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HOW TO SOLVE AN INFINITE SIMULTANEOUS SYSTEM OF QUADRATIC EQUATIONS

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Abstract. In the present paper we shall introduce several operators on the reproducing kernel spaces. And using them we shall find a solution of an infinite system of quadratic equations (1.1). In particular we shall convert problem for finding an approximate solution of infinite system of quadratic equations into problem for minimizing nonnegative biquadratic polynomial.

1. Introduction

Throughout this paper we shall concern with an infinite system of quadratic equations

$$\begin{cases} \sum_{i=1}^{\infty} a_{ii,k}x_i^2 + \sum_{i\neq j}^{\infty} a_{ij,k}x_ix_j = b_k, \ (k \in \mathbb{N}) \\ x_1 = x_0(\text{constant}), \end{cases} \tag{1.1}$$

where $X = (x_1, x_2, \cdots) \in \ell^2$, $b = (b_1, b_2, \cdots) \in \ell^2$, $\sum_{i,j=1}^{\infty} (a_{ij,k})^2 < +\infty, (k \in \mathbb{N})$, and $a_{ij,k} = a_{ji,k}$ for all $i, j \in \mathbb{N}$. If we informally introduce $\infty \times \infty$ symmetric matrix

$$A_k = (a_{ij,k})_{\infty\times\infty}, \ (k \in \mathbb{N}),$$

then (1.1) can be transformed into

$$\begin{cases} (X, A_kX^T)_{\ell^2} = b_k, \ (k \in \mathbb{N}) \\ x_1 = x_0 \end{cases} \tag{1.2}$$

and vice versa, where $(\cdot,\cdot)_{\ell^2}$ denotes the standard inner product in ℓ^2-space.

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The purpose of the present paper is to find a solution of an infinite system of quadratic equations (1.1). In particular we shall convert problem for finding an approximate solution of infinite system of quadratic equations into problem for minimizing nonnegative biquadratic polynomial.

2. Preliminaries

The reproducing kernel space $W_2^1[0,1]$ is defined as the set of functions

$$W_2^1[0,1] = \{u(t) | u \text{ is absolutely continuous and } u, u' \in L^2[0,1]\},$$

equipped with the inner product

$$(u,v)_{W_2^1} = \int_0^1 (u(t)v(t) + u'(t)v'(t))\,dt$$

and with norm

$$\|u\|^2_{W_2^1} = (u,u)_{W_2^1}.$$

The reproducing kernel of $W_2^1[0,1]$ can be given by

$$R_\eta(t) = \frac{1}{2(e^2-1)}\left(e^{t+\eta} + e^{2-(t+\eta)} + e^{|t-\eta|} + e^{2-|t-\eta|}\right), \tag{2.3}$$

for each $t, \eta \in [0,1]$, which satisfies the reproducing property

$$(u(t), R_\eta(t))_{W_2^1} = u(\eta) \tag{2.4}$$

for every $u \in W_2^1[0,1]$.

Let $D = [0,1] \times [0,1]$. The reproducing kernel space $W(D)$ is defined as the set of functions

$$W(D) = \left\{ u(s,t) | u \text{ is complete continuous}, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s \partial t} \in L^2(D) \right\},$$

equipped with the inner product

$$(u,v)_{W(D)} = \int\int_D \left(uv + \frac{\partial u}{\partial s}\frac{\partial v}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial s \partial t}\frac{\partial^2 v}{\partial s \partial t} \right) dsdt$$

and with norm

$$\|u\|^2_{W(D)} = (u,u)_{W(D)}.$$

The reproducing kernel of $W(D)$ can be given by

$$K_{(\xi,\eta)}(s,t) = R_\xi(s) R_\eta(t), \tag{2.5}$$

where $R_\xi(t)$ is given by (2.3) [1].

3. Linear operators on reproducing kernel spaces

In this section we shall introduce several operators, which will be needed in the later parts of our discussion. Throughout the present paper we shall choose and fix a countable dense subset

$$T = \{t_1, t_2, \cdots\} \tag{3.6}$$

of the interval $[0, 1]$, and put

$$\phi_i(t) \overset{\text{def}}{=} R_{t_i}(t), \ i \in \mathbb{N}. \tag{3.7}$$

Lemma 3.1. *The sequence of functions $\{\phi_i(t)\}_{i=1}^{\infty}$ constitutes a complete system of $W_2^1[0, 1]$.*

Proof. Let $u(t) \in W_2^1[0, 1]$. Since $(u(\cdot), \phi_i(\cdot))_{W_2^1} = u(t_i)$ for each $i \in \mathbb{N}$, we have $(u(\cdot), \phi_i(\cdot)) = 0$ if and only if $u(t_i) = 0$ if and if $u(t) = 0$, which proves our assertion. □

Using Gram-Schmidt process, we orthonormalize $\{\phi_i(t)\}_{i=1}^{\infty}$ to obtain an orthonormal system $\{\bar{\phi}_i(t)\}_{i=1}^{\infty}$ for $W_2^1[0, 1]$,

$$\bar{\phi}_i(t) \overset{\text{def}}{=} \sum_{l=1}^{i} \alpha_{il}\phi_l(t),$$

where α_{il} are the orthonormal coefficients.

We shall define an operator $\rho : \ell^2 \longrightarrow W_2^1[0, 1]$ by

$$\rho X \overset{\text{def}}{=} \sum_{i=1}^{\infty} x_i\bar{\phi}_i(t) \tag{3.8}$$

for each $X = (x_1, x_2, \cdots) \in \ell^2$. It is easy to show that ρ is one-to-one and norm preserving. It is noteworthy that (1.2) can be converted into

$$\big(u(\cdot), (\rho A_k\rho^{-1}u)(\cdot)\big)_{W_2^1} = b_k, \ (k \in \mathbb{N}) \tag{3.9}$$

where $u(t) = \rho X$.

Again we shall define an operator $\tilde{A}_k : W_2^1[0, 1] \longrightarrow W_2^1[0, 1]$ by

$$\left(\tilde{A}_k u\right)(t) \overset{\text{def}}{=} \left(\rho A_k\rho^{-1}u\right)(t), \quad k \in \mathbb{N}$$

for each $u(t) \in W_2^1[0, 1]$. Thus (3.9) can be converted into

$$\left(u(\cdot), (\tilde{A}_k u)(\cdot)\right)_{W_2^1} = b_k, \quad k \in \mathbb{N}$$

that is,

$$\int_0^1 \left[u(t)(\tilde{A}_k u)(t) + u'(t)(\tilde{A}_k u)'(t)\right] dt = b_k.$$

Let I and D be the identity and differential operators on $W_2^1[0,1]$ respectively. For each $k \in \mathbb{N}$, we shall define an operator $H_k : W(D) \longrightarrow \mathbb{R}$ by

(3.10)
$$H_k v \stackrel{\text{def}}{=} \int_0^1 \left[\left(\tilde{A}_k^{(\cdot)} I^{(*)} + (D\tilde{A}_k)^{(\cdot)} D^{(*)}\right) v(*,\cdot)\right](t)dt, \quad v \in W(D),$$

where " $\cdot$ " and " $*$ " denote the variables corresponding to function respectively.

4. Operator equation associated with (1.2)

We shall introduce an operator $L : W(D) \to W_2^1[0,1]$ defined by

$$(Lv)(t) \stackrel{\text{def}}{=} \rho((H_1 v, H_2 v, \cdots)) = \sum_{k=1}^{\infty} (H_k v)\bar{\phi}_k(t), \quad v \in W(D) \tag{4.11}$$

where H_k and ρ are given by (3.10) and (3.8) respectively. In fact, since L is a composition of bounded linear operators I, $\tilde{A}_k$, D, ρ and integral, we have L is also bounded linear operator.

Lemma 4.1. *Let $X = (x_1, x_2, \cdots) \in \ell^2$, and let $\rho X = u(t)$. Then we have*

$$u(t_1) = x_1 \|\phi_1\|_{W_2^1}. \tag{4.12}$$

Proof. Since $\{\bar{\phi}_i(t)\}_{i=1}^{\infty}$ is an orthonormal system of $W_2^1[0,1]$, we have, by virtue of (2.4), (3.7), and (3.8),

$$\begin{aligned} u(t_1) &= (u(\cdot), \phi_1(\cdot))_{W_2^1} \\ &= \|\phi_1\| \left(u(\cdot), \bar{\phi}_1(\cdot)\right)_{W_2^1} \\ &= \|\phi_1\| \left(\sum_{i=1}^{\infty} x_i \bar{\phi}_i(\cdot), \bar{\phi}_1(\cdot)\right)_{W_2^1} \\ &= x_1 \|\phi_1\|_{W_2^1}. \end{aligned}$$

□

THEOREM 4.2. *Let* $X = (x_1, x_2, \cdots) \in \ell^2$ *and* $x_1 \stackrel{\text{def}}{=} x_0$*, and let* $\rho X = u(t)$*. Then* X *is a solution of* (1.2) *if and only if* $u(s)u(t) \in W(D)$ *is a solution of*

$$(Lv)(t) = f(t), \tag{4.13}$$

where $f(t) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} b_k \bar{\phi}_k(t)$.

Proof. Suppose that X is a solution of (1.2). Then we have

$$\begin{aligned}
&(Lu(*)u(\cdot))(t)\\
&= \rho((H_1 u(*)u(\cdot), H_2 u(*)u(\cdot), \cdots))\\
&= \rho\left(\left(\int_0^1 \left[u(t)(\rho A_1 \rho^{-1} u)(t) + u'(t)(\rho A_1 \rho^{-1} u)'(t)\right] dt, \cdots\right)\right)\\
&= \rho\left(\left((u(\cdot), (\rho A_1 \rho^{-1} u)(\cdot))_{W_2^1}, \cdots\right)\right)\\
&= \rho\left(\left((X, A_1 X^T)_{\ell^2}, \cdots\right)\right)\\
&= \rho\left((b_1, \cdots)\right)\\
&= f(t).
\end{aligned}$$

Conversely suppose that $u(s)u(t)$ is a solution of (4.13). Then we have

$$\rho((u(t), (\rho A_1 \rho^{-1} u)(t))_{W_2^1}, \cdots)) = \rho((b_1, b_2, \cdots)).$$

Since ρ is one-to-one and norm preserving, we obtain, by virtue of Lemma 4.1,

$$((X, A_1 X^T), \cdots) = (b_1, \cdots), \quad x_1 = x_0.$$

Hence our assertion is proved. □

5. Direct sum decomposition of $W(D)$

Using the adjoint operator L^* of L defined by (4.11), we define $\psi_i(s,t)$ by

$$\psi_i(s,t) \stackrel{\text{def}}{=} (L^* \phi_i)(s,t) = (L^* R_{t_i})(s,t), \ (i \in \mathbb{N}). \tag{5.14}$$

LEMMA 5.1. *A function* $\psi_i(s,t)$*, defined above, can be expressed by*

$$\psi_i(s,t) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij,k} \bar{\phi}_i(s) \bar{\phi}_j(t) \right) \bar{\phi}_k(t_i), \ (i \in \mathbb{N}) \tag{5.15}$$

Proof. By virtue of (2.4), (2.5), (3.7), (3.10), and (4.11), we have

$$\begin{aligned}\psi_i(s,t) &= ((L^*\phi_i)(*,\cdot), R_s(*)R_t(\cdot))_{W(D)}\\ &= (\phi_i(\diamond), (LR_s(*)R_t(\cdot))(\diamond))_{W_2^1}\\ &= (LR_s(*)R_t(\cdot))(t_i)\\ &= \sum_{k=1}^{\infty}\left(R_s(\cdot), (\tilde{A}_k R_t)(\cdot)\right)_{W_2^1}\bar{\phi}_k(t_i)\\ &= \sum_{k=1}^{\infty}\left(\rho A_k \rho^{-1} R_t\right)(s)\bar{\phi}_k(t_i).\end{aligned}$$

On the other hand, since

$$R_t(s) = \sum_{k=1}^{\infty}(R_t(\cdot), \bar{\phi}_k(\cdot))_{W_2^1}\bar{\phi}_k(s) = \sum_{k=1}^{\infty}\bar{\phi}_k(t)\bar{\phi}_k(s)$$

we have

$$\rho^{-1}R_t(s) = (\bar{\phi}_1(t), \bar{\phi}_2(t), \cdots) \in \ell^2,$$

hence

$$(\rho A_k \rho^{-1} R_t)(s) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij,k}\bar{\phi}_j(t)\right)\bar{\phi}_i(s).$$

Thus we obtain the desired result

$$\psi_i(s,t) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij,k}\bar{\phi}_j(t)\bar{\phi}_i(s)\right)\bar{\phi}_k(t_i).$$

□

By Gram-Schmidt process, we obtain an orthonormal system

$$\{\bar{\psi}_i(s,t)\}_{i=1}^{\infty}$$

of $W(D)$ such that

$$\bar{\psi}_i(s,t) \stackrel{\text{def}}{=} \sum_{k=1}^{i}\beta_{ik}\psi_k(s,t), \tag{5.16}$$

where β_{ik} are orthonormal coefficients.

Let S be the closure of span $\big(\{\bar{\psi}_i(s,t)\}_{i=1}^{\infty}\big)$ and let $S^{\perp}$ be the orthogonal complement of S in $W(D)$. We choose a countable dense subset $B = \{(s_1,t_1),(s_2,t_2),\cdots\}$ of D. It is easy to show that

$$\rho_j(s,t) \overset{\text{def}}{=} R_{s_j}(s)R_{t_j}(t), \quad j \in \mathbb{N}$$

constitutes a basis of the space $W(D)$. Again we orthonormalize

$$\{\bar{\psi}_1, \bar{\psi}_2, \cdots, \rho_1, \rho_2, \cdots\}$$

to obtain

$$\bar{\rho}_j(s,t) = \frac{\rho_j(s,t) - \sum_{k=1}^{\infty}(\rho_j,\bar{\psi}_k)\bar{\psi}_k - \sum_{m=1}^{j-1}(\rho_j,\bar{\rho}_m)\bar{\rho}_m}{\|\rho_j(s,t) - \sum_{k=1}^{\infty}(\rho_j,\bar{\psi}_k)\bar{\psi}_k - \sum_{m=1}^{j-1}(\rho_j,\bar{\rho}_m)\bar{\rho}_m\|_{W(D)}}, \; j \in \mathbb{N},$$

that is,

$$\bar{\rho}_j(s,t) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \beta_{jk}\bar{\psi}_k(s,t) + \sum_{m=1}^{j} \beta^*_{jm}\rho_m(s,t), \; j \in \mathbb{N}. \tag{5.17}$$

Hence we have $W(D) = S \oplus S^{\perp}$, and $\{\bar{\psi}_1, \bar{\psi}_2, \cdots, \bar{\rho}_1, \bar{\rho}_2, \cdots\}$ constitutes an orthonormal basis for $W(D)$.

6. A separated type solution of $(Lv)(t) = f(t)$

Theorem 4.2 tells us that finding a solution of (1.2) is equivalent to finding a separated type solution of (4.13).

THEOREM 6.1. *Let* $\lambda = (\lambda_1, \lambda_2, \cdots)$ *and* $(\alpha_{1k}, \alpha_{2k}, \cdots)$ *be arbitrary constant in* ℓ^2 *for each* $k \in \mathbb{N}$. *With the same notation of* (5.16),

$$v(s,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \alpha_{ik} f(t_k)\bar{\psi}_k(s,t) + \sum_{j=1}^{\infty} \lambda_j \bar{\rho}_j(s,t) \tag{6.18}$$

is a solution of (4.13), *where* $f(t) \overset{\text{def}}{=} \sum_{k=1}^{\infty} b_k \bar{\phi}_k(t)$.

Proof. Taking L of both sides of (6.18), we have

$$(Lv)(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \alpha_{ik} f(t_k)(L\bar{\psi}_i)(t) + \sum_{j=1}^{\infty} \lambda_j (L\bar{\rho}_j)(t).$$

Let T be the same set as (3.6). For every $t_l \in T$, we have

$$(L\bar{\rho}_j)(t_l) = (L\bar{\rho}_j, \phi_l)_{W_2^1} = (\bar{\rho}_j, L^*\phi_l)_{W(D)} = (\bar{\rho}_j, \psi_l)_{W(D)} = 0.$$

Hence

$$\begin{aligned}(Lv)(t_l) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)(L\bar{\psi}_i)(t_l)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)(L\bar{\psi}_i,\phi_l)_{W_2^1}\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)(\bar{\psi}_i,\psi_l)_{W(D)}.\end{aligned}$$

Multiplying both sides of the above equality by β_{nl} and summing with respect to $l, (1 \le l \le n)$, we have, in the view of (5.16),

$$\begin{aligned}\sum_{l=1}^{n}\beta_{nl}(Lv)(t_l) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)(\bar{\psi}_i,\bar{\psi}_n)_{W_2^1}\\ &= \sum_{k=1}^{n}\alpha_{nk}f(t_k).\end{aligned}$$

We claim that$(Lv)(t_m) = f(t_m)$ holds for all $m \in \mathbb{N}$. For $n = 1$, it is easy to show that $(Lv)(t_1) = f(t_1)$. For induction, we assume that $(Lv)(t_n) = f(t_n)$ holds for $n \le m$. Since

$$\sum_{l=1}^{m+1}\alpha_{m+1,l}(Lv)(t_l) = \sum_{k=1}^{m+1}\alpha_{m+1,k}f(t_k)$$

and

$$\sum_{l=1}^{m}\alpha_{m+1,l}f(t_l) + \alpha_{m+1,m+1}(Lv)(t_{m+1}) = \sum_{k=1}^{m+1}\alpha_{m+1,k}f(t_k),$$

we have

$$(Lv)(t_{m+1}) = f(t_{m+1}).$$

Hence $(Lv)(t_m) = f(t_m)$ holds for every $t_m \in T$. Since T is dense in $[0, 1]$, we conclude $(Lv)(t) = f(t)$ holds for all $t \in [0, 1]$. Therefore our assertion is proved. □

LEMMA 6.2. *If $v(s, t)$ of* (6.18) *is expressible as a separated type $u(s)u(t)$, then we have*

(i) $v(t_1, t) = x_0\|\phi_1\|u(t)$
(ii) $v(t, t) = u^2(t)$

THEOREM 6.3. *If $v(s,t)$ of (6.18) is expressible as a separated type $u(s)u(t)$, then we have*

$$u(t) = \frac{1}{x_0\|\phi_1\|}\left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)\bar{\psi}_i(t_1,t) + \sum_{j=1}^{\infty}\lambda_j\bar{\rho}_j(t_1,t)\right], \tag{6.19}$$

where $f(t) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} b_k\bar{\phi}_k(t)$.

Proof. We have, by virtue of (6.18),

$$u(s)u(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}f(t_k)\bar{\psi}_i(s,t) + \sum_{j=1}^{\infty}\lambda_j\bar{\rho}_j(s,t).$$

Putting $s = t_1$ and dividing both sides of the above by $u(t_1)$, we have the required result by Lemma 6.1. □

Remark : If we take partial sum of (6.19) to get an approximation $u_{nm}(t)$ of $u(t)$, then we have

$$u_{nm}(t) = \frac{1}{x_0\|\phi_1\|}\left[\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(t_k)\bar{\psi}_i(t_1,t) + \sum_{j=1}^{m}\lambda_j\bar{\rho}_j(t_1,t)\right]$$

for each $m,n \in \mathbb{N}$. In order to obtain u_{nm}, we have to determine the values of $\lambda_1,\cdots,\lambda_m$. To do so, it suffices to find $\lambda_1,\cdots,\lambda_m$ so that they may minimize $G \stackrel{\text{def}}{=} \|v_{nm}(t,t) - u_{nm}^2(t)\|^2_{W_2^1}$, where $v_{nm}(t,t)$ is a partial sum of (6.18) in correspondence with $u_{nm}(t)$. Fortunately G is a biquadratic polynomial with respect to $\lambda_1,\cdots,\lambda_m$, of which optimization problem is familiar to us. In the present paper we converted problem for finding an approximate solution of infinite system of quadratic equations into problem for minimizing biquadratic polynomial. Running Mathematica 4.2 for a concrete example, it can be easily confirmed that our result is effective.

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