A NEW BIHARMONIC KERNEL
FOR THE UPPER HALF PLANE

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ABSTRACT. We introduce a new biharmonic kernel for the upper half plane, and then study the properties of its relevant potentials, such as the convergence in the mean and the boundary behavior. Among other things, we shall see that Fatou's theorem is valid for these potentials, so that the biharmonic Poisson kernel resembles the usual Poisson kernel for the upper half plane.

1. Notation

Let $\mathbb{D}$ denote the unit disk and $\mathbb{T} = \partial\mathbb{D}$ denote its boundary in the complex plane $\mathbb{C}$. The upper half plane will be denoted by

$$\mathbb{C}_+ = \{x + iy \in \mathbb{C} : \quad y > 0\}.$$

Let Δ stand for the Laplace operator

$$\Delta = \Delta_z = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad z = x + iy$$

in the complex plane. The biharmonic Green function $\Gamma(z, \zeta)$ for the operator Δ^2 in the unit disk is the function

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log\left|\frac{z - \zeta}{1 - \bar\zeta z}\right|^2 + \left(1 - |z|^2\right)\left(1 - |\zeta|^2\right), \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

This function solves, for fixed $\zeta \in \mathbb{D}$, the boundary value problem

$$
\begin{aligned}
\Delta_z^2 \Gamma(z, \zeta) &= \delta_\zeta(z), & z \in \mathbb{D}, \\
\Gamma(z, \zeta) &= 0, & z \in \mathbb{T}, \\
\partial_{n(z)}\Gamma(z, \zeta) &= 0, & z \in \mathbb{T},
\end{aligned}
$$

where $\partial_{n(z)}$ stands for the inward normal derivative with respect to the variable $z \in \mathbb{T}$, and δ_ζ denotes the Dirac distribution concentrated at the point $\zeta \in \mathbb{D}$. For details see [2].

2. Introduction

A real-valued function u defined on an open subset of the complex plane is said to be *biharmonic* if $\Delta^2 u = 0$. We first introduce a biharmonic kernel function for $\mathbb{C}_+$. Then we shall make use of this kernel function to produce a class of biharmonic functions in the upper half plane. The main objective here is to study this family of biharmonic functions in more detail. Among other things, we shall see that Fatou's theorem concerning the almost everywhere existence of nontangential limits is valid, so that the biharmonic (Poisson) kernel resembles the usual one.

We now proceed to give an overview of the origin of the biharmonic Poisson kernel for the unit disk. This suggests a direct method of calculating the desired kernel function for the upper half plane.

Let u be a C^∞-smooth function in a neighborhood of the closed unit disk. Using Green's formula twice we obtain

$$
\begin{aligned}
u(z) \quad = \quad & \int_{\mathbb{D}} \Gamma(z,\zeta) \Delta^2 u(\zeta)\, dA(\zeta) \\
& -\frac{1}{2} \int_{\mathbb{T}} \partial_{n(\zeta)} \left(\Delta_\zeta \Gamma(z,\zeta) \right) u(\zeta)\, d\sigma(\zeta) \\
& +\frac{1}{2} \int_{\mathbb{T}} \Delta_\zeta \Gamma(z,\zeta) \partial_{n(\zeta)} u(\zeta)\, d\sigma(\zeta),
\end{aligned}
\tag{1}
$$

where $dA(\zeta)$ denotes the normalized area measure on the unit disk, and $d\sigma(\zeta)$ stands for the normalized arc-lengh measure on the unit circle. A computation shows that

$$
\Delta_\zeta \Gamma(z,\zeta) = G(z,\zeta) + H(z,\zeta), \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D},
$$

where $G(z,\zeta)$ is the Green function for the Laplace operator in the unit disk;

$$
G(z,\zeta) = \log \left| \frac{z-\zeta}{1-\bar{\zeta}z} \right|^2, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D},
$$

and the second term is given by

$$
H(z,\zeta) = \left(1 - |z|^2\right) \frac{1 - |\zeta z|^2}{|1 - \bar{\zeta}z|^2}, \qquad (z,\zeta) \in \mathbb{D} \times \overline{\mathbb{D}}.
$$

Moreover, another computation shows that

$$(2) \qquad F(z,\zeta) = -\frac{1}{2}\partial_{n(\zeta)}\Delta_\zeta\Gamma(z,\zeta)$$

$$= \frac{1}{2}\left\{\frac{(1-|z|^2)^2}{|z-\zeta|^2} + \frac{(1-|z|^2)^3}{|z-\zeta|^4}\right\}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{T}.$$

For a possibly non-smooth function u satisfying some growth conditions, the author and Hedenmalm [1] succeeded to find a Riesz-type representation formula in terms of the functions $H(z,\zeta)$ and $F(z,\zeta)$. More precisely, the formula (1) generalizes to

$$(3) \qquad u(z) = \int_{\mathbb{D}}\Gamma(z,\zeta)\,d\mu(\zeta) + \int_{\mathbb{T}}H(z,\zeta)\,d\lambda(\zeta)$$

$$+ \int_{\mathbb{T}}F(z,\zeta)\,d\nu(\zeta), \qquad z \in \mathbb{D},$$

where μ is a positive Borel measure on the unit disk, and ν and λ are two real-valued Borel measures on the unit circle.

For fixed $\zeta \in \mathbb{T}$, the function $F(z,\zeta)$ defined by (2) is biharmonic; in the sense that it satisfies the equation

$$\Delta_z^2 F(z,\zeta) = 0, \qquad z \in \mathbb{D}.$$

The function $F(z,\zeta)$ is known as the *biharmonic Poisson kernel* for the unit disk. In this article we intend to find the upper half plane analog of $F(z,\zeta)$, and then manage to prove the upper half plane version of Fatou's theorem. This generalizes the classical Fatou's theorem valid for the (harmonic) functions defined by the usual Poisson kernel (see for instance [3]) to biharmonic functions defined by the biharmonic Poisson kernel.

3. The biharmonic Poisson kernel for the upper half plane

Given the usual Poisson kernel for the unit disk, it is easy to find the Poisson kernel for the upper half plane (or any other simply connected region). What we need is a Moebius transformation which maps the given region onto the unit disk, then a change of variables does the job. Unfortunately, the biharmonic functions are not preserved under Moebius transformations; therefore this method does not work. Instead, we have to appeal to a direct computation of the desired kernel function.

We first recall (see for instance [4]) the biharmonic Green function for the upper half plane; this is the function

$$U(z,\zeta) = |z-\zeta|^2 \log \left| \frac{z-\zeta}{z-\bar{\zeta}} \right|^2 + 4\,\mathrm{Im}(z)\,\mathrm{Im}(\zeta), \qquad (z,\zeta) \in \mathbb{C}_+ \times \mathbb{C}_+,$$

which solves, for fixed $\zeta \in \mathbb{C}_+$, the boundary value problem

$$\begin{aligned}
\Delta_z^2 U(z,\zeta) &= \delta_\zeta(z), & z &\in \mathbb{C}_+, \\
U(z,\zeta) &= 0, & z &\in \mathbb{R}, \\
\partial_{n(z)} U(z,\zeta) &= 0, & z &\in \mathbb{R}.
\end{aligned}$$

We start by the following lemma:

LEMMA 3.1. *For fixed $z \in \mathbb{C}_+$ we have*

$$\Delta_\zeta U(z,\zeta) = \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|^2 + 2\,\mathrm{Re}\left(\frac{z - \bar{z}}{\zeta - \bar{z}} \right), \qquad \zeta \in \mathbb{C}_+.$$

Proof. Since $U(z,\zeta)$ is symmetric, we can write

$$U(z,\zeta) = |\zeta - z|^2 \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|^2 + 4\,\mathrm{Im}(\zeta)\,\mathrm{Im}(z), \qquad (z,\zeta) \in \mathbb{C}_+ \times \mathbb{C}_+.$$

It is enough to compute the Laplacian of the first term, since

$$\Delta_\zeta \Big((\mathrm{Im}(z)\,\mathrm{Im}(\zeta) \Big) = 0.$$

Writing

$$|\zeta - z|^2 \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|^2 = (\zeta - z)(\bar{\zeta} - \bar{z}) \log \frac{(\zeta - z)(\bar{\zeta} - \bar{z})}{(\zeta - \bar{z})(\bar{\zeta} - z)},$$

we see that

$$\frac{\partial}{\partial \zeta} \left(|\zeta - z|^2 \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|^2 \right) = (\bar{\zeta} - \bar{z}) \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|^2 + (\bar{\zeta} - \bar{z}) \frac{z - \bar{z}}{\zeta - \bar{z}}.$$

Applying the differential operator $\frac{\partial}{\partial \bar{\zeta}}$ to the expression above, we end up with the desired result. $\qquad \square$

From now on, we adhere to the following convention. For z and ζ in the upper half plane we write

$$\begin{aligned}
z &= x + iy, & y &> 0, \\
\zeta &= t + is, & s &> 0.
\end{aligned}$$

It now follows from Lemma (3.1) that

$$\Delta_\zeta U(z,\zeta) = \log\left(\frac{(t-x)^2 + (s-y)^2}{(t-x)^2 + (s+y)^2}\right) + \frac{4y(s+y)}{(t-x)^2 + (s+y)^2},$$

and by differentiating with respect to s we obtain

$$\frac{\partial}{\partial s}\Delta_\zeta U(z,\zeta) = \frac{2(s-y)}{(t-x)^2 + (s-y)^2} - \frac{2(s+y)}{(t-x)^2 + (s+y)^2}$$
$$+ 4y\frac{(t-x)^2 - (s+y)^2}{((t-x)^2 + (s+y)^2)^2}.$$

To get the outward normal derivative of $\Delta_\zeta U(z,\zeta)$ on the boundary of the upper half plane, $\mathbb{R}$, it suffices to put $s = 0$ in the above expression. Motivated by the biharmonic Poisson kernel of the upper half plane given by the equation (2), we shall make the following definition.

$$(4)\quad F(z,t) = -\frac{1}{4\pi}\frac{\partial}{\partial s}\Delta_\zeta U(z,\zeta)\bigg|_{s=0}$$
$$= \frac{1}{\pi}\left\{\frac{y}{y^2 + (x-t)^2} + y\frac{y^2 - (x-t)^2}{((y^2 + (x-t)^2)^2}\right\}, \quad z \in \mathbb{C}_+, \quad t \in \mathbb{R}.$$

In the following lemma, we shall see that for fixed $t \in \mathbb{R}$, the function $F(z,t)$ is biharmonic in its first variable.

LEMMA 3.2. *For fixed $t \in \mathbb{R}$, we have*

$$\Delta_z^2 F(z,t) = 0, \qquad z \in \mathbb{C}_+.$$

Proof. It is well-known that

$$\frac{y}{(x-t)^2 + y^2} = \text{Im}\left(\frac{1}{t-z}\right) \qquad z \in \mathbb{C}_+$$

is the usual (harmonic) Poisson kernel for the upper half plane. In particular, it is biharmonic. What remains is to verify that

$$K(x,y) = \frac{y^3 - y(x-t)^2}{((x-t)^2 + y^2))^2}$$

is biharmonic. A direct computation shows that the expression

$$\Delta^2 K(x,y) = \frac{\partial^4}{\partial x^4}K(x,y) + 2\frac{\partial^4}{\partial x^2 \partial y^2}K(x,y) + \frac{\partial^4}{\partial y^4}K(x,y)$$

vanishes identically for $z = x + iy$ in the upper half plane. $\square$

We shall refer to $F(z,t)$ defined by (4) as the *biharmonic Poisson kernel for the upper half plane*. Note that $F(z,t)$ can be written in the form

$$F(z,t) = F(x+iy,t) = F_y(x-t), \qquad z \in \mathbb{C}_+, \quad t \in \mathbb{R},$$

where

$$F_y(t) = \frac{1}{\pi} \left\{ \frac{y}{y^2+t^2} + y \frac{y^2-t^2}{(y^2+t^2)^2} \right\}, \qquad t \in \mathbb{R}.$$

We proceed the study of our kernel function by the following lemma.

LEMMA 3.3. *The integral of the biharmonic kernel function over the real line is 1;*

$$\int_{\mathbb{R}} F_y(t)\, dt = 1, \qquad y > 0.$$

Proof. It is easy to see that for $y > 0$ we have

$$\int_{-\infty}^{\infty} \frac{y^3 - yt^2}{(y^2+t^2)^2}\, dt = \left[\frac{yt}{y^2+t^2} \right]_{t=-\infty}^{t=\infty} = 0.$$

It now follows from the definition of $F_y(t)$ that

$$\int_{\mathbb{R}} F_y(t)\, dt = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{y^2+t^2}\, dt = \frac{1}{\pi} \left[\arctan\left(\frac{t}{y} \right) \right]_{t=-\infty}^{t=\infty} = 1.$$

$\square$

LEMMA 3.4. *For fixed $y > 0$, F_y is a positive and even function on $\mathbb{R}$ which is decreasing for $0 \le t < \infty$.*

Proof. It is clear that $F_y(t) = F_y(-t)$, that is F_y is an even function on $\mathbb{R}$. A computation shows that

$$\frac{d}{dt} F_y(t) = -\frac{8y^3 t}{\pi(y^2+t^2)^3} \le 0$$

for $y > 0$ and $t \ge 0$. Therefore F_y is decreasing on the interval $0 \le t < \infty$. Hence its maximum value is attained for $t = 0$, that is,

$$F_y(t) \le F_y(0) = \frac{2}{\pi y}, \qquad t > 0.$$

Since F_y is strictly decreasing on $0 < t < \infty$, and $F_y(t) \to 0$ as $t \to \infty$, we conclude that F_y is positive. $\square$

LEMMA 3.5. *For every $\delta > 0$ we have*
(a) $\sup_{|t|>\delta} F_y(t) \to 0$, as $y \to 0$,
(b) $\int_{|t|>\delta} F_y(t)\, dt \to 0$, $y \to 0$.

Proof. It follows from Lemma 3.4 that

$$\sup_{|t|>\delta} F_y(t) = F_y(\delta) = \frac{y}{\delta^2 + y^2} + \frac{y^3 - y\delta^2}{(\delta^2 + y^2)^2} \to 0, \quad y \to 0.$$

This proves part (a). Part (b) follows from the fact that

$$\int_\delta^\infty F_y(t)\,dt \to 0, \quad y \to 0,$$

which in turn is a consequence of the monotonicity of F_y. □

Lemmas 3.3–3.5 are the typical properties of an approximate identity (similar to the ones that the usual Poisson kernel and other such identities possess). Thus, it is natural to expect that the approximation properties like Proposition 4.1 and Theorem 4.2 should follow in the usual way. We shall see that this is indeed the case.

4. The convergence problem

For a function $f \in L^1(\mathbb{R})$ we define for every $z = x + iy$ in the upper half plane,

$$u(z) = F[f](z) = \int_{\mathbb{R}} F_z(t)f(t)\,dt = \int_{\mathbb{R}} F_y(x-t)f(t)\,dt = \big(F_y * f\big)(x).$$

Note that for fixed $y > 0$ we have

$$\|F_y\|_{L^\infty(\mathbb{R})} = \sup_{t \in \mathbb{R}} F_y(t) = \frac{2}{\pi y} < \infty.$$

Also we note that

$$\left| \frac{y^3 - yt^2}{(y^2 + t^2)^2} \right| = O\left(\frac{1}{1+t^2} \right), \quad t \to \infty.$$

It follows that $F_y \in L^q(\mathbb{R})$, $q < 1 < \infty$, so that for $f \in L^p(\mathbb{R})$ the convolution $F_y * f \in L^1(\mathbb{R})$, meaning that $u(z)$ is well-defined.

PROPOSITION 4.1. *Let $f \in L^p(\mathbb{R})$ for $1 \le p \le \infty$. Assume that f is continuous at a point $x_0 \in \mathbb{R}$ and $u = F[f]$. Then*

$$\lim_{(x,y) \to x_0} u(x,y) = f(x_0).$$

Proof. Recall that

$$u(z) = u(x, y) = \int_{\mathbb{R}} F_y(t) f(x - t) \, dt.$$

It follows from Lemma 3.3 that

$$u(z) - f(x_0) = \int_{\mathbb{R}} F_y(t) \left(f(x - t) - f(x_0) \right) dt.$$

Let $\epsilon > 0$ be given. By the continuity of f at x_0, we can find $\delta > 0$ such that

$$\int_{|t| < \delta} F_y(t) \Big| f(x - t) - f(x_0) \Big| \, dt < \frac{\epsilon}{2}.$$

According to Lemma 3.5, for a fixed δ, the function $F_y(t)$ converges uniformly to zero as y approaches to zero, so that

$$\int_{|t| \geq \delta} F_y(t) \left| f(x - t) - f(x_0) \right| \, dt < \frac{\epsilon}{2}.$$

Therefore, separating the integral over $\mathbb{R}$ to $\int_{|t| \leq \delta}$ and $\int_{|t| > \delta}$ the result follows.   □

THEOREM 4.2. (a) *Let* $1 \leq p < \infty$, *and* $f \in L^p(\mathbb{R})$. *Then*

$$\|F_y * f - f\|_{L^p(\mathbb{R})} \to 0, \qquad y \to 0.$$

(b) *If* $f \in L^\infty(\mathbb{R})$, *then* $F_y * f \to f$ *weak-star, as* $y \to 0$.

(c) *If* f *is bounded and uniformly continuous on* $\mathbb{R}$, *then* $F_y * f \to f$ *uniformly as* $y \to 0$.

Proof. We note that

$$(F_y * f)(x) - f(x) = \int_{\mathbb{R}} F_y(t) \left(f(x - t) - f(x) \right) dt.$$

Therefore, by the Minkowski's inequality for integrals we obtain

$$\|F_y * f - f\|_{L^p(\mathbb{R})} \leq \int_{\mathbb{R}} F_y(t) \|f_t(x) - f(x)\|_{L^p(dx)} \, dt,$$

where $f_t(x) = f(t - x)$. Let $\delta > 0$ be suitably chosen, we then write

$$\|F_y * f - f\|_{L^p(\mathbb{R})} \leq \int_{|t| \leq \delta} F_y(t) \|f_t - f\|_{L^p(dx)} dt + \int_{|t| > \delta} F_y(t) \|f_t - f\|_{L^p(dx)} dt.$$

Since translations are continuous in $L^p(\mathbb{R})$, it follows that

$$\|f_t - f\|_{L^p(\mathbb{R})} \to 0, \qquad t \to 0,$$

so that for small δ the first integral can be made less than $\epsilon/2$. As for the second integral, we note that for fixed δ,

$$\int_{|t|>\delta} F_y(t)\|f_t - f\|_{L^p(dx)}dt \le 2\|f\|_{L^p(\mathbb{R})}\int_{|t|>\delta} F_y(t)dt \to 0, \qquad y \to 0,$$

in accordance with Lemma 3.5. This proves part (a).

As for part (b) we assume that $g \in L^1(\mathbb{R})$. We then have

$$\left|\int_{\mathbb{R}} g(x)\Big(F_y * f\Big)(x)\,dx - \int_{\mathbb{R}} g(x)f(x)\,dx\right|$$

$$= \left|\int_{\mathbb{R}} g(x)\int_{\mathbb{R}} F_y(t)f(x-t)dt\,dx - \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)f(x)F_y(t)dt\,dx\right|$$

$$= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} F_y(t)\Big(f(x-t)-f(x)\Big)g(x)dt\,dx\right|$$

$$\le \int_{\mathbb{R}}\left|\int_{\mathbb{R}} F_y(t)\Big(f(x-t)-f(x)\Big)dt\right||g(x)|dx$$

$$= \|F_y * f - f\|_{L^1(|g|dx)} \to 0, \qquad y \to 0,$$

from which (b) follows. Finally, for the last part suppose that f is bounded and uniformly continuous on $\mathbb{R}$. As in part (a),

$$\|F_y * f - f\|_{L^\infty(\mathbb{R})} \le \int_{\mathbb{R}} F_y(t)\|f_t - f\|_{L^\infty(\mathbb{R})}dt.$$

Again, we may write the integral on the right as th sum of two integrals on $\{|t| \le \delta\}$ and $\{|t| > \delta\}$. By the uniform continuity of f, the first integral can be made small for small δ. The fact that the second integral approaches zero follows from the estimate

$$\int_{|t|>\delta} F_y(t)\|f_t - f\|_{L^\infty(\mathbb{R})}dt \le 2\|f\|_{L^\infty(\mathbb{R})}\int_{|t|>\delta} F_y(t)dt \to 0, \qquad y \to 0.$$

$\square$

COROLLARY 4.3. *For a bounded and uniformly continuous function f on $\mathbb{R}$, define*

$$u(x+iy) = \begin{cases} (F_y * f)(x), & y > 0 \\ f(x), & y = 0. \end{cases}$$

Then u is biharmonic on $\mathbb{C}_+$ and continuous on $\mathbb{C}_+ \cup \mathbb{R}$.

Proof. The corollary follows from part (c) of Theorem 4.2. $\square$

REMARK. Given a finite measure μ on the real line, we may define the F-integral of μ in a similar fashion:

$$u(x + iy) = F[d\mu](x + iy) = \int_{\mathbb{R}} F_y(x - t)d\mu(t) = \big(F_y * \mu\big)(x).$$

We mention in passing that both F-integrals of functions in L^p, and F-integrals of measures can not be characterized by norm inequalities, in contrast to the usual harmonic Poisson kernel.

5. The boundary behavior

This section is devoted to the study of boundary behavior of the F-integrals of functions $f \in L^p$. We shall see that the function $u = F[f]$ has nontangential limit $f(t)$ for almost all $t \in \mathbb{R}$.

Let f be a measurable function on $\mathbb{R}$. Recall that for $\lambda > 0$, the distribution function is defined by

$$m(\lambda) = \Big|\{x \in \mathbb{R} : \quad |f(x)| > \lambda\}\Big|.$$

Recall also the Chebychev's inequality

$$m(\lambda) \leq \frac{\|f\|_{L^p}^p}{\lambda}, \qquad 0 < p < \infty.$$

Let $|I|$ denote the length of an interval $I \subset \mathbb{R}$. The Hardy-Littlewood maximal function of $f \in L^1_{loc}(\mathbb{R})$ is

$$Mf(x) = \sup \frac{1}{|I|} \int_I |f(t)|dt,$$

where the supremum is taken over all intervals I containing x. For $\alpha > 0$ and $t \in \mathbb{R}$, we denote the cone in $\mathbb{C}_+$ with vertex t and angle $2 \arctan \alpha$ by

$$\Gamma_\alpha(t) = \{(x, y) : |x - y| < \alpha y, \quad 0 < y < \infty\}.$$

PROPOSITION 5.1. *Let $f \in L^1(\mathbb{R})$, and set*

$$u(x, y) = \int_{\mathbb{R}} F_y(s)f(x - s)ds.$$

Then

$$\sup_{(x,y)\in\Gamma_\alpha(t)} |u(x,y)| \leq A_\alpha Mf(t), \qquad t \in \mathbb{R},$$

where A_α is a constant depending only on α.

Proof. The proof parallels that of Theorem 4.2, as presented in [3]. The point is that F_y is a positive even function which is decreasing on the interval $(0, \infty)$, so that it can be written as a convex combination of functions of the form $\frac{1}{2h} \chi_{(-h,h)}(s)$. We then consider the sequence of step functions

$$h_n(s) = \sum_{j=1}^{N} a_j \chi_{(-x_j, x_j)}(s), \qquad a_j \geq 0,$$

which are monotone convergent to F_y, and satisfy

$$\int_{\mathbb{R}} h_n(s) ds = \sum_{j=1}^{N} 2x_j a_j \leq \int_{\mathbb{R}} F_y(s) ds = 1.$$

The rest of the proof goes essentially the same way as in [3]. We will omit the details here. □

Let u be a biharmonic function defined on the upper half plane. The function

$$u^*(t) = \sup_{z \in \Gamma_\alpha(t)} |u(z)|, \qquad t \in \mathbb{R},$$

is called the nontangential maximal function of u at $t \in \mathbb{R}$.

PROPOSITION 5.2. *Let $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, and set $u = F[f]$. Then $u^* \in L^p(\mathbb{R})$.*

Proof. According to the Hardy-Littlewood maximal theorm we know that

$$\|Mf\|_p \leq A_p \|f\|_p.$$

The proposition now follows from the preceding one. □

It is now time to state the main result of this section.

THEOREM 5.3. *Let $u = F[f]$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then for almost all $t \in \mathbb{R}$ we have*

$$\lim_{\Gamma_\alpha(t) \ni z \to t} u(z) = f(t).$$

Proof. Let $1 < p < \infty$, and $u(z) = F[f]$. Define

$$\delta_f(t) = \limsup_{\Gamma_\alpha(t) \ni z \to t} u(z) - \liminf_{\Gamma_\alpha(t) \ni z \to t} u(z), \qquad t \in \mathbb{R}.$$

It is clear from the definition of maximal function u^* and Proposition 5.2 that

$$\delta_f(t) \leq 2u^*(t) \leq 2A_\alpha M f(t).$$

According to the Hardy-Littlewood maximal theorem, $Mf(t)$, and hence $\delta_f(t)$ is finite for almost every $t \in \mathbb{R}$. It now follows from the Chebychev's inequality that

$$|t : \delta_f(t) > \epsilon| \leq \frac{\|\delta_f\|_p^p}{\epsilon^p} \leq \frac{2^p \|u^*\|_p^p}{\epsilon^p} \leq \frac{2^p A_\alpha^p}{\epsilon^p} \|Mf\|_p^p.$$

Finally, according to the Hardy-Littlewood theorem we have

$$\|Mf\|_p \leq A_p \|f\|_p,$$

therefore

$$|t : \delta_f(t) > \epsilon| \leq \left(\frac{2AA_\alpha \|f\|_p}{\epsilon}\right)^p = B_p \left(\frac{\|f\|_p}{\epsilon}\right)^p.$$

We now choose a continuous function g such that

$$\|f + g\|_p \leq \epsilon^2.$$

Since $\delta_f = \delta_{f+g}$, it follows that

$$|t : \delta_f(t) > \epsilon| = |t : \delta_{f+g}(t) > \epsilon| \leq B_p \left(\frac{2}{\epsilon} \|f + g\|_p\right)^p \leq C_p \epsilon^p \to 0, \quad \epsilon \to 0.$$

This implies that for almost every $t \in \mathbb{R}$ we have $\delta_f(t) = 0$, from which it follows that u has nontangential limit almost everywhere. The fact that this limit coincides with f almost everywhere, follows from

$$\|u(x,y) - f\|_p \to 0, \qquad y \to 0.$$

$\square$

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