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BMAP\SM\1 대기시스템의 정상 알고리즘 개발†

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Stable Algorithm for a *BMAP\SM\1* Queueing System

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대기행렬 모형은 통신시스템이나 통신망 구현에 가장 적합한 수리모형으로 알려져 있고, 이에 대한 연구가 상당히 많이 진행되고 있다. 본 논문에서는 재해가 발생될 수 있는 BMAP/SM/1 대기시스템으로, 재해가 발생했을 경우 시스템 복구가 즉시 이루어지지 않고 임의 시간 후 복구 되는 시스템을 고려대상으로 하고 있다. 시스템의 정보입력흐름은 상호종속 또는 그룹 입력이 허용되는 배치마코프 도착과정으로 가정하였고, 또한 서비스분포는 세미 마코프 프로세스를 따른다고 가정하였다. 아울러 시스템에 재해가 발생하면 모든 고객은 즉시 시스템을 떠나게 되고, 재해복구는 임의 시간 후에 이루어진다. 임베디드 마코프체인의 안전상태 확률분포가 유도를 위한 정상 알고리즘 개발이 이루어졌다.

Keywords : *BMAP/SM/1*, disasters, recovery, stable algorithm

1. Introduction

During real queueing system operation, the appearance of disasters is possible which causes customers loss and system operation disturbances. Simultaneous loss of all customers can be described by a disaster causing all customers to leave the system instantaneously. Such disaster is a special case of a so-called negative arrival that removes one customer or a batch of ones of random size from the queueing system.

The theory of negative arrivals has been originated and developed significantly by Gelenbe, see, e.g., (Gelenbe 1991).

A *BMAP/SM/1* queueing system with *MAP* input of disasters in the cases of instantaneous and non-instantaneous recovery of a server after a disaster arrival was investigated in (Dudin and Nishimura 1999) and (Dudin and Karolik 2001). The main results obtained for this queueing system (arbitrary time stationary queue length distribution, the performance characteristics) need calculation of the stationary state distribution of the Markov chain embedded at customer departure epochs. The algorithm presented in (Dudin and Karolik 2001) exploits the analyticity of the vector generating function of the stationary distribution in a unit disc of a complex plane. It includes calculation of roots of some function in a unit disc. In the case of high dimensionality of the stationary state probability vectors the roots determination problem can arise. The algorithm presented in Dudin and Karolik (2001) has one more shortage that can take place in numerical realization for some examples. The recursion for probability vectors calculation contains subtraction operation that leads to calculation error growth and the probability vectors corresponding to higher states can have negative entries.

In this paper we present the alternative algorithm for sta-

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tionary state distribution calculation for the system *BMAP/SM/1* with disasters in the case of non-instantaneous recovery of a server. Presented algorithm is stable in numerical realization and doesn't have mentioned shortages. The similar algorithm is obtained in (Dudin and Semenova 2004) for the case of instantaneous recovery of a server after a disaster arrival.

2. The System Model

We consider a single-server queueing system with unlimited waiting space. The input into the system is a *BMAP* (*Batch Markovian Arrival Process*). This input is directed by a continuous-time Markov chain v_t, $t \geq 0$ with a state space $\{0,1,\dots W\}$. The transitions of process v_t, $t \geq 0$, and arrivals of customers are performed according to a matrix generation function $D(z)=\sum_{k=0}^{\infty} D_k z^k$, $|z|<1$. A more detailed description of *BMAP* and assumptions about the matrix function $D(z)$ are given by (Lucantoni 1991). Denote by $\vec{\varphi}$ the stationary probability row vector of the Markove chain ν_t, $t \geq 0$. It is defined by equations:

$$\vec{\varphi}\, D(1)=\vec{0}\ ,\ \vec{\varphi}\ \vec{1}=1$$

Here $\vec{0}$ is a zero row vector and $\vec{1}$ is a unit column vector. The intensity λ of *BMAP* input(the fundamental rate) is calculated as $\lambda=\vec{\varphi}\, D'(1)=\vec{1}$. We assume that service process is of *SM*-type. It means that successful service times of customers are the sojourn times of semi-Markovian process m_t, $t \geq 0$. This process has a state space $\{1,\dots,M\}$and a semi-Markovian kernel $B(x)=||B_{m,m'}(x)||_{m,m'=\overline{1,M}}$. The function $B_{m,m'}(x)$ is the conditional distribution function of the sojourn time of the process m_t, $t \geq 0$ in a state m under the condition that the next state is m', m, $m'=\overline{1,M}$. We use the same assumptions about the kernel $B(x)$ which are given in (Lucantoni and Neuts 1994) and (Neuts 1989). It's assumed that the states of the service directing process m_t, $t \geq 0$ are changed according to the matrix $P=B(\infty)$ at service completion epochs regardless of whether service is completed successfully or is cancelled by a disaster appearance. Denote by b_1 the mean service time which is not interrupted by a disaster arrival. The value b_1 is define by the formula $b_1=\vec{\delta}\int_0^{\infty} t dB(t)\vec{1}$, where $\vec{\delta}$ is invariant row vector of the matrix P. The input of disasters is *MAP* (*Markovian Arrival Process*). *MAP* is a partial case of *BMAP* when ordinary arrivals are arrowed only. We assume that *MAP* is directed by a continuous-time Markov chain η_t, $t \geq 0$ with a state space $\{0,1,\dots,N\}$ and a matrix generating function $F(z)=F_0+F_1 z, |z|<1$. Following (Jain and Sigman 1996) we suppose that arrival of disaster at a busy period immediately removes all customers from the system and the server is recovered during a period having a general distribution function $G(t)$. If disaster arrives to the empty system or during a recovery period it's ignored by the system. We consider two cases of customers admission during a recovery period:

a) arriving batch of customers is admitted to the queue with probability q_a and is ignored with complementary probability $1-q_a$.
b) each customer of arriving batch is admitted to the queue with probability q_b and is ignored with complementary probability $1-q_b$.

In both cases we suppose that disasters arrived when the system is empty or during a period of a server recovery are ignored by the system.

3. Stationary Distribution of Embedded Markov Chain

Let t_n be the n^{th} epoch of customer departure from the system, $n \geq 1$. It's a service completion epoch or a disaster arrival epoch at a busy period. Introduce into consideration the following five-dimensional Markov chain:

$$\xi_n=\{i_n, c_n, v_n, \eta_n, m_n\},\ n \geq 1$$

where i_n is a queue length at the epoch t_n+0 , $i_n \geq 0$; v_n is the state of arrival directing process v_t, $t \geq 0$ at the epoch $t_n, v_n=\overline{0,W}$; η_n is the state of disaster directing process η_t, $t \geq 0$ at the epoch t_n+0, $\eta_n=\overline{0,N}$; m_n is the state of service directing process m_t, $t \geq 0$ at the epoch t_n+0, $m_n=\overline{1,M}$

$$c_n=\begin{cases}0, & if\ t_n\ is\ a\ successful\ service\ completion\ epoch\\ 1, & if\ t_n\ is\ a\ disaster\ arrival\ epoch,\ n \geq 0\end{cases}$$

Denote by

$$P\{(i,c,v,\eta,m)\rightarrow(l,c',v',\eta',m')\}= P\{i_{n+1}=l,c_{n+1}=c',v_{n+1}=v',\eta_{n+1}=\eta',m_{n+1}=m \mid i_n=l,c_n=c,v_n=v,\eta_n=\eta,m_n=m\}$$

the one step transitions probabilities of the Markov chain ξ_n, $n\geq 1$. Let this probabilities be listed in lexicographic order of the components $\{v,\eta,m\}$ increasing.

Introduce into consideration the matrices

$$P_{i,l}=\begin{pmatrix} P_{i,l}^{(0,0)} & P_{i,l}^{(0,1)} \\ P_{i,l}^{(1,0)} & P_{i,l}^{(1,1)} \end{pmatrix} \qquad i,l\geq 0$$

where the block $P_{i,l}^{(c,c')}$ is the matrix formed by the probabilities $P\{(i,c,v,\eta,m)\rightarrow(l,c',v',\eta',m')\}$.

The non-zero matrices $P_{i,l}$, i, $l\geq 0$ have the form

$$P_{0,0}=\begin{pmatrix} \Psi_1\Omega_0 & \Psi(1)S \\ H_0\Psi_1\Omega_0 & H_0\Psi(1)S+\sum_{i=1}^{\infty}H_iS \end{pmatrix}$$

$$P_{1,0}=\begin{pmatrix} \Omega_0 & S \\ O & O \end{pmatrix}$$

$$P_{i,0}=\begin{pmatrix} O & S \\ O & O \end{pmatrix} \quad i>1$$

$$P_{0,i}=\begin{pmatrix} \sum_{i=1}^{l+1}\Psi_i\Omega_{l-i+1} & O \\ \sum_{i=1}^{l+1}(H_0\Psi_1+H_i)\Omega_{l-i+1} & O \end{pmatrix} \quad l>0$$

$$P_{i,l}=\begin{pmatrix} \Omega_{l-i+1} & O \\ O & O \end{pmatrix} \quad i\geq max\{1,\, i-1\},\ i>1$$

In the case of $l<i-1$, $l\neq 0$, $P_{i,l}$ are the null matrices of size $2K$ where $K=(W+1)(N+1)M$

Here the matrices Ω_l, H_l, $l\geq 0$ are defined by the matrix expansions

$$\sum_{l=0}^{\infty}\Omega_l z^l=\beta(z)=\int_0^{\infty}e^{D(z)t}\otimes e^{F_1 t}\otimes dB(t) \quad (1)$$

$$\sum_{l=0}^{\infty}H_l z^l=H(z)=\int_0^{\infty}e^{(R(z)\oplus F(1))t}dG(t)\otimes I_M \quad (2)$$

$$R(z)=\begin{cases} q_a D(z)+(1-q_a)D(1), & \text{in the case } a \\ D(q_b(z-1)+1), & \text{in the case } b \end{cases}$$

The matrices S and Ψ_k, $k\geq 1$ and are calculated as

$$S=\int_0^{\infty}e^{D(1)t}\otimes(e^{F_0 t}F_1)\otimes(P-B(t))dt \quad (3)$$

$$\Psi_k=-[(D_0\oplus F(1))^{-1}(D_k\otimes I_{N+1})\otimes I_M \quad k\geq 1, \quad (4)$$

Where $\otimes$ and $\oplus$ are the symbols of the Kronecker product and the Kronecker sum, I denotes an identity matrix of corresponding size, O is a null matrix of size $K=(W+1)(N+1)M$. The matrices Ω_l and S describe transitions of the process $\zeta_n=\{v_n,\eta_n,m_n\}$, $n\geq 1$ during the service time. The matrix Ω_l corresponds to the l customers arrival and no disaster arrival during the service time, $l\geq 0$. The matrix S corresponds to the disaster arrival during the service time. The matrix Ψ_k describes transitions of the process ζ_n, $n\geq 1$ during the idle period (excluding the recovery period) which finishes by arrival of batch of k customers. The matrix H_l means accumulation of l customers during a recovery period, $l\geq 0$.

4. Stable Algorithm

Consider the following stationary state probabilities:

$$r(i,v,\eta,m)=\lim_{n\to\infty}P\{i_n=i,c_n=0,v_n=v,\eta_n=\eta,m_n=m\}$$

$$k(i,v,\eta,m)=\lim_{n\to\infty}P\{i_n=0,c_n=1,v_n=v,\eta_n=\eta,m_n=m\}$$

$$i\geq 0, v=\overline{0,W}, \eta=\overline{0,N}, m=\overline{1,M} \quad (5)$$

The limits (5) exist for any finite positive arrival and service rates due to the presence of disasters. Define the following vectors:

$$\vec{p}(i,v,\eta)=(p(i,v,\eta,1),\cdots,p(i,v,\eta,M))$$
$$\vec{p}(i,v)=(\vec{p}(i,v,0),\cdots,\vec{p}(i,v,N))$$
$$\vec{p}_i(\vec{p}(i,0),\cdots,\vec{p}(i,W))$$
$$\vec{k}(v,\eta)=(\vec{k}(v,\eta,1),\cdots,\vec{k}(v,\eta,M))$$
$$\vec{k}(v)=(\vec{k}(v,0),\cdots,\vec{k}(v,N))$$
$$\vec{k}=(\vec{k}(0),\cdots,\vec{k}(W))$$

Below we obtain the algorithm for calculating the stationary state probabilities in the form of vectors

$$\vec{\pi}_0=(\vec{p}_0,\vec{k}),\ \vec{\pi}_i=(\vec{p}_i,\vec{0}),\ i\geq 1 \quad (6)$$

Let $G^{(k)}$ be the matrix that characterizes transitions of components $\{v_n,\eta_n,m_n\}$ of the Markov chain ξ_n, $n\geq 1$in the time interval during which the state of the component i_n changes from $k+1$ to k and no disasters arrives, $k\geq 0$. The matrices $G^{(k)}$, $k\geq 0$ satisfy the following equation:

$$G^{(k)}=P_{k+1,k}+\sum_{i=k+1}^{\infty}P_{k+1,i}G^{(i-1)}G^{(i-2)}\cdots G^{(k)},k\geq 0 \quad (7)$$

We here assume that $P_{1,0}=\begin{pmatrix} \Omega_0 & S \\ O & O \end{pmatrix}$. For the system under consideration the value of the matrix $G^{(k)}$does not depend on k, $k\geq 0$ and is equal to

$$G=\begin{pmatrix} \hat{G} & O \\ O & O \end{pmatrix} \quad (8)$$

where $\widehat{G}$ is the solution of the matrix equation

$$\widehat{G}=\beta(\widehat{G})=\sum_{l=0}^{\infty}\Omega_l\widehat{G}^l. \qquad (9)$$

Formulas (8), (9) follows from (7) and the block form of matrices $P_{k,i}$, $i\geq k-1$. Algorithm for solving the equation (9) can be found in (Neuts 1989). Let $X^{(k)}$ be the matrix that characterizes the transitions of components $\{v_n,\eta_n,m_n\}$ of the Markov chain ξ_n, $n\geq 1$ in the time interval that starts from the state k of the component i_n and finishes by reaching the state 0 due to a disaster arrival without visiting the state $k-1$, $k\geq 1$.
The matrices $X^{(k)}$, $k\geq 1$ are defined by the equation

$$X^{(k)}=P_{k,0}+P_{k,k}X^{(k)}+\sum_{n=k+1}^{\infty}P_{k,n}\left(\sum_{i=1}^{n-1}G^{(n-1)}G^{(i-2)}\cdots G^{(i)}H^{(i)}+H^{(n)}\right),\quad k\geq 1$$

Here we assume that $P_{1,0}=\begin{pmatrix}O & S\\ O & O\end{pmatrix}$.
As $G^{(k)}$ the matrix $X^{(k)}$ does not depend on k and is equal to

$$X=\begin{pmatrix}O & (\widehat{G}-I)(\beta(1)-I)^{-1}S\\ O & O\end{pmatrix}$$

Theorem. Stationary probability vectors $\vec{\pi}_i$, $i\geq 0$ are calculated by the following way:

$$\vec{\pi}_i=\vec{\pi}_0\Phi_i,\ i\geq 1 \qquad (10)$$

where the matrices Φ_i, $i\geq 0$ are calculated recurrently

$$\Phi_0=I$$
$$\Phi_k=\sum_{i=0}^{k-1}\Phi_i Y_k^{(i)}(I-Y_k^{(k)})^{-1},\ k\geq 1 \qquad (11)$$

the matrices $Y_k^{(i)}$ are defined as

$$\overline{Y}_k^{(i)}=\sum_{l=k}^{\infty}P_{i,l}G^{l-k},i=\overline{0,k},\ k\geq 0 \qquad (12)$$

vector $\vec{\pi}_0$ satisfies the system

$$\vec{\pi}_0\left(I-Y_0^{(0)}-\sum_{k=0}^{\infty}P_{0,k}\sum_{i=0}^{k-1}G^iX\right)=\vec{0}$$
$$\vec{\pi}_0\sum_{i=0}^{\infty}\Phi_i\vec{e}=1 \qquad (13)$$

where $\vec{e}$ is a unit column vector.

Theorem is proved as Theorem 1 in (Dudin and Semenova 2004). The proof is based on the theory of censoring Markov chains; see (Kemeni et al. 1996). Non-singularity of the matrix $I-Y_k^{(k)}$ in (11) follows from the Hadamard Theorem. It follows from the Lederman theorem that entries of the matrix $(I-Y_k^{(k)})^{-1}$ are non-negative. Formulas (11) involve only sum and product of the matrices with non-negative entries. So the recursion (11) is numerically stable.

For numerical realization we calculate the first J vectors $\vec{\pi}_i$. Level J is defined from the inequality

$$\|\vec{\pi}_{J+1}-\vec{\pi}_J\|<\epsilon \qquad (14)$$

where ϵ is a given accuracy of calculations.

5. Numerical Examples

Let algorithm B be the algorithm elaborated in (Dudin and Karolik 2001) and algorithm A be one based on formulas (8)-(13). The aim of numerical experiments is comparison of algorithm A and B. It's assumed in (Dudin and Karolik 2001) that all customers either are accumulated or are lost during the recovery period. So we can only consider the cases $q_a=q_b=1$ or $q_a=q_b=0$ only.
We compare the following characteristics of algorithms:

- the stability of the both algorithms. In our experiments the stability is characterized by the number $\hat{N}$ of probability vectors calculated correctly. The correctness means that calculated vector has non-negative entries.
- the values of corresponding vectors calculated by the algorithms A and B correctly.
- the running time.

We will compare the vectors $\vec{p}_i$, $i\geq 0$. For algorithm B the vector $\vec{p}_i$ is determined from (6) as the first K entries of the vector $\vec{\pi}_i$.

Example 1. Let us consider the case $q_a=q_b=1$. It means that all arriving customers are accepted to the queue during a recovery period.
BMAP- input is defined by the matrices

$$D_0=\begin{pmatrix}-2.2 & 1.2\\ 4.8 & -7.8\end{pmatrix},\quad D_1=D_2=\begin{pmatrix}0.5 & 0\\ 0 & 1.5\end{pmatrix}$$

The intensity of *BMAP* is 1.2 and correlation coefficient is

0.013. The *MAP*-input of disasters is characterized by the matrices

$$F_0=\begin{pmatrix}-0.17 & 0.16\\ 0.27 & -0.35\end{pmatrix},\quad F_1=F_2=\begin{pmatrix}0.1 & 0\\ 0 & 0.08\end{pmatrix}$$

with intensity 0.092 and correlation coefficient is 0.0032. For *SM*-service we assume that the kernel $B(x)$ has the form

$$B(x)=\begin{pmatrix}0.65B_1(x) & 0.35B_1(x)\\ 0.45B_2(x) & 0.55B_2(x)\end{pmatrix} \quad\quad (15)$$

where

$$B_i(t)=\int_0^t \frac{\gamma_i(\gamma_i\tau)^{k_i-1}}{(k_i-1)!}e^{-\gamma_i\tau}d\tau,\ i=\overline{1,2} \quad\quad (16)$$

$\gamma_1=15$, $\gamma_2=20$, $k_1=3$, $k_2=4$. The mean service time is 0.20. Recovery period is distributed exponentially with the rate 0.5.

Denote by ρ the product of the *BMAP* fundamental rate and mean service time. In our example $\rho=0.42$.

<Table 1> contains the values $\vec{p}_i\vec{1}$,calculated by the algorithms A and B, the number $\hat{N}$ of vectors calculated correctly and running time T. We set $\epsilon=10^{-5}$ in condition (14) of algorithm B. We need to calculate non less 42 vectors $\vec{p}_i$ to provide the given accuracy ϵ.

<Table 1>

	Algorithm A	Algorithm B
i	$\vec{p}_i\vec{1}$	$\vec{p}_i\vec{1}$
0	0.334410031	0.334412237
1	0.276703645	0.276705467
2	0.143837023	0.143837968
3	0.087357481	0.087358054
4	0.047631483	0.047631795
5	0.028496848	0.028497033
6	0.017399227	0.017399327
7	0.011347005	0.011346807
8	0.007778331	0.007772587
$\hat{N}$	8	$\geq$ 50
T	00:15:36	00:10:40

It follows from <Table 1> that the corresponding values $\vec{p}_i\vec{1}$ coincide with accuracy 10^{-5}. If the service process is recurrent with distribution function $B_1(t)$defined by the equation (16), we can calculate the first 17 vectors $\vec{p}_i$ correctly by algorithm A.

Example 2. Now let us consider the case $q_a=q_b=0$. It means that all arriving customers are lost during a recovery period. The *BMAP*-input, distribution or recovery period and ε are the same as in example 1. The time interval between disaster arrivals is distributed exponentially with rate 0.9. The service times have exponential distribution with the rate 0.8. In this case $\rho=2.625$. Note that the system *BMAP/SM/1* without disasters has no stationary distribution for $\rho\geq 1$, see (Lucantoni and Neuts 1994). As mentioned above the stationary distribution can be calculated for the system under consideration due to disaster arrivals. Using algorithm A the first 13 vectors $\vec{p}_i$ are calculated correctly. About 145 vectors must be calculated to provide the accuracy 10^{-5} in algorithm B.

<Table 2>

	Algorithm A	Algorithm B
i	$\vec{p}_i\vec{1}$	$\vec{p}_i\vec{1}$
0	0.024153855	0.024153876
1	0.032344791	0.032344799
2	0.031966088	0.031966095
3	0.034250833	0.034250839
4	0.034330425	0.034330431
5	0.034635679	0.034635684
6	0.034226661	0.034226667
7	0.033672918	0.033672923
8	0.032828344	0.032828349
$\hat{N}$	13	$\geq$ 300
T	00:4:17	01:13:15

The values $\vec{p}_i\vec{1}$ calculated by algorithms A and B, the characteristics $\hat{N}$ and T are given in <Table 2>.

6. Conclusion

In this paper, we consider a *BMAP/SM/1* queueing system with Markovian arrival input of disasters. After a disaster arrival all customers leave the system instantaneously and a server is recovered during a random period of time. Possible correlation and group arrivals are taken into account by means of considering the *Batch Markovian Arrival Process* (*BMAP*) as input stream to the system. We consider both variants of accumulating and losing the customers arriving during a recovery period. Numerically stable algorithm for calculation of the stationary state distribution of embedded Markov chain is presented. Analyzing the numerical results the following conclusions can be made. Firstly, the stability of algorithm A depends on dimensions of vectors, the value ρ and type of service process(SM or recurrent). The algorithm B provides the stable calculations for all considered arrival, service, disasters and recovery parameters. Secondly, the values of vectors $\vec{p}_i$ calculated correctly by the both algorithms coincide with accuracy 10^{-k} when $\epsilon = 10^{-k}$in (14). Finally, if dimension of vectors $\vec{p}_i$ is up to 8 the algorithm A is more preferable than algorithm B because of less running time. The results can be exploited for capacity planning and performance evaluations of real life queues in case of correlated bursty input.

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