

## ON THE $r$ -TH HYPER-KLOOSTERMAN SUMS AND ITS HYBRID MEAN VALUE

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**ABSTRACT.** The main purpose of this paper is using the properties of Gauss sums, primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of the  $r$ -th hyper-Kloosterman sums  $Kl(h, k+1, r; q)$  and the hyper Cochrane sums  $C(h, q; m, k)$ , and give an interesting mean value formula.

### 1. Introduction

For any positive integer  $q$  and  $n$  and an arbitrary integer  $h$ , the general Dedekind sums  $S(h, n, q)$  is defined by

$$S(h, n, q) = \sum_{a=1}^q \overline{B}_n\left(\frac{a}{q}\right) \overline{B}_n\left(\frac{ah}{q}\right),$$

where

$$\overline{B}_n(x) = \begin{cases} B_n(x - [x]), & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

$B_n(x)$  is the Bernoulli polynomial, and  $\overline{B}_n(x)$  is the  $n$ -th Bernoulli periodic function, defined on the interval  $0 < x \leq 1$ . In [8] and [6], the second author has given some mean value properties of  $S(h, n, q)$ . In October 2000, during his visit to Xi'an, Professor Todd Cochrane introduced the following sums analogous to the Dedekind sums,

$$C(h, q) = \sum_{a=1}^q' \left( \left( \frac{\bar{a}}{q} \right) \right) \left( \left( \frac{ah}{q} \right) \right),$$

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where  $\sum'$  denotes the summation over all  $a$  such that  $(a, q) = 1$ ,  $a\bar{a} \equiv 1 \pmod{q}$ , and

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

He advised us to study the arithmetical properties and mean value distribution properties of  $C(h, q)$ . Yet we still know very little about this problem. In [3], the second author found that there exists some interesting connections between  $C(h, q)$  and Kloosterman sums

$$K(m, n; q) = \sum_{b=1}^q e\left(\frac{mb + n\bar{b}}{q}\right),$$

where  $e(y) = e^{2\pi iy}$ . For example, if  $q$  is a square-full number (i.e.,  $p \mid q$  if and only if  $p^2 \mid q$ ), then we have the following asymptotic formula

$$(1) \quad \sum_{h=1}^q K(h, 1; q)C(h, q) = \frac{-1}{2\pi^2} q\phi(q) + O\left(q \exp\left(\frac{3 \ln q}{\ln \ln q}\right)\right).$$

For general integer  $q \geq 3$ , the second author [4] proved the asymptotic formula

$$(2) \quad \sum_{h=1}^q K(h, 1; q)C(h, q) = \frac{-1}{2\pi^2} q\phi(q) \prod_{p \parallel q} \left(1 - \frac{1}{p(p-1)}\right) + O\left(q^{\frac{3}{2}+\epsilon}\right),$$

where  $\epsilon$  be any fixed positive number. Moreover, the second author [5] studied the hybrid mean value of the Cochrane sums and the  $r$ -th Kloosterman sums

$$K(m, n, r; q) = \sum_{b=1}^q e\left(\frac{mb^r + n\bar{b}^r}{q}\right),$$

and obtained the following asymptotic formula

$$(3) \quad \sum_{h=1}^q K(h, 1, r; p)C(h, q) = \frac{-1}{2\pi^2} p^2 + O\left(rp^{\frac{3}{2}} \ln^2 p\right).$$

For any general positive integers  $q \geq 3$  and  $r$  with  $l_r = (r, \phi(q)) > 1$ , he [6] proved that

$$(4) \quad \sum_{h=1}^q' K(h, 1, r; q) C(h, q) = \frac{-1}{2\pi^2} q\phi(q) \prod_{p \parallel q} \left(1 - \frac{1}{p(p-1)}\right) + O\left(l_r q^{\frac{3}{2}+\epsilon}\right).$$

In [7], Professor Mordell introduced the hyper-Kloosterman sums as following:

$$\begin{aligned} & Kl(h, k+1; q) \\ &= \sum_{a_1=1}^q' \sum_{a_2=1}^q' \cdots \sum_{a_k=1}^q' e\left(\frac{a_1 + a_2 + \cdots + a_k + h \cdot \bar{a}_1 \bar{a}_2 \cdots \bar{a}_k}{q}\right). \end{aligned}$$

About the hyper-Kloosterman sums, many scholars had studied about it before. Applications of the hyper-kloosterman sums were found in the estimation of Fourier coefficients of Maass forms [8] and the work on Selberg's eigenvalue conjecture [9]. On the other hand, Smith [10] had built some interesting connections between the hyper-Klooster-man sums and the Heibronn sums.

Following the spirit of [3], we found that there exists some interesting connections between the hyper-Kloosterman sums and the hyper Cochrane sums

$$\begin{aligned} & C(h, q; m, k) \\ &= \sum_{a_1=1}^q' \cdots \sum_{a_k=1}^q' \overline{B}_{m_1}\left(\frac{\bar{a}_1}{q}\right) \overline{B}_{m_k}\left(\frac{\bar{a}_k}{q}\right) \overline{B}_{m_{k+1}}\left(\frac{h \cdot a_1 \cdots a_k}{q}\right), \end{aligned}$$

and proved that the following mean value theorem

$$(5) \quad \begin{aligned} & \sum_{h=1}^q' Kl(h, k+1; q) C(h, q; m, k) \\ &= \frac{(-1)^{k+1} 2^k m_1! \cdots m_{k+1}! \phi(q) q^k}{(2\pi i)^{m_1+\cdots+m_{k+1}}} \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2}\right) \\ &+ O\left(q^{k+\epsilon}\right) \end{aligned}$$

holds for any odd numbers  $m_1, m_2, \dots, m_{k+1}$  (see [12]).

Similarly, we can define the  $r$ -th hyper-Kloosterman sums as following:

$$\begin{aligned} & Kl(h, k+1, r; q) \\ &= \sum_{a_1=1}^q \sum_{a_2=1}^q \cdots \sum_{a_k=1}^q e\left(\frac{a_1^r + a_2^r + \cdots + a_k^r + h \cdot \bar{a}_1^r \bar{a}_2^r \cdots \bar{a}_k^r}{q}\right). \end{aligned}$$

It is obvious that  $Kl(h, k+1, r; q) = Kl(h, k+1; q)$  if  $(r, \phi(q)) = 1$ , so we suppose  $(r, \phi(q)) > 1$ . In this paper, we shall use the properties of Gauss sums, primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value of the  $r$ -th hyper-Kloosterman sums  $Kl(h, k+1, r; q)$  and the hyper Cochrane sums  $C(h, q; m, k)$ , and give an interesting mean value formula. That is, we shall prove the following:

**THEOREM.** *Let any positive integers  $r$  and  $q \geq 3$  with  $l_r = (r, \phi(q)) > 1$ . Then for any odd numbers  $m_1, m_2, \dots, m_{k+1}$ , we have the asymptotic formula*

$$\begin{aligned} & \sum_{h=1}^q Kl(h, r, k+1; q) C(h, q; m, k) \\ &= \frac{(-1)^{k+1} 2^k m_1! \cdots m_{k+1}! \phi(q) q^k}{(2\pi i)^{m_1+\cdots+m_{k+1}}} \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2}\right) \\ &+ O\left(l_r q^{k+\frac{1}{2}+\epsilon}\right). \end{aligned}$$

Taking  $q = p$ , an odd prime in our Theorem, we may immediately deduce the following

**COROLLARY.** *For any odd prime  $p$  and odd numbers  $m_1, m_2, \dots, m_{k+1}$ , we have*

$$\begin{aligned} & \sum_{h=1}^p Kl(h, r, k+1; p) C(h, p; m, k) \\ &= \frac{(-1)^{k+1} 2^k m_1! \cdots m_{k+1}! p^{k+1}}{(2\pi i)^{m_1+\cdots+m_{k+1}}} + O\left(l_r p^{k+\frac{1}{2}+\epsilon}\right). \end{aligned}$$

## 2. Some lemmas

To prove the theorem, we need the following several lemmas. For convenience, first we may define the  $r$ -th Gauss sums as

$$G(n, \chi, r; q) = \sum_{b=1}^q \chi(b) e\left(\frac{nb^r}{q}\right).$$

It is clear that  $G(n, \chi, 1; q) = G(n, \chi)$  is the classical Gauss sums and we have the following

LEMMA 1. For any positive integer  $q$ , let  $\chi$  be a non-primitive character modulo  $q$  and  $\chi_q \Leftrightarrow \chi_{q^*}$ . If  $(n, q) > 1$ , we have

$$G(n, \chi) = \begin{cases} \bar{\chi}^*\left(\frac{n}{(n, q)}\right) \chi^*\left(\frac{q}{q^*(n, q)}\right) \mu\left(\frac{q}{q^*(n, q)}\right) \\ \times \phi(q) \phi^{-1}\left(\frac{q}{(n, q)}\right) \tau(\chi^*), & q^* = \frac{q_1}{(n, q_1)}; \\ 0, & q^* \neq \frac{q_1}{(n, q_1)}, \end{cases}$$

where  $q_1$  is the greatest divisor of  $q$  that has the same prime factors as  $q^*$ ,  $\tau(\chi) = G(1, \chi)$ , and  $\mu(n)$  is the Möbius function.

If  $(n, q) = 1$ , then we have

$$G(n, \chi) = \bar{\chi}^*(n) \chi^*\left(\frac{q}{q^*}\right) \mu\left(\frac{q}{q^*}\right) \tau(\chi^*).$$

*Proof.* See [11]. □

LEMMA 2. Let  $\chi$  be a character modulo  $q$ , generated by the primitive character  $\chi_m$  modulo  $m$ . Then we have the identity

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right) \mu\left(\frac{q}{m}\right) \tau(\chi_m).$$

*Proof.* See [11]. □

LEMMA 3. Let any positive integers  $h$  and  $q \geq 3$  with  $(h, q) = 1$ . Then for any odd numbers  $m_1, m_2, \dots, m_{k+1}$ , we have

$$\begin{aligned} & C(h, q; m, k) \\ &= \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1 + \cdots + m_{k+1}} \phi(q)} \\ &\quad \times \sum_{\substack{\chi \pmod{q} \\ \chi(-1) = -1}} \bar{\chi}(h) \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \cdots \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\}, \end{aligned}$$

where  $\chi$  denotes a Dirichlet character modulo  $q$  with  $\chi(-1) = -1$ .

*Proof.* From the orthogonality relation for character sums modulo  $q$  we have

$$\begin{aligned} C(h, q; m, k) &= \sum_{a_1=1}^q' \bar{B}_{m_1} \left( \frac{\bar{a}_1}{q} \right) \cdots \sum_{a_k=1}^q' \bar{B}_{m_k} \left( \frac{\bar{a}_k}{q} \right) \bar{B}_{m_{k+1}} \left( \frac{a_1 \cdots a_k h}{q} \right) \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \left\{ \sum_{a_1=1}^q \chi(a_1) \bar{B}_{m_1} \left( \frac{a_1}{q} \right) \right\} \\ &\quad \times \cdots \times \left\{ \sum_{a_{k+1}=1}^q \chi(a_{k+1}) \bar{B}_{m_{k+1}} \left( \frac{a_{k+1} h}{q} \right) \right\}. \end{aligned}$$

Note that

$$\bar{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r \neq 0}}^{+\infty} \frac{e(xr)}{r^n},$$

and for any integer  $h$  with  $(h, q) = 1$ ,  $G(hn, \chi) = \bar{\chi}(h)G(n, \chi)$ . Then we have

$$\begin{aligned} C(h, q; m, k) &= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \left\{ \sum_{a_1=1}^q \left( -\frac{m_1!}{(2\pi i)^{m_1}} \right) \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \chi(a_1) \frac{e(\frac{r_1 a_1}{q})}{r_1^{m_1}} \right\} \\ &\quad \times \cdots \times \left\{ \sum_{a_{k+1}=1}^q \left( -\frac{m_{k+1}!}{(2\pi i)^{m_{k+1}}} \right) \sum_{\substack{r_{k+1}=-\infty \\ r_{k+1} \neq 0}}^{+\infty} \chi(a_{k+1}) \frac{e(\frac{r_{k+1} a_{k+1} h}{q})}{r_{k+1}^{m_{k+1}}} \right\} \\ &= \frac{(-1)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1 + \cdots + m_{k+1}} \phi(q)} \sum_{\chi \pmod{q}} \left\{ \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \frac{1}{r_1^{m_1}} \sum_{a_1=1}^q \chi(a_1) e\left(\frac{r_1 a_1}{q}\right) \right\} \\ &\quad \times \cdots \times \left\{ \sum_{\substack{r_{k+1}=-\infty \\ r_{k+1} \neq 0}}^{+\infty} \frac{1}{r_{k+1}^{m_{k+1}}} \sum_{a_{k+1}=1}^q \chi(a_{k+1}) e\left(\frac{r_{k+1} a_{k+1} h}{q}\right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \sum_{\chi \pmod{q}} \left\{ \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \\
&\quad \times \cdots \times \left\{ \sum_{\substack{r_{k+1}=-\infty \\ r_{k+1} \neq 0}}^{+\infty} \frac{G(hr_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\} \\
&= \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \bar{\chi}(h) \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \\
&\quad \times \cdots \times \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\}.
\end{aligned}$$

This proves Lemma 3.  $\square$

LEMMA 4. For any positive integers  $q$  and  $r > 1$ , let  $l_r = (r, \phi(q))$  and  $\chi_1$  be a  $l_r$ -th-order character modulo  $q$ . Then for any character  $\chi$  modulo  $q$ , we have the identities

$$\sum_{h=1}^q \bar{\chi}(h) K l(h, k+1, 1; q) = \tau^{k+1}(\bar{\chi})$$

and

$$\sum_{h=1}^q \bar{\chi}(h) K l(h, k+1, r; q) = \tau(\bar{\chi}) \left( \tau(\bar{\chi}) + \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i) \right)^k.$$

*Proof.* From the properties of Gauss sums we have

$$\begin{aligned}
&\sum_{h=1}^q \bar{\chi}(h) K l(h, k+1, r; q) \\
&= \sum_{h=1}^q \bar{\chi}(h) \sum'_{a_1=1}^q \sum'_{a_2=1}^q \cdots \sum'_{a_k=1}^q e\left(\frac{a_1^r + a_2^r + \cdots + a_k^r + h \cdot \bar{a}_1^r \bar{a}_2^r \cdots \bar{a}_k^r}{q}\right) \\
&= \sum_{h=1}^q \bar{\chi}(h) e\left(\frac{h}{q}\right) \sum'_{a_1=1}^q \sum'_{a_2=1}^q
\end{aligned}$$

$$\begin{aligned}
& \times \cdots \times \sum_{a_k=1}^q' \bar{\chi}(a_1^r a_2^r \cdots a_k^r) e\left(\frac{a_1^r + a_2^r + \cdots + a_k^r}{q}\right) \\
& = \tau(\bar{\chi}) \left( \sum_{b=1}^k \bar{\chi}(b^r) e\left(\frac{b^r}{q}\right) \right)^k \\
& = \tau(\bar{\chi}) G^k(1, \bar{\chi}^r, r; q).
\end{aligned}$$

It is clear that  $G(1, \bar{\chi}, 1; q) = \tau(\bar{\chi})$ , so we have

$$\sum_{h=1}^q \bar{\chi}(h) K_l(h, k+1, 1; q) = \tau^{k+1}(\bar{\chi}).$$

On the other hand, let  $\chi_1$  be a  $l_r$ th-order character modulo  $q$ . Then for any integer  $1 \leq a \leq q$  with  $(a, q) = 1$ , we have the identity

$$1 + \chi_1(a) + \cdots + \chi_1^{l_r-1}(a) = \begin{cases} l_r, & \text{if } a \text{ is a } l_r \text{th residue mod } q; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
G(1, \bar{\chi}^r, r; q) & = \sum_{b=1}^k \bar{\chi}(b^r) e\left(\frac{b^r}{q}\right) \\
& = \sum_{b=1}^q \left(1 + \chi_1(b) + \cdots + \chi_1^{l_r-1}(b)\right) \bar{\chi}(b) e\left(\frac{b}{q}\right) \\
& = \tau(\bar{\chi}) + \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i).
\end{aligned}$$

This proves Lemma 4.  $\square$

**LEMMA 5.** *Let  $q$  and  $r$  be integers with  $q \geq 2$  and  $(r, q) = 1$ ,  $\chi$  be a Dirichlet character modulo  $q$ . Then we have the identities*

$$\sum_{\chi \pmod{q}}^* \chi(r) = \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right),$$

where  $\sum_{\chi \pmod{q}}^*$  denotes the summation over all primitive characters modulo  $q$  and  $J(q)$  denotes the number of primitive characters modulo  $q$ .

*Proof.* This is Lemma 3 of [3].  $\square$

LEMMA 6. For any positive integers  $r$  and  $q \geq 3$  with  $l_r = (r, \phi(q)) > 1$ , let  $q = uv$ , where  $(u, v) = 1$ ,  $u$  be a square-full number or  $u = 1$ ,  $v$  be a square-free number,  $\chi_1$  be a  $l_r$ th-order character modulo  $q$ . For any positive integers  $t_1, t_2, \dots, t_{k+1}$ , if we let

$$\begin{aligned} \Phi &= \sum_{d|v} \sum_{d_1| \frac{v}{d}} \cdots \sum_{d_{k+1}| \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_k) \mu\left(\frac{v}{dd_{k+1}}\right)}{d_1^{t_1} \cdots d_{k+1}^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_{k+1}}\right)} \\ &\quad \times \sum_{\substack{\chi \pmod{ud} \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_k) \chi\left(\frac{v}{dd_{k+1}}\right) \\ &\quad \times L(t_1, \bar{\chi}) \cdots L(t_{k+1}, \bar{\chi}) \sum_{a=1}^{ud} \chi(a) e\left(\frac{a}{ud}\right) \\ &\quad \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum_{m=1}^{ud} \chi_1^i(lud+m) \bar{\chi}(m) e\left(\frac{lud+m}{q}\right), \end{aligned}$$

then we have the following estimate

$$\Phi \ll l_r q^{\frac{1}{2}+\epsilon}.$$

*Proof.* Without loss of generality we can assume  $t = \min(t_1, t_2, \dots, t_{k+1})$ . We now let

$$r_{k+1}(n) = \sum_{d_1 d_2 \cdots d_{k+1} = n} d_1^{t-t_1} d_2^{t-t_2} \cdots d_{k+1}^{t-t_{k+1}}.$$

Then for any parameter  $N \geq ud$  and non-principal character  $\chi$  modulo  $ud$ , applying Abel's identity we have

$$\begin{aligned} L(t_1, \bar{\chi}) \cdots L(t_{k+1}, \bar{\chi}) &= \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) r_{k+1}(n)}{n^t} \\ &= \sum_{1 \leq n \leq N} \frac{\bar{\chi}(n) r_{k+1}(n)}{n^t} + t \int_N^{\infty} \frac{A(y, \bar{\chi})}{y^{t+1}} dy, \end{aligned}$$

where  $A(y, \bar{\chi}) = \sum_{N < n \leq y} \bar{\chi}(n) r_{k+1}(n)$ .

Note that by using induction, we can prove the following estimate

$$\sum_{\substack{\chi \text{ mod } ud \\ \chi(-1)=-1}} |A(y, \bar{\chi})| \ll y^{1-(2/2^{k+1})+\epsilon} \phi^{3/2}(ud)$$

(see [12]).

Hence we have

$$\begin{aligned} & \sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{k+1}|\frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_k) \mu(\frac{v}{dd_{k+1}})}{d_1^{t_1} \cdots d_{k+1}^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_{k+1}}\right)} \\ & \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum_{m=1}^{ud} \chi_1^i(lud+m) \\ & \times e\left(\frac{lud+m}{q}\right) \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right) \\ & \times \sum_{\substack{\chi \text{ mod } ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_k m) \chi(d_{k+1} am) t \int_N^\infty \frac{A(y, \bar{\chi})}{y^{t+1}} dy \\ & \ll l_r q^{1+\epsilon} \int_N^\infty \frac{t}{y^{t+1}} \left( \sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=-1}} |A(y, \bar{\chi})| \right) dy \\ & \ll l_r q^{1+\epsilon} \int_N^\infty \frac{y^{1-\frac{1}{2^k}+\epsilon} \phi^{3/2}(q)}{y^{t+1}} dy \\ & \ll \frac{l_r q^{\frac{5}{2}+\epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}}. \end{aligned}$$

Combining the above we have

$$\begin{aligned} \Phi &= \sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{k+1}|\frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_k) \mu(\frac{v}{dd_{k+1}})}{d_1^{t_1} \cdots d_{k+1}^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_{k+1}}\right)} \\ & \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right) \sum_{1 \leq n \leq N} \frac{r_{k+1}(n)}{n^t} \\
& \times \sum_{\substack{\chi \text{ mod } ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_k nm) \chi(d_{k+1} am) \\
(6) \quad & + O\left(\frac{l_r q^{\frac{5}{2}+\epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}}\right).
\end{aligned}$$

Note that for any integer  $a$  with  $(a, q) = 1$ , from Lemma 5 we have

$$\begin{aligned}
\sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=-1}}^* \chi(a) &= \frac{1}{2} \sum_{\chi \text{ mod } q}^* (1 - \chi(-1)) \chi(a) \\
&= \frac{1}{2} \sum_{\chi \text{ mod } q}^* \chi(a) - \frac{1}{2} \sum_{\chi \text{ mod } q}^* \chi(-a) \\
&= \frac{1}{2} \sum_{u|(q,a-1)} \mu\left(\frac{q}{u}\right) \phi(u) - \frac{1}{2} \sum_{u|(q,a+1)} \mu\left(\frac{q}{u}\right) \phi(u).
\end{aligned}$$

Therefore by formula (6) and  $\sum_{d|n} f(d) = \sum_{d|n} f\left(\frac{n}{d}\right)$  we have

$$\begin{aligned}
\Phi &= \sum_{d|v} \sum_{d_1 \mid \frac{v}{d}} \\
&\times \cdots \times \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{d_1^{t_1} \cdots d_k^{t_k} (\frac{v}{dd_{k+1}})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
&\times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right) \\
&\times \sum_{1 \leq n \leq N} \frac{r_{k+1}(n)}{n^t} \sum_{\substack{\chi \text{ mod } ud \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_k nm) \chi(d_{k+1} am) \\
&+ O\left(\frac{l_r q^{\frac{5}{2}+\epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{d|v} \sum_{d_1|\frac{v}{d}} \\
&\quad \times \cdots \times \cdots \sum_{d_{k+1}|\frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{d_1^{t_1} \cdots d_k^{t_k} (\frac{v}{dd_{k+1}})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
&\quad \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right) \\
&\quad \times \sum_{\substack{1 \leq n \leq N \\ (n, ud)=1}} \frac{r_{k+1}(n)}{n^t} \sum_{\substack{s|ud \\ \frac{d_1 \cdots d_k n}{d_{k+1} a} \equiv 1(s)}} \mu\left(\frac{ud}{s}\right) \phi(s) \\
&\quad - \frac{1}{2} \sum_{d|v} \sum_{d_1|\frac{v}{d}} \\
&\quad \times \cdots \times \sum_{d_{k+1}|\frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{d_1^{t_1} \cdots d_k^{t_k} (\frac{v}{dd_{k+1}})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
&\quad \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \sum_{a=1}^{ud} e\left(\frac{am}{ud}\right) \\
&\quad \times \sum_{\substack{1 \leq n \leq N \\ (n, ud)=1}} \frac{r_{k+1}(n)}{n^t} \sum_{\substack{s|ud \\ \frac{d_1 \cdots d_k n}{d_{k+1} a} \equiv -1(s)}} \mu\left(\frac{ud}{s}\right) \phi(s) \\
&\quad + O\left(\frac{l_r q^{\frac{5}{2}+\epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}}\right) \\
&= \frac{1}{2} \sum_{d|v} \sum_{d_1|\frac{v}{d}} \\
&\quad \times \cdots \times \sum_{d_{k+1}|\frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{d_1^{t_1} \cdots d_k^{t_k} (\frac{v}{dd_{k+1}})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
&\quad \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \sum_{\substack{a=1 \\ d_1 \cdots d_k | d_{k+1} a}}^{ud} e\left(\frac{am}{ud}\right) \\
&\quad \times \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \sum_{l=0}^{\frac{d_1 \cdots d_k N}{d_{k+1} a s}} \frac{r_{k+1}\left(\frac{(ls+1)ad_{k+1}}{d_1 \cdots d_k}\right)}{\left(\frac{(ls+1)ad_{k+1}}{d_1 \cdots d_k}\right)^t} - \frac{1}{2} \sum_{d|v} \sum_{d_1|\frac{v}{d}}
\end{aligned}$$

$$\begin{aligned}
& \times \cdots \times \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{d_1^{t_1} \cdots d_k^{t_k} (\frac{v}{dd_{k+1}})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
& \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \sum'_{a=1}^{ud}_{d_1 \cdots d_k \mid d_{k+1}a} e\left(\frac{am}{ud}\right) \\
& \times \sum_{s \mid ud} \mu\left(\frac{ud}{s}\right) \phi(s) \sum_{l=0}^{\frac{d_1 \cdots d_k N}{d_{k+1} a^s}} r_{k+1}\left(\frac{(ls-1)ad_{k+1}}{d_1 \cdots d_k}\right) \\
& + O\left(\frac{l_r q^{\frac{5}{2}+\epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}}\right).
\end{aligned}$$

From the definition of  $t$ , we know that all the terms in the above have the highest order only in the case that  $t=t_1=\cdots=t_{k+1}$ , so we have

$$\begin{aligned}
\Phi & \ll \sum_{d \mid v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{(\frac{v}{d})^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
& \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) e\left(\frac{lud+m}{q}\right) \\
& \times \sum'_{a=1}^{ud}_{d_1 \cdots d_k \mid d_{k+1}a} e\left(\frac{lud+m}{q}\right) \frac{r_{k+1}\left(\frac{d_{k+1}a}{d_1 \cdots d_k}\right)}{a^{t_{k+1}}} \sum_{s \mid ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
& + O\left(\sum_{d \mid v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k}{\phi^k(q)\phi(ud)} \cdot l_r \cdot q \right. \\
& \quad \left. \times \sum'_{a=1}^{ud} \frac{1}{a^{t_{k+1}}} \cdot \sum_{s \mid ud} \phi(s) \sum_{l=1}^{\frac{d_1 \cdots d_k N}{d_{k+1} a^s}} \frac{N^{\epsilon_1}}{(ls+1)^{t_{k+1}}} \right) \\
& + O\left(\sum_{d \mid v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k}{\phi^k(q)\phi(ud)} \cdot l_r \cdot q \right. \\
& \quad \left. \times \sum'_{a=1}^{ud} \frac{1}{a^{t_{k+1}}} \cdot \sum_{s \mid ud} \phi(s) \sum_{l=1}^{\frac{d_1 \cdots d_k N}{d_{k+1} a^s}} \frac{N^{\epsilon_1}}{(ls-1)^{t_{k+1}}} \right)
\end{aligned}$$

$$\begin{aligned}
& + O \left( \frac{l_r q^{\frac{5}{2} + \epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}} \right) \\
& \ll \sum_{d|v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{\left(\frac{v}{d}\right)^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
& \times \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum'_{m=1}^{ud} \chi_1^i(lud+m) \\
& \times e\left(\frac{lud+m}{q}\right) \sum_{\substack{a=1 \\ d_1 \cdots d_k \mid d_{k+1}a}}^{ud'} e\left(\frac{\frac{q}{ud} \cdot a(lud+m)}{q}\right) \\
& \times \frac{r_{k+1}(d_{k+1}a/d_1 \cdots d_k)}{a^{t_{k+1}}} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) \\
& + O \left( \frac{l_r q^{\frac{5}{2} + \epsilon}}{(\sqrt{N})^{2t+2^{1-k}-2}} \right) + O(l_r N^{\epsilon_1}),
\end{aligned}$$

where we have used the estimate  $r_{k+1}(n) \ll n^{\epsilon_1}$ .

Now taking  $N = q^{5 \times 2^{k-1}}$  in the above, and note the identity  $J(u) = \phi^2(u)/u$  if  $u$  is a square-full number, we can immediately obtain the following

$$\begin{aligned}
\Phi & \ll \sum_{d|v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_{k+1})}{\left(\frac{v}{d}\right)^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
& \times \sum_{i=1}^{l_r-1} \sum_{b=1}^q \chi_1^i(b) e\left(\frac{b}{q}\right) \sum_{\substack{a=1 \\ d_1 \cdots d_k \mid d_{k+1}a}}^{ud'} e\left(\frac{\frac{av}{d}b}{q}\right) \frac{r_{k+1}(d_{k+1}a/d_1 \cdots d_k)}{a^{t_{k+1}}} \\
& \times \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) + O(l_r q^\epsilon) \\
& \ll \sum_{d|v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k J(ud) \mu(d_1) \cdots \mu(d_{k+1})}{\left(\frac{v}{d}\right)^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_k}\right) \phi\left(\frac{qdd_{k+1}}{v}\right)} \\
& \times \sum_{\substack{a=1 \\ d_1 \cdots d_k \mid d_{k+1}a}}^{ud'} \frac{r_{k+1}(d_{k+1}a/d_1 \cdots d_k)}{a^{t_{k+1}}} \sum_{i=1}^{l_r-1} G(1 + av/d, \chi_1^i) + O(l_r q^\epsilon)
\end{aligned}$$

$$=: \Psi + O(l_r q^\epsilon).$$

Since  $\chi_1$  is a  $l_r$ -th-order character modulo  $q$  and  $i < l_r$ ,  $\chi_1^i$  can not be principal character modulo  $q$ . From the properties of Gauss sums we get

$$|G(1 + av/d, \chi_1^i)| \ll (1 + av/d, q) q^{\frac{1}{2} + \epsilon}.$$

So we have

$$\begin{aligned} \Psi &\ll \sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{k+1}|\frac{v}{d}} \frac{(ud)^{k+1}}{\phi^{k+1}(ud)} \cdot l_r q^{\frac{1}{2} + \epsilon} \cdot \sum_{a=1}^{ud} \frac{(1 + av/d, q)}{(av/d)^{t_{k+1}}} \\ &\ll \sum_{d|v} l_r q^{\frac{1}{2} + \epsilon} \sum_{s|q} \sum_{\substack{a=1 \\ s|1+av/d}}^{ud} \frac{s}{(av/d)^{t_{k+1}}} \\ &\ll l_r q^{\frac{1}{2} + \epsilon} \sum_{s|q} \sum_{1 \leq l \leq \frac{q+1}{s}} \frac{s}{(ls - 1)^{t_{k+1}}} \ll l_r q^{\frac{1}{2} + \epsilon}. \end{aligned}$$

This completes the proof of Lemma 6.  $\square$

### 3. Proof of the theorem

In this section, we shall complete the proof of the theorem. For any positive integers  $r$  and  $q \geq 3$  with  $l_r = (r, \phi(q)) > 1$ , let  $\chi_1$  be a  $l_r$ -th-order character modulo  $q$ . Then from Lemma 4 we have

$$\begin{aligned} (7) \quad &\sum_{h=1}^q \bar{\chi}(h) Kl(h, k+1, r; q) \\ &= \sum_{a_1=1}^q \sum_{a_2=1}^q \cdots \sum_{a_k=1}^q \sum_{h=1}^q \bar{\chi}(h) e\left(\frac{a_1^r + a_2^r + \cdots + a_k^r + h \cdot \bar{a}_1^r \bar{a}_2^r \cdots \bar{a}_k^r}{q}\right) \\ &= \tau(\bar{\chi}) \left( \tau(\bar{\chi}) + \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i) \right)^k. \end{aligned}$$

Hence from Lemma 3 and 4 we have

$$\begin{aligned}
& \sum_{h=1}^q' Kl(h, k+1, r; q) C(h, q; m, k) \\
&= \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \left( \sum_{h=1}^q' \bar{\chi}(h) Kl(h, k+1, r; q) \right) \\
&\quad \times \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \times \cdots \times \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\} \\
&= \sum_{h=1}^q Kl(h, k+1; q) C(h, q; m, k) + \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \\
&\quad \times \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} C_k^1 \cdot \tau^k(\bar{\chi}) \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i) \\
&\quad \times \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \times \cdots \times \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\} \\
&\quad + \frac{(-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \\
&\quad \times \sum_{j=2}^k \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} C_k^j \cdot \tau^{k-j+1}(\bar{\chi}) \left( \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i) \right)^j \\
&\quad \times \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \times \cdots \times \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\} \\
&\equiv \sum_{h=1}^q Kl(h, k+1; q) C(h, q; m, k) + E_1 + E_2,
\end{aligned} \tag{8}$$

where  $C_k^j = \frac{k!}{(k-j)!j!}$ .

We now let  $q = uv$ , where  $(u, v) = 1$ ,  $u$  be a square-full number or  $u = 1$ ,  $v$  be a square-free number. Note that  $\chi^*(\frac{q}{m}) \mu(\frac{q}{m}) \neq 0$  if and

only if  $m = ud$ , where  $d \mid v$ . Then from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
E_1 &= \frac{k \cdot (-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi(-1)=-1}} \tau^k(\bar{\chi}) \sum_{i=1}^{l_r-1} \tau(\bar{\chi} \chi_1^i) \\
&\quad \times \left\{ \sum_{r_1=1}^{\infty} \frac{G(r_1, \chi)}{r_1^{m_1}} \right\} \times \cdots \times \left\{ \sum_{r_{k+1}=1}^{\infty} \frac{G(r_{k+1}, \chi)}{r_{k+1}^{m_{k+1}}} \right\} \\
&= \frac{k \cdot (-2)^{k+1} m_1! \cdots m_{k+1}!}{(2\pi i)^{m_1+\cdots+m_{k+1}} \phi(q)} \sum_{d \mid v} \sum_{\substack{\chi \pmod{ud} \\ \chi(-1)=-1}}^* \\
&\quad \left\{ \sum_{d_1 \mid \frac{v}{d}} \frac{\phi(q) \chi\left(\frac{v}{dd_1}\right) \mu\left(\frac{v}{dd_1}\right) \tau(\chi) L(m_1, \bar{\chi})}{d_1^{m_1} \phi\left(\frac{q}{d_1}\right)} \right\} \\
&\quad \times \cdots \times \left\{ \sum_{d_{k+1} \mid \frac{v}{d}} \frac{\phi(q) \chi\left(\frac{v}{dd_{k+1}}\right) \mu\left(\frac{v}{dd_{k+1}}\right) \tau(\chi) L(m_{k+1}, \bar{\chi})}{d_{k+1}^{m_{k+1}} \phi\left(\frac{q}{d_{k+1}}\right)} \right\} \\
&\quad \times \bar{\chi}^k\left(\frac{v}{d}\right) \mu^k\left(\frac{v}{d}\right) \tau^k(\bar{\chi}) \\
&= \frac{k \cdot (-2)^{k+1} m_1! \cdots m_{k+1}! \phi^k(q)}{(2\pi i)^{m_1+\cdots+m_{k+1}}} \\
&\quad \times \sum_{d \mid v} \sum_{d_1 \mid \frac{v}{d}} \cdots \sum_{d_{k+1} \mid \frac{v}{d}} \frac{u^k d^k \mu(d_1) \cdots \mu(d_k) \mu\left(\frac{v}{dd_{k+1}}\right)}{d_1^{t_1} \cdots d_{k+1}^{t_{k+1}} \phi\left(\frac{q}{d_1}\right) \cdots \phi\left(\frac{q}{d_{k+1}}\right)} \\
&\quad \times \sum_{\substack{\chi \pmod{ud} \\ \chi(-1)=-1}}^* \bar{\chi}(d_1 \cdots d_k) \chi\left(\frac{v}{dd_{k+1}}\right) L(t_1, \bar{\chi}) \cdots L(t_{k+1}, \bar{\chi}) \\
&\quad \times \sum_{a=1}^{ud} \chi(a) e\left(\frac{a}{ud}\right) \sum_{i=1}^{l_r-1} \sum_{l=0}^{q/ud-1} \sum_{m=1}^{ud} \chi_1^i(lud + m) \bar{\chi}(m) e\left(\frac{lud + m}{q}\right),
\end{aligned}$$

where we have used the identities that

$$\bar{\chi}\left(\frac{v}{d}\right) = \bar{\chi}\left(\frac{v}{dd_1}\right) \bar{\chi}(d_1), \quad \mu\left(\frac{v}{d}\right) = \mu\left(\frac{v}{dd_1}\right) \mu(d_1),$$

and

$$\tau(\chi^*) \tau(\bar{\chi}^*) = -m,$$

where  $\chi^*$  is a primitive character modulo  $m$  and  $\chi(-1) = -1$ . So from Lemma 6 we have the estimate

$$(9) \quad E_1 \ll l_r q^{k+\frac{1}{2}+\epsilon}.$$

Using the similar method of proving Lemma 6, we may obtain the following estimate

$$(10) \quad E_2 \ll l_r q^{k+\frac{1}{2}+\epsilon}.$$

Therefore from (5), (8), (9) and (10), we may immediately obtain

$$\begin{aligned} & \sum_{h=1}^q' K l(h, k+1, r; q) C(h, q; m, k) \\ &= \sum_{h=1}^q' K l(h, k+1; q) C(h, q; m, k) + O\left(l_r q^{k+\frac{1}{2}+\epsilon}\right) \\ &= \frac{(-1)^{k+1} 2^k m_1! \cdots m_{k+1}! \phi(q) q^k}{(2\pi i)^{m_1+\cdots+m_{k+1}}} \prod_{p \parallel q} \left(1 - \frac{p^k - 1}{p^k(p-1)^2}\right) \\ & \quad + O\left(l_r q^{k+\frac{1}{2}+\epsilon}\right). \end{aligned}$$

This completes the proof of the theorem.

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