

BAYESIAN APPROACH TO MEAN TIME BETWEEN FAILURE USING THE MODULATED POWER LAW PROCESS

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ABSTRACT. The Renewal process and the Non-homogeneous Poisson process (NHPP) process are probably the most popular models for describing the failure pattern of repairable systems. But both these models are based on too restrictive assumptions on the effect of the repair action. For these reasons, several authors have recently proposed point process models which incorporate both renewal type behavior and time trend. One of these models is the Modulated Power Law Process (MPLP). The Modulated Power Law Process is a suitable model for describing the failure pattern of repairable systems when both renewal-type behavior and time trend are present. In this paper we propose Bayes estimation of the next failure time after the system has experienced some failures, that is, Mean Time Between Failure for the MPLP model. Numerical examples illustrate the estimation procedure.

1. PRELIMINARIES

The the renewal process and Nonhomogeneous Poisson Process (NHPP) are models that are often used to describe the random occurrence of events in time. For a renewal process, the times between failure are independent and identically distributed(iid). Since the times between failure are iid, a repaired unit is always brought to a like-new condition. For this reason, the renewal process cannot be used to model a system experiencing deterioration or reliability improvement. The renewal process has also been called the *good-as-new* model.

For an NHPP, the probability of an event in a small interval of time, $(t, t + \Delta t)$, depends only on t and not on the previous pattern of events. The limit,

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{an event in } (t, t + \Delta t))}{\Delta t}, \quad t > 0,$$

is defined the intensity function of an NHPP. If the Poisson process is used to model the failure times of a repairable system, then the fact that λ depends on t and not on the previous pattern of failures means that a failed unit is in exactly the same condition after a repair as it was just before the failure. For this reason, the NHPP is often called the *same-as-old* model when it is applied to a repairable system. The homogeneous

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Poisson Process (HPP), for which the intensity function is a constant and the times between failures are independent and identically exponentially distributed, is a special case of both.

If the intensity function has the form

$$\lambda(t) = \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1}, t > 0,$$

then the process is called the power law process. This process has been called the Weibull process. Typically, the parameters β and θ are unknown and must be estimated from data from one or more systems. Crow (1974) developed many of the inference properties of the power law process. Rigdon and Basu (1989) presented a review of properties of the power law process.

Unfortunately, renewal process and NHPP are based on too restrictive assumptions on the effect of the repair action. For these reasons, several authors have recently proposed point process models which incorporate both renewal type behavior and time trend. The Modulated Power Law Process (MPLP) is compromise between the Renewal process and the NHPP with power intensity function, since its failure probability at a given time t depends both on the age t of a system and on the distance of t from the last failure time. Thus the MPLP is a suitable model for describing the failure pattern of repairable systems when both renewal-type behavior and time trend are present.

Lakey and Rigdon (1992) have introduced the MPLP which is special case of the inhomogeneous gamma process introduced by Berman (1981). Lakey and Rigdon (1992) propose Maximum likelihood point estimators of the three parameters of MPLP. Black and Rigdon (1996) describe an algorithm for obtaining the MLEs of the three parameters and derive the asymptotic variance of these point estimates.

In reliability analysis, it is quite important to be able to determine the next failure time after the system has experienced some failures during the development and test process. In order words, the mean time to the next failure at the n -th observed failure time t_n , or the Mean Time Between Failures at t_n , $MTBF(t_n)$, is of significant interest.

In this paper we derive Mean Time Between Failure (MTBF) for the MPLP model and propose Bayes estimation of Mean Time Between Failure (MTBF) at the n -th failure time. Numerical examples illustrate the estimation procedure.

2. Mean Time Between Failure for Modulated Power Law Process

Suppose that a system failure occurs not at every shock but at every k -th shock, where k is a positive integer and suppose that shocks occur according to the NHPP with intensity function:

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{a shock in } (t, t + \Delta t))}{\Delta t}, t > 0.$$

If for example $k = 3$, then the system failure occur at every third shock. Thus, a failed and repaired system would be in better condition than it was in before the failure

occurrence, since 3 other shocks must occur in order to observe the next failure. Even if the explanation given previously in terms of shocks does not carry over when k is not an integer, the MPLP can be still defined for any positive value of k .

Let $T_1 < T_2 < \dots < T_n$ denote the first n failure times of a failure truncated MPLP. The conditional reliability function of T_i given T_{i-1} , is;

$$\begin{aligned} R(t_i|t_{i-1}) &= \Pr(T_i > t|T_{i-1} = t_{i-1}) \\ &= \Pr(\text{no failure in the interval } [t_{i-1}, t]) \\ &= \Pr(N(t) - N(t_{i-1}) \leq k - 1) \\ &= \sum_{j=0}^{k-1} \frac{(U(t) - U(t_{i-1}))^j}{j!} \exp \left[-U(t) + U(t_{i-1}) \right], \end{aligned}$$

where $U(t) = \int_0^t \lambda(u)du$ is mean value function of NHPP. Thus the conditional pdf of T_i given T_{i-1} can be easily computed from the above conditional reliability function as follows;

$$f(t_i|t_{i-1}) = \frac{1}{\Gamma(k)} \frac{\beta}{\theta} \left(\frac{t_i}{\theta}\right)^{\beta-1} \left[\left(\frac{t_i}{\theta}\right)^\beta - \left(\frac{t_{i-1}}{\theta}\right)^\beta \right]^{k-1} \exp \left[-\left(\frac{t_i}{\theta}\right)^\beta + \left(\frac{t_{i-1}}{\theta}\right)^\beta \right].$$

Then the joint pdf of failure times $T_1 < T_2 < \dots < T_n$, is;

$$f(t_1, t_2, \dots, t_n | \beta, \theta, k) = \frac{1}{(\Gamma(k))^n \theta^{n\beta k}} \exp \left[-\left(\frac{t_n}{\theta}\right)^\beta \right] \prod_{i=1}^n t_i^{\beta-1} \prod_{i=1}^n (t_i^\beta - t_{i-1}^\beta)^{k-1}.$$

When $k = 1$ the MPLP reduces to the PLP, when $\beta = 1$ the process becomes a Gamma renewal process, when $k = 1$ and $\beta = 1$ the MPLP reduces to the HPP. Thus β is a measure of the system improvement or deterioration over the system life, whereas k is a measure of the improvement of worsening introduced by the repair actions.

In reliability analysis, one of important characteristics is the next failure time after the system has experienced some failures. In order words, the mean time to the next failure at the n -th observed failure time t_n . Now we derive Mean Time Between Failure (MTBF) for the MPLP model. Let $T_1 < T_2 < \dots < T_n < \dots$ be the successive system failure times. Then the MTBF at the n -th failure time t_n is defined as;

$$MTBF(t_n) = E[T_{n+1} - T_n | T_n = t_n].$$

In order to express the $MTBF(t_n)$ more explicitly, we need the conditional distribution of T_{n+1} given $T_n = t_n$. From (1), the conditional pdf of T_{n+1} , given $T_n = t_n$, is

$$f_{n+1}(t | t_1, \dots, t_n) = \frac{1}{\Gamma(k)} \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} \left[\left(\frac{t}{\theta}\right)^\beta - \left(\frac{t_n}{\theta}\right)^\beta \right]^{k-1} \exp \left[-\left(\frac{t}{\theta}\right)^\beta + \left(\frac{t_n}{\theta}\right)^\beta \right]$$

Then MTBF at $T_n = t_n$ for the MPLP is given by

$$MTBF(t_n) = E[T_{n+1} - T_n | T_n = t_n]$$

$$\begin{aligned}
&= \int_{t_n}^{\infty} t f_{n+1}(t|t_1, \dots, t_n) dt - t_n \\
&= \int_{t_n}^{\infty} t \frac{1}{\Gamma(k)} \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} \left[\left(\frac{t}{\theta}\right)^{\beta} - \left(\frac{t_n}{\theta}\right)^{\beta} \right]^{k-1} \exp \left[-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_n}{\theta}\right)^{\beta} \right] dt - t_n
\end{aligned}$$

Using integral by parts, we get

$$MTBF(t_n) = \frac{\beta}{\Gamma(k)} \int_{t_n}^{\infty} \left(\frac{t}{\theta}\right)^{\beta} \left[\left(\frac{t}{\theta}\right)^{\beta} - \left(\frac{t_n}{\theta}\right)^{\beta} \right]^{k-1} \exp \left[-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_n}{\theta}\right)^{\beta} \right] dt - t_n.$$

Let $y = (t/\theta)^{\beta} - (t_n/\theta)^{\beta}$, then

$$(1.3) \quad MTBF(t_n) = \frac{\theta}{\Gamma(k)} \int_0^{\infty} y^{k-1} \left[y + \left(\frac{t_n}{\theta}\right)^{\beta} \right]^{\frac{1}{\beta}} \exp(-y) dy - t_n.$$

3. Bayes Estimation

Let $t_1 < t_2 < \dots < t_n$ denote the first n failure times of a failure truncated MPLP sample. From (2), the likelihood function, given failure times $t_1 < t_2 < \dots < t_n$, is;

$$L(\beta, \theta, k|t_1, t_2, \dots, t_n) = \frac{1}{(\Gamma(k))^n \theta^n \beta^k} \exp \left[-\left(\frac{t_n}{\theta}\right)^{\beta} \right] \prod_{i=1}^n t_i^{\beta-1} \prod_{i=1}^n (t_i^{\beta} - t_{i-1}^{\beta})^{k-1}.$$

Using arguments in Calabria and Pulcini (1997), we have Jeffrey's (1961) priors as follows;

$$\begin{aligned}
\pi(\beta) &= \frac{1}{\beta} \\
\pi(\theta) &= \frac{1}{\theta} \\
\pi(k) &= \sqrt{\psi'(k)},
\end{aligned}$$

where $\psi'(k) = d^2 \ln \Gamma(k) / dk^2$ is the tri-gamma function. Using the *a priori* independence assumption, the non-informative joint prior is

$$\pi(\beta, \theta, k) = \frac{\sqrt{\psi'(k)}}{\beta \theta}$$

and the corresponding joint posterior pdf results in;

$$\pi(\beta, \theta, k|t_1, \dots, t_n) = \frac{1}{C} \frac{\sqrt{\psi'(k)}}{(\Gamma(k))^n \theta^n \beta^{k+1}} \exp \left[-\left(\frac{t_n}{\theta}\right)^{\beta} \right] \prod_{i=1}^n t_i^{\beta-1} \prod_{i=1}^n (t_i^{\beta} - t_{i-1}^{\beta})^{k-1}$$

where the denominator

$$C = \int_0^{\infty} \int_0^{\infty} g(0, 0) dk d\beta$$

and

$$g(a, b) = k^a \frac{\beta^{n-2+b} \sqrt{\psi'(k)} \Gamma(nk)}{t_n^{n\beta k} (\Gamma(k))^n} \prod_{i=1}^n t_i^{\beta-1} \prod_{i=1}^n (t_i^\beta - t_{i-1}^\beta)^{k-1}.$$

The marginal posterior pdf of β is

$$\pi(\beta|t_1, \dots, t_n) = \frac{1}{C} \int_0^\infty g(0, 0) dk,$$

and the posterior mean is

$$(1.4) \quad E(\beta|t_1, \dots, t_n) = \frac{1}{C} \int_0^\infty \int_0^\infty g(0, 1) dk d\beta.$$

The marginal posterior pdf of θ is

$$\pi(\theta|t_1, \dots, t_n) = \frac{1}{C} \frac{1}{\theta} \int_0^\infty \beta^{n-1} \prod_{i=1}^n t_i^{\beta-1} \exp \left[- \left(\frac{t_n}{\theta} \right)^\beta \right] \int_0^\infty \frac{\sqrt{\psi'(k)}}{(\Gamma(k))^n} \frac{1}{\theta^{n\beta k}} \prod_{i=1}^n (t_i^\beta - t_{i-1}^\beta)^{k-1} dk d\beta$$

and the posterior mean is

$$E(\theta|t_1, \dots, t_n) = \frac{1}{C} \int_0^\infty \beta^{n-2} \prod_{i=1}^n t_i^{\beta-1} \int_0^\infty \frac{\sqrt{\psi'(k)} \Gamma(nk - 1/\beta) \prod_{i=1}^n (t_i^\beta - t_{i-1}^\beta)^{k-1}}{(\Gamma(k))^n t_n^{n\beta k - 1}} dk d\beta.$$

Finally, the marginal posterior pdf of k is

$$\pi(k|t_1, \dots, t_n) = \frac{1}{C} \int_0^\infty g(0, 0) d\beta,$$

and the posterior mean is

$$(1.6) \quad E(k|t_1, \dots, t_n) = \frac{1}{C} \int_0^\infty \int_0^\infty g(1, 0) dk d\beta.$$

We now propose a Bayes estimator of $MTBF(t_n)$ from (3):

$$(1.7) \quad \widehat{MTBF}_B(t_n) = \frac{\hat{\theta}}{\Gamma(\hat{k})} \int_0^\infty y^{\hat{k}-1} \left[y + \left(\frac{t_n}{\hat{\theta}} \right)^{\hat{\beta}} \right]^{\frac{1}{\hat{\beta}}} \exp(-y) dt - t_n.$$

where $\hat{\theta}$, $\hat{\beta}$ and \hat{k} are the Bayes estimator of θ , β and k , obtained from (4), (5) and (6), under square error lose, respectively.

4. Examples

Two numerical examples illustrate the proposed estimation procedure. The first is the failure times of an aircraft generator taken from Duane (1964). These failure times have been read from a plot in Duane's paper by Black and Rigdon (1996). For this system there were $n = 14$ failures, and these failure times are shown in Table 4.1.

Table 4.1. Failure Times for Aircraft Generator in Duane (1964)

10	55	166	205	341	488	567	731	1308	2050	2453	3115	4017	4596
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The point estimates for the parameters are $\hat{\theta} = 1.30$, $\hat{\beta} = 0.419$ and $\hat{k} = 4.20$. Thus, from (7), the estimate of MTBF is $\widehat{MTBF}_B(4596) = 653$ and the estimated mean time to the next failure at $t = 4596$ is 5249.

The second example consists of the failure times from the second aircraft airconditioning unit from Proschan (1963). The failure times are shown in Table 4.2.

Table 4.2. Failure Times of Aircraft Airconditioning Equipment given by Proschan (1963)

413	427	485	522	622	687	696	865	1496	1532	1733
1851	1885	1916	1934	1952	2019	2076	2145	2167	2201	

The point estimates for the parameters are $\hat{\theta} = 340.4$, $\hat{\beta} = 2.15$ and $\hat{k} = 1.4$. Thus $\widehat{MTBF}_B(2201) = 67$, that is, the mean time to the next failure at $t = 2201$ is 2268.

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