alpha^2}{2} & \frac{\alpha^{-2}+\alpha^2}{2} \end{pmatrix} \cong \begin{pmatrix} \alpha^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}. \tag{7.4}$$

## 8. The Goldman's length parameters

The set of positive hyperbolic elements of $\mathbf{SL}(3,\mathbb{R})$ is denoted by $\mathbf{Hyp}_+$. Goldman [5] defined the length parameters $\ell, m$ on $\mathbf{Hyp}_+$ as $\ell(B) = \log\left(\frac{\nu}{\lambda}\right)$, $m(B) = 3\log(\mu)$, where $B$ is a positive hyperbolic element represented by the diagonal matrix (4.2).

In this paper we will modify Goldman's length parameters $\ell, m$ in order to maintain the consistency with the Fenchel-Nielsen's length parameter $\ell$. The *modified Goldman's length parameters* $\ell, m$ are

$$\ell(B) = \frac{1}{2}\log\left(\frac{\nu}{\lambda}\right), \quad m(B) = \frac{3}{4}\log(\mu) \tag{8.1}$$

with $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$.

For a hyperbolic manifold $M$, let $\Omega$ be the developing image in $\mathbb{H}^2$ and $A$ an element of the holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$. The translation length $\ell(A)$ of $A$ is defined by

$$\ell(A) = \inf_{z\in\Omega} d_P(z, A(z)),$$

where $d_P$ is the Poincaré metric on $\Omega$. Then the translation length $\ell(A)$ of $A$ is achieved if and only if $z$ lies on the principal line of $A$ which is the line joining the repelling and attracting fixed point of $A$. From Beardon's book [1], we get the relation

$$\left|\frac{\mathrm{tr}(A)}{2}\right| = \cosh\left(\frac{\ell(A)}{2}\right) \tag{8.2}$$

between the translation length and trace of $A$.

Since $\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})$ and $|\mathrm{tr}(A)| = \alpha + \alpha^{-1}$ for $\alpha > 1$, Equation (8.2) becomes

$$\begin{aligned}
\frac{\ell(A)}{2} &= \cosh^{-1}\left(\frac{\alpha + \alpha^{-1}}{2}\right) \\
&= \log\left(\frac{\alpha + \alpha^{-1}}{2} + \sqrt{\frac{\alpha^2 + 2 + \alpha^{-2}}{4} - 1}\right) \\
&= \log\left(\frac{\alpha + \alpha^{-1}}{2} + \frac{|\alpha - \alpha^{-1}|}{2}\right) \\
&= \log\left(\frac{\alpha + \alpha^{-1}}{2} + \frac{\alpha - \alpha^{-1}}{2}\right) = \log(\alpha).
\end{aligned}$$

Therefore the Fenchel-Nielsen's length parameter $\ell$ can be defined as

$$\ell(A) = \log(\alpha^2) \tag{8.3}$$

for a hyperbolic element $A \in \mathbf{PSL}(2, \mathbb{R})$ represented by the diagonal matrix (4.1) with $\alpha > 1$.

THEOREM 8.1. *The modified Goldman's length parameter $\ell$ is an isometric extension of the Fenchel-Nielsen's length parameter $\ell$.*

*Proof.* Let $B = \varphi(A)$. Since the length parameter $\ell$ is invariant under the conjugation, consider the identifications $\lambda = \alpha^{-2}$, $\mu = 1$ and $\nu = \alpha^2$ in (7.4). Then we have

$$\ell(B) = \frac{1}{2}\log\left(\frac{\nu}{\lambda}\right) = \frac{1}{2}\log\left(\frac{\alpha^2}{\alpha^{-2}}\right) = \frac{1}{2}\log(\alpha^4) = \log(\alpha^2) = \ell(A).$$

Therefore the modified Goldman's length parameter $\ell(B)$ is exactly the same parameter to the Fenchel-Nielsen's length parameter $\ell(A)$. □

We can extend the concept of translation length of hyperbolic structures to that of convex real projective structures. For any $B \in \mathbf{Hyp}_+$ there exist three non-collinear fixed points and a $B$-invariant line in $\mathbb{RP}^2$. We shall refer that the repelling, saddle, attracting fixed points

$\mathrm{Fix}_-(B), \mathrm{Fix}_0(B), \mathrm{Fix}_+(B)$ are the fixed points corresponding to the eigenvectors in $\mathbb{R}^3$ for the smallest eigenvalue $\lambda$, middle eigenvalue $\mu$, the largest eigenvalue $\nu$, respectively. The principal line $\sigma(B)$ is the line joining the repelling and attracting fixed points of $B$.

Each positive hyperbolic element $B$ can be uniquely decomposed as $HV$ up to $\mathbf{SL}(3,\mathbb{R})$-conjugation, where

$$H = \begin{pmatrix} \lambda\sqrt{\mu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu\sqrt{\mu} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1/\sqrt{\mu} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1/\sqrt{\mu} \end{pmatrix}. \tag{8.4}$$

We call $H$ the *horizontal* factor and $V$ the *vertical* factor of $B \in \mathbf{Hyp}_+$. $H$ will be also called the *pure hyperbolic* factor of $B$.

Consider the horizontal factor $H$ of $B$. For any point $a = [x, 0, s]^t$ in the principal line $\sigma(B)$ such that $x \neq 0, s \neq 0$, the modified Hilbert distance $d_{H'}$ between $a$ and $H(a)$ is

$$\begin{aligned} d_{H'}(a, H(a)) &= \frac{1}{2}\log\,[\mathrm{Fix}_-(B),\, a,\, H(a),\, \mathrm{Fix}_+(B)] \\ &= \frac{1}{2}\log\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} \lambda\sqrt{\mu}\,x \\ 0 \\ \nu\sqrt{\mu}\,s \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right] \\ &= \frac{1}{2}\log\frac{\left(\infty - \frac{\lambda}{\nu}\frac{x}{s}\right)\left(\frac{x}{s} - 0\right)}{\left(\infty - \frac{x}{s}\right)\left(\frac{\lambda}{\nu}\frac{x}{s} - 0\right)} = \frac{1}{2}\log\left(\frac{\nu}{\lambda}\right) = \ell(B). \end{aligned}$$

We call $\ell(B) = d_{H'}(a, H(a))$ the *horizontal translation length* and it is the length of the boundary component represented by $B$.

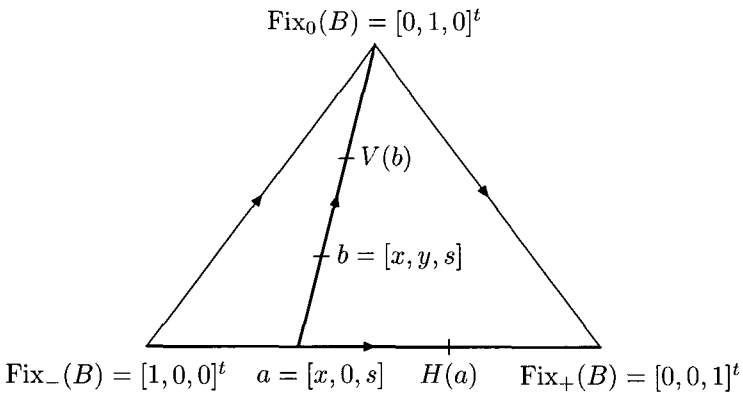



FIGURE 4. The horizontal and vertical translation lengths

Consider the vertical factor $V$ of $B$. Then the stationary set is the principal line $\sigma(B)$ and the saddle fixed point $\mathrm{Fix}_0(B)$. Without loss of generality we assume $\mu > 1$. Then for any $b = [x, y, s]^t$ in the segment joining $[x, 0, s]^t$ and $[0, 1, 0]^t$, the point $V(b)$ goes toward $[0, 1, 0]^t$ since $\mu > 1$. Then the modified Hilbert distance $d_{H'}$ between $b$ and $V(b)$ is

$$\begin{aligned}
d_{H'}(b, V(b)) &= \frac{1}{2}\log\left[[x, 0, s]^t,\ b,\ V(a),\ \mathrm{Fix}_0(B)\right] \\
&= \frac{1}{2}\log\left[\begin{bmatrix} x \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} x \\ y \\ s \end{bmatrix}, \begin{bmatrix} \frac{x}{\sqrt{\mu}} \\ y\mu \\ \frac{x}{\sqrt{\mu}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right] \\
&= \frac{1}{2}\log\left[\frac{x}{0}, \frac{x}{y}, \frac{\frac{x}{\sqrt{\mu}}}{y\mu}, \frac{0}{1}\right] \\
&= \frac{1}{2}\log\left[\frac{1}{0}, \frac{1}{y}, \frac{1}{y\mu^{\frac{3}{2}}}, \frac{0}{1}\right] \\
&= \frac{1}{2}\log\frac{\left(\infty - y^{-1}\mu^{-\frac{3}{2}}\right)\left(y^{-1} - 0\right)}{\left(\infty - y^{-1}\right)\left(y^{-1}\mu^{-\frac{3}{2}} - 0\right)} \\
&= \frac{1}{2}\log\left(\mu^{\frac{3}{2}}\right) = \frac{3}{4}\log(\mu) = m(B).
\end{aligned}$$

We call $m(B) = d_{H'}(b, V(b))$ the *vertical translation length.* Therefore $B = HV \in \mathbf{SL}(3, \mathbb{R})$ moves a point in $\Omega$ vertically by $\frac{3}{4}\log(\mu)$ and horizontally by $\frac{1}{2}\log(\frac{\nu}{\lambda})$. We can easily compute the following relations.

$$\begin{aligned}
\ell(H) &= \frac{1}{2}\log\left(\frac{\nu\sqrt{\mu}}{\lambda\sqrt{\mu}}\right) = \frac{1}{2}\log\left(\frac{\nu}{\lambda}\right) = \ell(B), \\
m(H) &= \frac{3}{4}\log(1) = 0.
\end{aligned}$$

The above equations imply the horizontal translation lengths of $B$ and $H$ are the same. If a positive hyperbolic element $B \in \mathbf{SL}(3, \mathbb{R})$ is derived from a hyperbolic element, then $m(B) = 0$. Therefore the length parameter $m$ measures the deviation of convex real projective structures from hyperbolic structures.

ACKNOWLEDGEMENTS. I want to thank W. Goldman for introducing me to this subject and Jin-Hwan Cho for an advice to convert metapost graphics to eps files. I also thank the referee who points out some errors and mistakes.

## References

[1] A. F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, 1983.

[2] S. Choi and W. M. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 657–661.

[3] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), no. 2, 200–225.

[4] ———, *Geometric structures on manifolds and varieties of representations*, Geometry of group representations (Boulder, CO, 1987), 169–198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988.

[5] ———, *Convex real projective structures on compact surfaces*, J. Differential Geom. **31** (1990), no. 3, 791–845.

[6] D. Johnson and J. J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), 48–106, Progr. Math., 67, Birkhäuser Boston, Boston, MA, 1987.

[7] H. C. Kim, *Matrix presentations of the Teichmüller space of a pair of pants*, J. Korean Math. Soc. **42** (2005), no. 3, 553–569.

[8] S. Kobayashi, *Invariant distances for projective structures*, Symposia Mathematica, Vol. XXVI (Rome, 1980), pp. 153–161, Academic Press, London-New York, 1982.

[9] N. Kuiper, *On convex locally projective spaces*, Convegno Int. Geometria Diff., Italy, 1954, 200–213, Edizioni Cremonese, Roma, 1954.

[10] K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Oxford Science Publications, Oxford University Press, New York, 1998.

[11] J. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[12] D. Sullivan and W. Thurston, *Manifolds with canonical coordinates charts: some examples*, Enseign. Math.(2) **29** (1983), no. 1-2, 15–25.

[13] W. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, 35. Princeton University Press, 1997.

Department of Mathematics Education
Korea University
Seoul 136-701, Korea
*E-mail*: hongchan@korea.ac.kr