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A Batch Arrival Queue with Bernoulli Vacation Schedule under Multiple Vacation Policy*

Gautam Choudhury**
Mathematical Sciences Division ,
Institute of Advanced Study in Science and Technology,
Paschim Boragaon, Guwahati–781033, Assam, India

Kailash C. Madan
Department of Mathematical Sciences,
Ahlia University, PO Box 10878, Manama, Kingdom of Bahrain

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ABSTRACT

We consider an $M^X/G/1$ queueing system with Bernoulli vacation schedule under multiple vacation policy, where after each vacation completion or service completion the server takes sequence of vacations until a batch of new customer arrive. This generalizes both $M^X/G/1$ queueing system with multiple vacation as well as $M/G/1$ Bernoulli vacation model. We carryout an extensive analysis for the queue size distributions at various epochs. Further attempts have been made to unify the results of related batch arrival vacation models.

Keywords: $M^X/G/1$ Queue, Vacation Time, Bernoulli Schedule Vacation, Multiple Vacation Policy, Queue Size

1. INTRODUCTION

The queueing system under the special consideration with respect to idle period (refereed to as vacation) is not new. Levy and Yechiali [20] were first to consider such a model under the assumption that the server takes a sequence of vacations until it finds at least one unit is waiting in the system at the end of a vacation,

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** Corresponding author, Email: choudhuryg@yahoo.com

known as multiple vacation policy (MVP). One of the most remarkable result that concern with such types of models is Stochastic Decomposition result, which allows the system to be analyzed by considering separately the distribution of the queue size with no vacations and the additional queue size due to vacation. This important result was first established by Fuhrmann and Cooper [9] for the *M*/*G*/1 queue with generalized vacations. The case where length of each vacation has constant (fixed) length of duration '*T*' is so called *T-Policy*, introduced by Heyman [11] in the study of a control operating policy. The literature on control of queueing system is rich and varied. For a survey see Tadj and Choudhury [27].

Perhaps due to practical applications in many real life situtations there have been considerable amount of works done on *T-Policy* models during the last decade. Gakis *et al.* [10] studied the distributions of idle and busy periods in several controlable *M*/*G*/1 queues including *T-Policy*. Recently, Artalejo and Lopez Herroro [1] investigate the busy period distribution of *T-Policy* model further through entropy maximization principle and corrected the result of [10] for purely Markovian models. Tadj [26] studied the queue size distribution of the T-policy model for quorum queueing system. His analysis is bassed on combination of embedded Markov chain and Semi-regenerative process.

Another more general class of model related to multiple vacation model is Bernoulli vacation model, where after each service completion the server may take vacation with probability p ($0 \leq p \leq 1$) and starts a new service with probability (1-p). The decision about taking a vacation after each service completion or vacation completion are independent. This type of model was introduced by Kelsion and Servi [12] for GI/G/1 queueing system extended in a few following papers [13, 14, 23, 25] for M/G/1 queueing system under MVP. Recently, Madan and Choudhury [21] investigate the Bernoulli vacation model with single vacation for a batch arrival queueing system under restricted admissibility policy, according to which C_1 (say) percent of arrival batches are accepted during busy period and C_2 (say) percent of arrival batches are accepted during vacation periods.

Presently, most of the studies have been devoted to batch arrival vacation models under different vacation policy because of its interdisciplinary character. Numerous researchers, including Baba [2], Lee and Srinivasan [17], Rosenberg and Yechiali [24], Teghem [29], Lee *et al.* [18, 19], Choudhury [5, 6] and Yechialli [31] and many other studied batch arrival queueing system under different vacation policies.

However, in this paper we propose to study such an $M^X/G/1$ Bernoulli vacation model, where concept of multiple vacation policy is also introduced. Our objective in present paper is not only generalize *M*/*G*/1 Bernoulli vacation model for batch arrival queueing system, but also to unify the results of many related vacation models.

As possible extension of our model we mention the possibility of assuming batch services and control policies (see recent papers [16, 31] and references therein) as auxiliary tools leading to development of more versatile queueing models with applications to transportation, production systems and digital communication systems.

The rest of the paper is organized as follows. In section 2, we describe the mathematical model of the present paper. Section 3 deals with queue size distribution at a random epoch and at a departure epoch. The queue size distribution at busy period initiation epoch and busy period distribution are discussed in section 4 and section 5. The queue size distribution due to idle period process is investigated in section 6. In section 7 we obtain mean queue sizes. Finally queue waiting time distribution has been derived in section 8.

2. MATHEMATICAL MODEL

We consider an $M^X/G/1$ queueing system in which arrival occurs according to a compound Poisson process with batches of random size X. An active server goes on serving the units until the system becomes empty. The service discipline is assumed to be FCFS. After each service completion the server may have a option to go for a short vacation (*Phase-II vacation*) with probability p and start a new service with probability q (= 1-p). However, if the system becomes idle, after a service completion or a *Phase-II vacation* completion, the server takes a primary vacation (*Phase-I vacation*), if there is no unit in the system. This process will be repeated until it finds at least a batch of customer in the queue i.e. we are considering the case of multiple vacations. In general, if the server finds a batch of customers upon return from the Phase-I vacation, it always starts the service of the first arrival. The decision about taking a Phase-I vacation after each service completion or *Phase-II vacation* completion are independent. Further, it is assumed that service time (B) random variable and vacation time (V) random variable (either in *Phase-I vacation* or in *Phase-II vacation*) are independent of each other and that of arrival process. Thus the time required by a customer to complete a service cycle, which we may call as a modified service time and is given by

$$G = \begin{cases} B + V & with \quad probability \quad p \\ B & with \quad probability \quad q = (1-p) \end{cases}$$

In fact, the concept of modified service time was first introduced by Keilson and Servi [12] for an $GI/G/1$ queueing system and subsequently by others (e.g. see [25]) for $M/G/1$ queueing systems. The policy adopted in our model may be termed as multiple vacation policy. It should be noted here that, if $p = 1$, this vacation policy results in a vacation after each service completion (e.g. see Takagi [28]). Similarly, if $p = 0$ this reduces to the $M^X/G/1$ queue with multiple vacations considered by Baba [2] (also see [6]). Using Kendall's notation, the model consider here is an $M^X/G/1(BS)/V_M$ queue, where V_M represents vacation time with multiple vacations and BS denotes Bernoulli schedule.

3. QUEUE SIZE DISTRIBUTION AT A RANDOM EPOCH

In this section, we first set up the system state equations for its stationary (random) queue size (including the one being served, if any) distribution by treating the elapsed service time and the elapsed vacation time as supplementary variables. Then we solve the equations and derive the probability generating function (*PGF*) for it. We now define following notations and probabilities:

λ	batch arrival rate
X	arrival size of a batch (a random variable) $a_k = Prob\ \{X = k\};\ k = 1, 2, 3, \ldots$
$B(x)[V(x)]$	Probability distribution function of B $[V]$
$B^*(s)[V^*(s)]$	Laplace Stieltjes transform of B $[V]$
$E(B^r)[E(V^r)]$	r-th moment of B $[V]$
$E[X_{(r)}] = E[\prod_{i=1}^{r}(X-i+1)]$	r-th factorial moment of X

Further, it may be noted that $V(0) = 0$, $V(\infty) = 1$, $B(0) = 0$ and $B(\infty) = 1$ and that $V(x)$ and $B(x)$ are continuous at $x = 0$, such that

$$\upsilon(x)dx = \frac{dV(x)}{1-V(x)}, \qquad \mu(x)dx = \frac{dB(x)}{1-B(x)}$$

are the first order differential functions (hazard rate function) of V and B respectively.

Let $N_Q(t)$ be the queue size at time 't' and $B^0(t)$ be the elapsed service time at time 't'. Further, we assume that $V^0(t)$ be the elapsed vacation time at time 't' and define the following random variable:

$$Y(t) = \begin{cases} 0, if \quad the \quad server \quad is \quad on \quad Phase-I \quad vacation \quad at \quad time \quad 't' \\ 1, if \quad the \quad server \quad is \quad on \quad Phase-II \quad vacation \quad at \quad time \quad 't' \\ 2, if \quad the \quad server \quad is \quad busy \quad at \quad time \quad 't' \end{cases}$$

Thus the supplementary variables $B^0(t)$ and $V^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), Y(t)\}$ and define the following limiting probabilities

$$P_{0,n}(x)dx = \lim_{t\to\infty} Prob \ [N_Q(t) = n, \ Y(t) = 0; \ \ x < V^0(t) \le x + dx], \ \ x > 0, \ \ n \ge 0,$$

$$P_{1,n}(x)dx = \lim_{t\to\infty} Prob \ [N_Q(t) = n, \ \ Y(t) = 1; \ x < V^0(t) \le x + dx], \ \ x > 0, \ \ n \ge 0,$$

and $Q_n(x) = \lim_{t\to\infty} Prob \ [N_Q(t) = n, \ Y(t) = 2; \ \ x < B^0(t) \le x + dx], \ \ x > 0, \ \ n \ge 1.$

Also, we define

$$P_{i,n} = \int_0^\infty P_{i,n}(x)dx \quad \text{for} \quad i = 0,1, \ n \ge 1 \text{ and } \ Q_n = \int_0^\infty Q_n(x)dx \ ; n \ge 1.$$

Then the Kolmogorov forward equations, to govern the system under the steady state conditions (e.g. see Cox [8]) can be written as follows:

$$\frac{d}{dx}P_{0,n}(x) + [\lambda + v(x)]P_{1,n}(x) = \lambda \sum_{k=1}^{n} a_k P_{0,n-k}(x); \ \ x > 0, \ n \ge 0, \tag{3.1}$$

$$\frac{d}{dx}P_{1,n}(x) + [\lambda + v(x)]P_{1,n}(x) = \lambda \sum_{k=1}^{n} a_k P_{1,n-k}(x); \ \ x > 0, n \ge 0, \tag{3.2}$$

$$\frac{d}{dx}Q_n(x) + [\lambda + \mu(x)]Q_n(x) = \lambda \sum_{k=1}^{n} a_k Q_{n-k}(x); \ \ x > 0, n \ge 1, \tag{3.3}$$

$$\lambda P_{0,0} = \sum_{i=0}^{1} \int_0^\infty v(x)P_{i,0}(x)dx + q\int_0^\infty \mu(x)Q_1(x)dx, \tag{3.4}$$

where $P_{i,-1}(x) = 0$ for i = 0, 1 occurring in equations (3.1) and (3.2) and $Q_0(x) = 0$ occurring in equation (3.3).

These set of equations are to be solved under the following boundary conditions at

$x = 0$:

$$P_{0,n}(0) = \lambda \delta_{0,n} P_{0,0}; \quad n \geq 0 \tag{3.5}$$

$$P_{1,n}(0) = p \int_0^\infty \mu(x) Q_{n+1}(x) dx; \quad n \geq 0 \tag{3.6}$$

$$Q_n(0) = \sum_{i=0}^{1} \int_0^\infty v(x) P_{i,n}(x) dx + q \int_0^\infty \mu(x) Q_{n+1}(x) dx \quad ; \quad n \geq 1, \tag{3.7}$$

where $\delta_{i,j} = \begin{cases} 1; & if \quad i = j \\ 0; & if \quad i \neq j \end{cases}$ denotes Kronecker's delta.

To solve these equations the normalizing condition is given by

$$\sum_{i=0}^{1} \sum_{n=0}^{\infty} \int_0^\infty P_{i,n}(x) dx + \sum_{n=1}^{\infty} \int_0^\infty Q_n(x) dx = 1. \tag{3.8}$$

Let us define the following PGF's:

$$P_i(x;z) = \sum_{n=0}^{\infty} z^n P_{i,n}(x); \quad (: |z| \leq 1, x > 0 :) \text{ for } i = 0, 1$$

$$P_i(0;z) = \sum_{n=0}^{\infty} z^n P_{i,n}(0); \quad (: |z| \leq 1 :) \text{ for } i = 0, 1$$

$$Q(x;z) = \sum_{n=1}^{\infty} z^n Q_n(x); \quad (: |z| \leq 1, x > 0 :)$$

$$Q(0;z) = \sum_{n=1}^{\infty} z^n Q_n(0); \quad (: |z| \leq 1 :). \text{ and } X(z) = \sum_{n=1}^{\infty} z^n a_n \quad (: |z| \leq 1 :).$$

Now proceeding in the usual manner with the equations (3.1)-(3.3), we obtain

$$P_i(x;z) = P_i(0;z)[1 - V(x)] e^{-\lambda(1-X(z))x}, \; x > 0, \quad \text{for } i = 0, 1 \tag{3.9}$$

and $$Q(x;z) = Q(0;z)[1 - B(x)] e^{-\lambda(1-X(z))x}, \; x > 0. \tag{3.10}$$

Now multiplying equation (3.5) by appropriate powers of z and then taking summation over all values of 'n', we get

$$P_0(0,z) = \lambda P_{0,0}; \tag{3.11}$$

and therefore from equation (3.9) for $i = 0$, we have

$$P_0(z) = \int_0^\infty P_0(x;z)dx = \frac{P_{0,0}[1 - V^*(\lambda - \lambda X(z))]}{[1 - X(z)]} \tag{3.12}$$

where $V^*(\lambda - \lambda X(z)) = \int_0^\infty e^{-\lambda(1-X(z))x} dV(x)$ is the z-transform of V.

Again multiplying equation (3.6) by appropriate powers of z and then taking summation over all possible values of 'n', we get

$$P_1(0;z) = pQ(0;z)B^*(\lambda - \lambda X(z))z^{-1}; \tag{3.13}$$

where $B^*(\lambda - \lambda X(z)) = \int_0^\infty e^{-\lambda(1-X(z))x} dB(x)$ is the z-transform of B.

Similarly multiplying equation (3.7) by appropriate powers of z and then taking summation over all possible values of 'n' and then utilizing (3.4), (3.10) and (3.13), we get on simplification

$$Q(0,z) = \frac{\lambda z P_{0,0}[1 - V^*(\lambda - \lambda X(z))]}{[\{q + pV^*(\lambda - \lambda X(z))\}B^*(\lambda - \lambda X(z)) - z]} \tag{3.14}$$

and therefore from equation (3.9) on utilizing (3.13) and (3.14), we get for $i = 1$

$$P_1(z) = \int_0^\infty P_1(x;z)dx = \frac{pP_{0,0}[1 - V^*(\lambda - \lambda X(z))]^2 B^*(\lambda - \lambda X(z))}{[1 - X(z)][\{q + pV^*(\lambda - \lambda X(z))\}B^*(\lambda - \lambda X(z)) - z]} \tag{3.15}$$

Finally from equations (3.10) and (3.14), we have

$$Q(z) = \int_0^\infty Q(x;z)dx = \frac{zP_{0,0}[1 - V^*(\lambda - \lambda X(z))][1 - B^*(\lambda - \lambda X(z))]}{[1 - X(z)][\{q + pV^*(\lambda - \lambda X(z))\}B^*(\lambda - \lambda X(z)) - z]}. \tag{3.16}$$

The unknown constant $P_{0,0}$ can be determined by using the normalizing condition (3.8), which is equivalent to $P_0(1) + P_1(1) + Q(1) = 1$. Thus we get

$$P_{0,0} = \frac{(1 - \rho^*)}{\lambda E(V)}; \tag{3.17}$$

where $\rho^* = \rho + p\lambda E(X)E(V)$ and $\rho = \lambda E(X)E(B)$ is the utilization factor of the system.

Note that equation (3.17) represents the steady state probability that the server is idle but available in the system. Also, from equation (3.17) we have $\rho^* < 1$, which is the stability condition under which the steady state solution exists .Consequently , the system state probabilities can be obtained from (3.12), (3.15) and (3.16) on utilizing (3.17). Thus we have

Prob [the server is on *Phase-I vacation*] $= P_0(1) = (1-\rho^*)$,
Prob [the server is on *Phase-II vacation*] $= P_1(1) = p\lambda E(X)E(V)$, and
Prob [the server is *busy*]$= Q(1) = \rho$ respectively.

Let $P_Q(z) = P_0(z) + zP_1(z) + Q(z)$ be the PGF of the queue size distribution at a random epoch, then

$$P_Q(z) = \frac{(1-\rho^*)(1-z)[1-V^*(\lambda-\lambda X(z))][q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{E(V)[\lambda-\lambda X(z)][\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z]}; \quad (3.18)$$

which is consistent with the result obtained by Servi [24] for single unit arrival case and with equation (3.11) of Choudhury [5] for $p = 0$.
Note that, the stochastic decomposition property for this model can be demonstrated easily by showing

$$P_Q(z) = \left[\frac{1-V^*(\lambda-\lambda X(z))}{E(V)(\lambda-\lambda X(z)}\right]\left[\frac{(1-\rho^*)(1-z)[q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z}\right]$$
$$= \xi(z)\, P_Q(M^x/G/1(BS);\, z) \quad (3.20)$$

where $P_Q\,(M^x/G/1(BS);\, z)$, the second factor in the right hand side of equation (3.20), is the PGF of the stationary queue size distribution of an $M^x/G/1$ queue with a single vacation under Bernoulli vacation schedule. This can be obtained from Pollaczeck Khinchine transform formula by replacing original service time distribution by our modified service distribution i.e $G^*(s) = \{q+pV^*(s)\}B^*(s)$ (in terms of *LST*) and thus we have

$$P_Q(M^x/G/1(BS);z) = \frac{(1-\rho^*)(1-z)[q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z}$$

and $\xi(z) = \dfrac{[1-V^*(\lambda-\lambda X(z))]}{E(V)[\lambda-\lambda X(z))]}$; the first factor in the right hand side of (3.20) is the PGF of the number of customers that arrive during the residual life of the va-

cation time (e.g. see Choudhury [5]).
Further utilizing the relationship between departure point queue size distribution and queue size distribution at a random epoch (see section-3 of Choudhury [5]), we may write

$$\pi(z) = \frac{[1-X(z)]}{E(X)(1-z)} P_Q(z)$$
$$= \frac{(1-\rho^*)[1-V^*(\lambda-\lambda X(z))][q+p\theta V^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\lambda E(X)E(V)[\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z]}; \qquad (3.21)$$

where $\pi(z)$ is the *PGF* of the queue size distribution at a departure epoch.
Now if the vacation time is deterministic with a constant (fixed) period of length 'T', then this will be the case of *T-Policy* (see Heyman [11]). Thus for this model, we have $V^*(\lambda-\lambda X(z)) = e^{-\lambda T(1-X(z))}$, $E(V) = T$ and $\rho^* = \rho + pE(X)\lambda T$ and therefore from equation (3.21), we get

$$\pi(z) = \frac{(1-\rho^*)[1-e^{-\lambda T(1-X(z))}][q+pe^{-\lambda T(1-X(z))}]B^*(\lambda-\lambda X(z))}{\lambda E(X)T[\{q+pe^{-\lambda T(1-X(z))}\}B^*(\lambda-\lambda X(z))-z]}; \qquad (3.22)$$

which is the PGF of the queue size distribution at a departure epoch of *an $M^X/G/1$ queue with T-Policy under Bernoulli schedule.*
In particular, if $p = 0$, then equation (3.22) reduces to

$$\pi(z) = \frac{(1-\rho)[1-e^{-\lambda T(1-X(z))}]B^*(\lambda-\lambda X(z))}{\lambda E(X)T[B^*(\lambda-\lambda X(z))-z]};$$

which is the PGF of the queue size distribution at a departure epoch of an $M^X/G/1$ queue with *T-Policy*. Note that for *Prob*$\{X=1\} = 1$ i.e. for single unit arrival case this agrees with equation (15) of Tadj [26].

Remark 3.1. It is important to note here that the stationary queue size distribution at a random epoch of this $M^X/G/1(BS)/V_M$ queue given by the equation (3.20) decomposes in to distributions of two independent random variables viz. –

1. The stationary queue size distribution of an $M^X/G/1$ queue with a single vacation under Bernoulli schedule (represented by the second factor) and
2. The number of customers that arrive during the residual life of the vacation time, which occur during the *Phase-I vacation* (represented by the first term).

4. QUEUE SIZE DISTRIBUTION AT BUSY PERIOD INITIATION EPOCH

In this section we derive the PGF of the queue size distribution at busy period initiation epoch. To derive it we define $\alpha_n (n \geq 1)$ as the steady state probability that an arbitrary (tagged) customer finds a batch of 'n' customer in the queue (including those are in service, if any) at busy period initiation epoch (or completion epoch of the idle period). Then conditioning number of units within the arriving batches during the Phase-I vacation and utilizing the argument of *PASTA* we may write following state equation

$$\alpha_n = \sum_{k=1}^{n} f_k a_n^{(k)}; \quad n \geq 1 \tag{4.1}$$

where $f_k = Prob$ {'k' individual units arrive (and are accepted) with Phase-I vacation}

$$= \sum_{i=0}^{\infty} (g_0)^i g_k = \frac{g_k}{(1-g_0)} \quad ; k \geq 1,$$

$g_j = Prob$ { 'j ' units arrive during the vacation time 'V' }$= \int_0^{\infty} \frac{e^{-\lambda x}(\lambda x)^j}{j!} dV(x)$

$A_n = X_1 + X_2 + \cdots + X_n; X_i$'s are i.i.d. random variables and have the same distribution as X.

$a_j^{(n)} = Prob \ \{A_n = j\}$ is the n-fold convolution of $\{a_j\}$ with itself and $a_j^{(0)} = 1$.

Let $\alpha(z) = \sum_{n=1}^{\infty} z^n \alpha_n$ be the PGFs of $\{\alpha_n \quad n \geq 1\}$, then from equation (4.1), we have

$$\alpha(z) = \frac{[V^*(\lambda - \lambda X(z)) - V^*(\lambda)]}{(1 - V^*(\lambda))}; \tag{4.2}$$

which is the PGF of the queue size distribution at busy period initiation epoch.
Let $E(\alpha)$ be the expected number of arrivals during the idle period, then we have

$$E(\alpha) = \sum_{n=1}^{\infty} n\alpha_n = \alpha'(1) = \frac{\lambda E(X)E(V)}{(1 - V^*(\lambda))} \tag{4.3}$$

Now utilizing Little's formula in (4.3), we get

$$\frac{E(\alpha)}{\lambda E(X)} = \frac{E(V)}{(1 - V^*(\lambda))} = E(T_0)(say);$$

which is the expression for expected length of the idle period of this model.

In particular, if we take vacation time as deterministic with a fixed period of length ' T ' (i.e. for the case of *T-Policy* model), then

$$g_n = \frac{e^{-\lambda T}(\lambda T)^n}{n!}; n \geq 0$$

and therefore from equation (4.3) and (4.2), we have

$$\alpha(z) = \frac{[e^{-\lambda T(1-X(z))} - e^{-\lambda T}]}{(1 - e^{-\lambda T})} \quad \text{and} \quad E(\alpha) = \frac{\lambda E(X)T}{(1 - e^{-\lambda T})}.$$

5. BUSY PERIOD DISTRIBUTION

Baba [2] obtained the LST of the busy period distribution for an $M^X/G/1$ queue with multiple vacations. Now utilizing his argument, we can also obtain the LST of the busy distribution from equation (4.2). We now define busy period as the length of time interval that keeps the server busy without interruption. This continues up to the instant when the system becomes empty again. This means that our busy period includes *Phase-II vacation period*. This type of busy period is known as delay busy period and studied by Conway *et al.* [7] and Miller [22].

Let T_B and $T_B^*(s)$ be the busy period random variable and its LST respectively, then utilizing the argument of Baba [2] in equation (4.2), we get

$$T_B^*(s) = \frac{[V^*(\lambda - \lambda X(\theta^*(s))) - V^*(\lambda)]}{(1 - V^*(\lambda))};$$

where $\theta^*(s)$ is the well known LST of the busy period distribution of an $M^X/G/1$ queue started with one unit by taking our modified service time as service time and this is given by

$$\theta^*(s) = G^*(s + \lambda - \lambda X(\theta^*(s))) \quad \text{and} \quad G^*(s) = (q + pV^*(s))B^*(s).$$

The mean busy period is given by

$$E(T_B) = -\frac{T_B^*(s)}{ds}\Bigg]_{s=0} = \frac{\rho E(V)}{(1-\rho^*)(1-V^*(\lambda))} + \frac{p\lambda E(X)\{E(V)\}^2}{(1-\rho^*)(1-V^*(\lambda))} \tag{5.1}$$

In particular , when $p = 0$, $\rho^* = \rho$ and therefore equation (5.1) (minor correction of equation (31) of [2]) reduces to

$$E(T_B) = \frac{\rho E(V)}{(1-\rho)(1-V^*(\lambda))};$$

which is consistent with equation (4.15) of Yechialli [30] for single unit arrival case. For the case of *T-Policy* model, we have $V^*(\lambda) = e^{-\lambda T}$ and $E(V) = T$ and therefore equation (5.1) yields

$$E(T_B) = \frac{\rho T}{(1-\rho-p\lambda TE(X))(1-e^{-\lambda T})} + \frac{p\lambda E(X)T^2}{(1-\rho-p\lambda TE(X))(1-e^{-\lambda T})};$$

which is consistent with equation (22) of Artalejo and Lopez-Herrero [1] for $p = 0$ and $E(X) = 1$ i.e for single unit arrival case. It should be noted here that in this context [1] have obtained some useful results including explicit expression for probability density function of busy period distribution for an $M/M/1$ queue under *T-policy*, *N-Policy* and *D-Policy*.

6. QUEUE SIZE DISTRIBUTION DUE TO IDLE PERIOD

In this section our objective is to obtain the stationary queue size distribution due to the idle period process. To obtain it let us define $\{\psi_n ; n \geq 0\}$ as the steady state probability that a batch of 'n' customer arrived before a tagged customer during the forward recurrence time (residual life) of the idle period in which the tagged customer is chosen randomly from the arriving batch that turns up at the busy period initiation epoch. Now since the batch of arriving customers are associated with the tagged customer which is chosen randomly from the arriving batches that turns up at the busy period initiation epoch and therefore by virtue of "*stationary renewal process*" (see [15], page-94), we may write

$$\psi_n = \sum_{k=n+1}^{\infty} \frac{\gamma_k}{k} \quad ; \quad n = 0, 1, 2, \cdots$$

where $\{\gamma_k ; k \geq 1\}$ is the probability that the k-th. batch that starts a busy period to which the tagged arrival belongs is chosen randomly with probability $(1/k)$. This can be obtained directly from equation (4.1) by applying length biasing argument of renewal theory. Thus we get

$$\gamma_n = \frac{n\alpha_n}{\sum_{n=1}^{\infty} n\alpha_n} = \frac{(1-V^*(\lambda))n\alpha_n}{\lambda E(X)E(V)}\ ; n = 1, 2, \cdots \tag{6.1}$$

Let $\psi(z)$ be the PGF of $\{\psi_n ; n \geq 0\}$, then we have

$$\psi(z) = \frac{[1-V^*(\lambda-\lambda X(z))]}{\lambda E(X)E(V)(1-z)} = H(z)\xi(z)\,; \tag{6.2}$$

where $H(z) = \frac{[1-X(z)]}{E(X)(1-z)}$ is the PGF of the number of units placed before an arbitrary (tagged) customer in a batch in which the tagged customer arrives. This number is given as a backward recurrence time in discrete time renewal process, where successive renewal points are generated by the arrival size random variable. This is due to randomness nature of the arrival size random variable.

The expression (6.2) is the PGF of the queue size distribution of idle period due to *Phase-I vacation*. Because of the *PASTA* property this is equivalent to the PGF of the number of customers that arrive during an interval from the beginning of the idle period to a random point in the idle period. More specifically, we may call it queue size distribution due to the idle period. Note that for single unit arrival case our equation (6.2) is consistent with the result obtained in Takagi [28].

Now let us consider the case of *T-Policy* model, where equation (6.1) becomes

$$\gamma_n = \frac{n}{\lambda E(X)T}\sum_{k=1}^{n} \frac{e^{-\lambda T}(\lambda T)^k}{k!} a_n^{(k)}\ ; n = 1, 2, \ldots$$

and therefore from equation (6.2), we have

$$\psi(z) = \frac{[1-e^{-\lambda T(1-X(z))}]}{\lambda E(X)T(1-z)}; \tag{6.3}$$

which is the PGF of the queue size distribution due to the idle period of an $M^X/G/1$ queue with *T-Policy*. Note that for *Prob* $\{X = 1\} = 1$ i.e. for single unit arrival case our equation (6.3) is consistent with formula (18) of Tadj [26] (also see Bruneel [3]).

Remark 6.1. The PGF of the queue size distribution due to idle period of this $M^X/G/1(BS)/V_M$ queue given by equation (6.2) decomposes in to PGFs of distributions of two independent random variables viz.-

1. The queue size distribution due to residual life of vacation time (represented by the second term).
2. The queue size distribution due to random nature of the arrival size random variable (represent by the first term).

7. MEAN QUEUE SIZE

Our next objective is to obtain the mean queue sizes at different point of time. Let L_0 be the mean queue size due to idle period process, then

$$L_0 = \psi'(1) = \lambda E(X)E(V_R) + E(X_R); \tag{7.1}$$

where $E(V_R) = \dfrac{E(V^2)}{2E(V)}$ is the mean residual vacation time

and $E(X_R) = \dfrac{E[X(X-1)]}{2E(X)}$ is the mean residual batch size.

Again, if we denote L_Q as mean queue size at a random point of time then

$$L_Q = P_Q'(1) = \rho^* + \frac{\lambda^2[E(B^2) + 2pE(V)E(B) + pE(V^2)]E^2(X)}{2(1-\rho^*)} + \frac{\lambda[E(B) + pE(V)]E[X(X-1)]}{2(1-\rho^*)} + \lambda E(X)E(V_R);$$

which is consistent with the result obtained by Choudhury [6] for $p = 0$.
Further, mean queue size at a departure epoch L_S (say), of this model is found to be

$$L_S = \pi'(1) = \rho^* + \frac{\lambda^2[E(B^2) + 2pE(V)E(B) + pE(V^2)]E^2(X)}{2(1-\rho^*)} + \frac{\lambda[E(B) + pE(V)]E[X(X-1)]}{2(1-\rho^*)} + L_0. \tag{7.2}$$

In particular, if we take $p = 0$ and $E(V) = T$ and $E(V^2) = T^2$ then from equation

(7.2), we have

$$L_S = \rho + \frac{\lambda^2 E(B^2) E^2(X)}{2(1-\rho)} + \frac{E(X_R)}{(1-\rho)} + \frac{\lambda E(X) T}{2};$$

which is the expression for an $M^X/G/1$ queue under *T-Policy*. Note that for *Prob* $\{X = 1\} = 1$, the above result agrees with equation (9) of Heyman [11] (also see Tadj [26]).

8. WAITING TIME DISTRIBUTION

To obtain the waiting time distribution in the queue, we first derive the waiting time of the first customer in an arriving batch, W_1 (say) and use $W_1^*(s)$ to denote LST of W_1.

Now if we identify a batch with a single customer, then its service time is just the modified service time of customers constituting the batch. In this case, the batch will have as its batch size $X(z) = z$. The mean arrival rate will be λ and LST of the modified service time of the batch will replace $G^*(s) = (q + pV^*(s))B^*(s)$ by $X(G^*(s))$. Using the information and the results by Chaudhry and Templeton [4] (see Chapther 3) , from equation (3.21) we have

$$\pi(z) = \frac{(1-\rho^*)[1 - V^*(\lambda - \lambda z)] X[G^*(\lambda - \lambda z)]}{\lambda E(V)[X[G^*(\lambda - \lambda z)] - z]} \tag{8.1}$$

If the waiting time of each batch is independent of the part of arrival process following the arrival time of the batches left behind a departing batch are those that arrive during the time it spends in the queue and in service. it follows that (see Fuhrmann and Cooper [9]).

$$\pi(z) = W_1^*(\lambda - \lambda z) X[G^*(\lambda - \lambda z)] \tag{8.2}$$

Now putting $s = (\lambda - \lambda z)$ in (8.2) and utilizing (8.1) in (8.2), we get finally

$$W_1^*(s) = \frac{(1-\rho^*)[1 - V^*(s)]}{E(V)[s - \lambda + \lambda X(G^*(s))]}. \tag{8.3}$$

Next, let W be the waiting time of an arbitrary customer in a batch and denote by $W^*(s)$ the LST of W. If $j \geq 1$ is the position of the customer within arrival batch, then

$$W = W_1 + \sum_{i=1}^{j-1} G_i^/ \; ; \; j \geq 1, \tag{8.4}$$

where $G_i^/$ denotes the difference between modified service time and inter arrival time of the i customer in the batch.

If χ_j is the probability of an arbitrary customer being the j-th position of an arriving batch , then applying the results of Chaudhry and Templeton [4] (see Chapter 3), we may write

$$P_r[\sum_{i=1}^{j-1} G_i^/ \leq t] = \sum_{j=1}^{\infty} \chi_j G(t)^{(j-1)*} \;;$$

where $G(t) = P_r[G_i^/ \leq t]$ and $\chi_j = (1 - \sum_{i=1}^{j-1} a_i) / E(X)$.

Consequently taking LST of (8.4), we get on simplification

$$\begin{aligned} W^*(s) &= E[e^{-sW}]. \\ &= E[e^{-sW_1}].E[e^{-s\sum_{i=1}^{j-1} G_i^/}] \\ &= \frac{W_1^*(s)}{E(X)} \cdot \frac{[1 - X(G^*(s))]}{[1 - G^*(s)]} \end{aligned}$$

and therefore LST of the waiting time distribution in the queue for this model is given by

$$W^*(s) = \frac{[1 - V^*(s)][1 - X(G^*(s))]}{E(X)E(V)[s - \lambda + \lambda X(G^*(s))][1 - G^*(s)]}. \tag{8.5}$$

Note that for $p = 0$ our equation (8.5) is consistent with the result obtained by Baba [2]. However our method of derivation is completely different from him.

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