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ON COTYPE AND SUMMING PROPERTIES FOR BANACH SPACE OPERATORS

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Abstract. We characterize Gaussian cotype X operators acting between Banach spaces, where X is a Banach sequence space. Further we give an extensive presentation of results on the connections between cotype and summing operators.

1. Introduction

The theory of type and cotype reflects the interplay between geometry and probability in Banach spaces. In particular, considerable effort has been expended on the precise determination of the cotype nature of certain classes of operators in analysis. The tightness of the relationship between cotype and summing operators has been spotlighted.

B. Maurey [5] described Rademacher cotype q operators acting on Banach lattices as follows : Let $2 < q < \infty$. The following are equivalent statements about an operator T from a Banach lattice L to a Banach space F.

(i) T is $(q, 1)$-summing.
(ii) T is (q, r)-concave for all $1 \leq r < q$.
(iii) T is of Rademacher cotype q.
(iv) There is a constant C such that for all choices of finitely many disjoint vectors $x_1, \cdots, x_n$ from L we have

$$(\sum_1^n \|Tx_k\|^q)^{1/q} \leq C \cdot \| \sum_1^n x_k \|.$$

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Afterwards an alternative approach to this kind of problem was proposed by G. Pisier [9]. He proved that a $(q, 1)$-summing operator acting from a $C(K)$ space to a Banach space admits a factorization through the Lorentz space $L_{q,1}(\mu)$ for some probability measure μ on K.

Thus concerning operators acting from a $C(K)$ space to a Banach space, the prototype of $(2, 1)$-summing operators is the canonical embedding from $C([0, 1])$ to the Lorentz space $L_{2,1}$. M. Talagrand [12] showed that this canonical embedding is not even of Gaussian cotype 2.

Treading the same path of ideas S.J. Montgomery-Smith [6] implicitly rediscovered Pisier's criteria and proved that for a probability measure μ on K the canonical embedding from a $C(K)$ space to the Lorentz-Orlicz space $L_{t^2 \log t,2}(\mu)$ is of Gaussian cotype 2.

The following Talagrand's characterization of Gaussian cotype q operators acting on $C(K)$ spaces [13] built on previous work of S.J. Montgomery-Smith [6], [7].

Let $2 < q < \infty$. The following statements about the operator T from a $C(K)$ space to a Banach space F are equivalent.

(i) T is of Gaussian cotype q.

(ii) T factors through the Lorentz-Orlicz space $L_{t^q(\log t)^{q/2},1}(\mu)$ for some probability measure μ on K.

(iii) For each sequence (x_k) in $C(K)$ satisfying $\| \sum_k |x_k| \|_\infty \leq 1$, there exists a constant C such that

$$\sum_k \frac{\|Tx_k\|^q}{\left(\log(\frac{C}{\|Tx_k\|})\right)^{q/2}} \leq C.$$

Talagrand's result was complemented by M. Junge [2]. He proved that an operator T from a $C(K)$ space to a Banach space F is of Gaussian cotype q if and only if

$$\Big(\sum_k \frac{\|Tx_k\|^q}{(\log(k+1))^{q/2}}\Big)^{1/q} \leq C \, \Big(\int_0^1 \| \sum_k r_k(t)x_k\|^2 dt\Big)^{1/2},$$

for all sequences (x_k) in $C(K)$ with $(\|Tx_k\|)$ decreasing.

In this paper we survey the behaviour of cotype operators in connection with the summability property. Here, we present M. Junge's approach to this subject [2].

The concepts of (X, r)-summing and cotype X operators are extended notions of (q, r)-summing and cotype q operators to the setting of Banach sequence space X.

We first establish a description of Gaussian cotype X operators acting between Banach spaces in terms of $(X, 2)$-summing operators, which is a generalization of the result due to N. Tomczak-Jaegermann [14]. And then we provide usable necessary condition which implies that an operator is of Gaussian cotype X.

Next by using Junge's proof, we extend the above mentioned result of Maurey to the framework of maximal Banach sequence space.

Finally we see how the Gaussian cotype X operators on $C(K)$ spaces are linked with the Rademacher cotype X operators on $C(K)$ spaces.

2. Definitions and Notation

We give some of the definitions and notation to be used. Throughout this paper E and F denote Banach spaces with duals E^* and F^* respectively.

By a Banach sequence space we mean a real Banach lattice on the set of positive integers.

A Banach sequence space X is called symmetric if

$$\|(x_n)\|_X = \|(x_n^*)\|_X,$$

where (x_n^*) denotes the decreasing rearrangement of the sequence (x_n).

A Banach sequence space X is said to be maximal if the unit ball of X is closed in the pointwise convergence topology induced by the space ω of all real sequences.

The Köthe dual X^+ of a Banach sequence space X is

$$X^+ = \{\sigma = (\sigma_n) \in \omega : \sum |\sigma_n \tau_n| < \infty \quad \text{for all} \quad \tau = (\tau_n) \in X\}.$$

Note that X^+ is a Banach sequence space under the norm

$$\|\sigma\|_+ = \sup\{\sum |\sigma_n \tau_n| : \|\tau\|_X \leq 1\},\ \sigma \in X^+.$$

We let $\mathcal{M}(X,Y)$ denote the space of multipliers from a Banach sequence space X to a Banach sequence space Y, that is $\mathcal{M}(X,Y)$ consists of all scalar sequences σ such that the associated multiplication operator M_σ is defined and bounded from X to Y. $\mathcal{M}(X,Y)$ is a Banach sequence space equipped with the norm $\|\sigma\|_{\mathcal{M}(X,Y)} = \sup\{\|\sigma\tau\|_Y : \|\tau\|_X \le 1\}$.

For $1 \le p < \infty$, a Banach lattice L is called p-convex if there is a constant C such that irrespective of the finite collection of vectors $x_1, \cdots, x_n \in L$,we have $\|(\sum_{k=1}^n |x_k|^p)^{1/p}\| \le C \cdot (\sum_{k=1}^n \|x_k\|^p)^{1/p}$. The least such C is denoted by $K^p(L)$.

If L is p-convex and $K^p(L) = 1$ then $L_{(p)} = \{x : |x|^{1/p} \in L\}$ endowed with the norm $\|x\|_{L_{(p)}} = \||x|^{1/p}\|_L^p$, $x \in L_{(p)}$, is a Banach lattice.

Let $1 \le p < \infty$ and let X be a Banach sequence space. An operator T from a Banach lattice L to a Banach space F is called (X,p)-concave if there is a constant C such that for any choice of finitely many vectors $x_1, \cdots, x_n \in L$,we have $\|\sum_{k=1}^n \|Tx_k\| e_k\|_X \le C \cdot \|(\sum_{k=1}^n |x_k|^p)^{1/p}\|$. We write $K_{X,p}(T)$ for the least constant C that works.

Notation

1. The sequence of unit vectors in ℓ_∞ is denoted by (e_n).
2. $\mathcal{B}(E,F)$ denotes the set of all bounded linear operators from E into F.
3. M_σ denotes the multiplication operator induced by σ.
4. $C(K)$ denotes the space of all continuous functions defined on a compact Hausdorff space K.
5. The closed unit ball of E is denoted by B_E.
6. For $1 < p < \infty$, the conjugate of p is denoted by p', i.e.,$1/p + 1/p' = 1$.

For a sequence (x_n) in E we write

$$\|(x_n)\|_p^{\text{weak}} = \sup\{(\sum_n |\langle x^*, x_n\rangle|^p)^{1/p} : x^* \in B_{E^*}\}.$$

Let $1 \le q \le p < \infty$. For $T \in \mathcal{B}(E,F)$, we set

$$\nu_{q,p}(T) = \inf\{(\sum_n \|x_n^*\|^q)^{1/q} \cdot \|(y_n)\|_{p'}^{\text{weak}}\},$$

where the infimum is taken over all representations $T = \sum_n x_n^* \otimes y_n$ with (x_n^*) in E^* and (y_n) in F. We say that $T \in \mathcal{B}(E,F)$ is (q,p)-nuclear if it can be written in the form $T = \sum_n T_n$, with $T_n \in \mathcal{B}(E,F)$ such that $\sum_n \nu_{q,p}(T_n) < \infty$. Define the (q,p)-nuclear norm of T by $\hat{\nu}_{q,p}(T) = \inf\{\sum_n \nu_{q,p}(T_n)\}$, where the infimum is taken over all finite representations of T as above.

Let $\{g_k\}$ be a sequence of identically distributed independent standard Gaussian random variables on a probability space (Ω, Σ, P).

The sequence of Rademacher functions $(r_n(t))$ on [0,1] is defined by $r_n(t) = \text{sign}\ (\sin 2^n \pi t)$ and is a sequence of independent identically distributed random variables taking the values ± 1 with probability 1/2.

An operator $T \in \mathcal{B}(E,F)$ is called γ-summing if there is a constant C such that regardless of the natural number n and regardless of the choice of $x_1, \cdots, x_n$ in E, we have

$(\int_\Omega \| \sum_{k=1}^n g_k(\omega) T x_k \|^2 dP(\omega))^{1/2} \leq C\ \|(x_k)_1^n\|_2^{\text{weak}}$.

The infimum of such C is denoted by $\pi_\gamma(T)$. We shall write $\Pi_\gamma(E,F)$ for the set of all γ-summing operators from E to F.

Let $1 \leq p < \infty$ and let X be a Banach sequence space. For any operator $T \in \mathcal{B}(E,F)$ we define $\pi_{X,p}^n(T) = \inf C$, where the infimum is taken over all constants C such that for any vectors $x_1, \cdots, x_n$ in E, $\| \sum_1^n \|T x_k\|\, e_k \|_X \leq C \cdot \|(x_k)_1^n\|_p^{\text{weak}}$. We say that an operator T is (X,p)-summing if $\pi_{X,p}(T) = \sup_n \pi_{X,p}^n(T) < \infty$.

Let X be a Banach sequence space. For an operator $T \in \mathcal{B}(E,F)$, $\text{rc}_X^n(T)$ is the smallest constant C such that for any vectors $x_1, \cdots, x_n$ in E we have

$$\| \sum_1^n \|T x_k\| e_k \|_X \leq C \cdot (\int_0^1 \| \sum_1^n r_k(t) x_k \|^2 dt)^{1/2}.$$

An operator T is said to be of Rademacher cotype X if $\text{rc}_X(T) = \sup_n \text{rc}_X^n(T) < \infty$.

Let X be a Banach sequence space. For any operator $T \in \mathcal{B}(E,F)$ we define $\text{gc}_X^n(T) = \inf C$, where the infimum is taken over all constants C such that for any vectors $x_1, \cdots, x_n$ in E,

$$\| \sum_1^n \|T x_k\| e_k \|_X \leq C \cdot (\int_\Omega \| \sum_{k=1}^n g_k(\omega) x_k \|^2 dP(\omega))^{1/2}.$$

We say that an operator T is of Gaussian cotype X if $\mathrm{gc}_X(T) = \sup_n \mathrm{gc}_X^n(T) < \infty$.

Let $f : (0, \infty) \to (0, \infty)$ be a continuous function with $f(1) = 1$ and $\sup_{s>0} \frac{f(ts)}{f(s)} < \infty$ for every $t > 0$. The Lorentz-Marcinkiewicz sequence space $\ell_{f,q}$ consists of all bounded sequences of scalars $\sigma = (\sigma_n)$ having a finite quasi-norm

$$\|\sigma\|_{f,q} = \begin{cases} \left(\sum_n (f(n)\sigma_n^*)^q \, n^{-1}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_n \left[f(n)\sigma_n^*\right] & \text{if } q = \infty. \end{cases}$$

For $0 < p, q \leq \infty$, $-\infty < \upsilon < \infty$ and $f(t) = t^{1/p}(\log(t+1))^{\upsilon}$ we get the Lorentz-Zygmund sequence space which is denoted by

$$(\ell_{p,q}(\log \ell)^{\upsilon}, \ \|\cdot\|_{p,q,\upsilon}).$$

In particular, for $\upsilon = 0$ we obtain the Lorentz sequence space $(\ell_{p,q}\,, \|\cdot\|_{p,q})$.

The n-th approximation number of $T \in \mathcal{B}(E, F)$ is defined by

$$a_n(T) = \inf\{\|T - S\| : S \in \mathcal{B}(E, F), \operatorname{rank}(S) < n\}.$$

The n-th Weyl number of $T \in \mathcal{B}(E, F)$ is defined by

$$x_n(T) = \sup\{\, a_n(TU) : U \in \mathcal{B}(\ell_2, E), \|U\| \leq 1\}.$$

For $T \in \mathcal{B}(E, F)$, the n-th Weyl number relative to π_γ is defined by

$$x_n(T|\pi_\gamma) = \sup\{\, a_n(TU) : U \in \Pi_\gamma(\ell_2, E), \pi_\gamma(U) \leq 1\}.$$

3. Results

Let us start with the problem which gives a characterization of Gaussian cotype X operators by means of $(X, 2)$-summing operators.

Theorem 1. *Let $2 \leq q < \infty$ and let X be a Banach sequence space. An operator $T \in \mathcal{B}(E, F)$ is of Gaussian cotype X if and only if $TS \in \mathcal{B}(\ell_2, F)$ is $(X, 2)$-summing for all $S \in \Pi_\gamma(\ell_2, E)$. In this case* $\mathrm{gc}_X^n(T) = \sup\{\, \pi_{X,2}^n(TS) : S \in \Pi_\gamma(\ell_2, E), \ \pi_\gamma(S) \leq 1\}$.

Proof. Suppose first that $T \in \mathcal{B}(E,F)$ is of Gaussian cotype X. Then given any operator $S \in \Pi_\gamma(\ell_2, E)$ with $\pi_\gamma(S) \le 1$, we have

$$\|\sum_1^n \|TSx_k\|\, e_k\|_X \le \mathrm{gc}_X^n(T)\cdot\Big(\int_\Omega \|\sum_{k=1}^n g_k(\omega)Sx_k\|^2 dP(\omega)\Big)^{1/2}$$
$$\le \mathrm{gc}_X^n(T)\cdot\pi_\gamma(S)\cdot\|(x_k)_1^n\|_2^{\mathrm{weak}} \le \mathrm{gc}_X^n(T)\cdot\|(x_k)_1^n\|_2^{\mathrm{weak}}$$

for any vectors $x_1,\cdots,x_n$ in ℓ_2.
This assures us that $\sup\{\pi_{X,2}^n(TS) : S \in \Pi_\gamma(\ell_2,E), \pi_\gamma(S) \le 1\} \le \mathrm{gc}_X^n(T)$.

For the other implication, we assume that $TS \in \mathcal{B}(\ell_2, F)$ is $(X,2)$-summing for all $S \in \Pi_\gamma(\ell_2,E)$. For any given $\epsilon > 0$, we select vectors $x_1,\cdots,x_n$ in E such that $(\int_\Omega \|\sum_{k=1}^n g_k(\omega)x_k\|^2 dP(\omega))^{1/2} = 1$ and $\mathrm{gc}_X^n(T) \le (1+\epsilon)\|\sum_1^n \|Tx_k\|e_k\|_X$. Define an operator $S \in \mathcal{B}(\ell_2^n, E)$ by $S = \sum_1^n e_k \otimes x_k$. Since $Se_k = x_k$ for $k = 1,\cdots,n$, we get $\pi_\gamma(S) = 1$. Our hypothesis guarantees that

$$\|\sum_1^n \|TSe_k\|\, e_k\|_X \le \pi_{X,2}^n(TS)\cdot\|(e_k)_1^n\|_2^{\mathrm{weak}} \le \pi_{X,2}^n(TS).$$

Consequently $\mathrm{gc}_X^n(T) \le \sup\{\pi_{X,2}^n(TS) : S \in \Pi_\gamma(\ell_2,E),\ \pi_\gamma(S) \le 1\}$. □

In the following we find a necessary condition for an operator to be of Gaussian cotype X.

THEOREM 2. *Let X be a Banach sequence space. If an operator $T \in \mathcal{B}(E,F)$ is of Gaussian cotype X then $(x_n(T|\pi_\gamma)) \in X$.*

Proof. Let $S \in \Pi_\gamma(\ell_2,E)$ with $\pi_\gamma(S) \le 1$. Since $T \in \mathcal{B}(E,F)$ is of Gaussian cotype X, an appeal to theorem 1 establishes that $TS \in \mathcal{B}(\ell_2,F)$ is $(X,2)$-summing. From lemma 2.7.1. of [8] we know that for every $\epsilon > 0$, there exists an orthonormal family $\{x_1,\cdots,x_n\}$ in ℓ_2 such that $a_k(TS) \le (1+\epsilon)\,\|TSx_k\|$ for $k = 1,\cdots,n$. Therefore we have

$$\|\sum_1^n a_k(TS)e_k\|_X \le (1+\epsilon)\,\|\sum_1^n \|TSx_k\|\, e_k\|_X$$
$$\le (1+\epsilon)\,\pi_{X,2}(TS)\,\|(x_k)_1^n\|_2^{\mathrm{weak}} = (1+\epsilon)\,\pi_{X,2}(TS).$$

This yields that $(x_n(T|\pi_\gamma)) \in X$. □

Next we establish a quotient formula for (X,p)-summing operators, which is an improvement of the result due to M. Defant and M. Junge [1].

LEMMA 1. *Let $1 \le r \le q < \infty$ and let Y be a maximal symmetric Banach sequence space. If $X = \mathcal{M}(Y,\ell_q)$ then for any operator $T \in \mathcal{B}(E,F)$ we have*

$$\pi^n_{X,r}(T) = \sup\{\pi^n_{q,r}(M_\sigma RT) | R \in \mathcal{B}(F,\ell_\infty), M_\sigma \in \mathcal{B}(\ell_\infty,\ell_\infty), \|R\| \le 1, \sigma \in B_Y\}.$$

Proof. Take any vectors $x_1,\cdots,x_n$ in E. A quick peek at the definition of $\|(\|Tx_k\|)_1^n\|_X$ ensures that for any $\epsilon > 0$ there exists a sequence $\sigma \in B_Y$ satisfying $\|\sum_1^n \|Tx_k\|e_k\|_X \le (1+\epsilon)(\sum_1^n |\|Tx_k\|\sigma_k|^q)^{1/q}$. Furthermore there is a sequence $(y_k^*)_1^n$ in B_{F^*} with $\langle y_k^*, Tx_k\rangle = \|Tx_k\|$ for $k = 1,\cdots,n$. Define an operator $R \in \mathcal{B}(F,\ell_\infty)$ via $R = \sum_1^n y_k^* \otimes e_k$. It is obvious that $\|R\| = 1$ and we get

$$\frac{1}{1+\epsilon}\|\sum_1^n \|Tx_k\|e_k\|_X \le (\sum_1^n |\langle y_k^*, Tx_k\rangle\sigma_k|^q)^{1/q}$$
$$\le (\sum_1^n \|M_\sigma RTx_k\|_\infty^q)^{1/q} \le \pi^n_{q,r}(M_\sigma RT)\,\|(x_k)_1^n\|_r^{\text{weak}}.$$

Hence $\pi^n_{X,r}(T) \le (1+\epsilon)\pi^n_{q,r}(M_\sigma RT)$. This gives us the upper estimate.

To obtain the lower estimate we choose a sequence $\sigma \in B_Y$ and an operator $R \in \mathcal{B}(F,\ell_\infty)$ with $\|R\| \le 1$. It is enough to show that for every natural number m with $n < m$, $\pi^n_{q,r}(P_m M_\sigma RT) \le \pi^n_{X,r}(T)$, where $P_m = \sum_1^m e_k \otimes e_k \in \mathcal{B}(\ell_\infty, \ell_\infty^m)$. Taking note of the fact that the (q,r)-summing norm and the (q',r')-nuclear norm are in trace duality we derive that there is an operator $S \in \mathcal{B}(\ell_\infty^m, E)$ such that $\nu_{q',r'}(S) \le 1$ and $\pi^n_{q,r}(P_m M_\sigma RT) \le (1+\epsilon)\operatorname{tr}(SP_m M_\sigma RT)$. Thanks to Maurey's result [1], it is no loss to assume that $S = \sum_1^N \alpha_k B_k M_{\tau^k}\tilde{P}_k$, where $(\alpha_k)_1^N$ is a sequence of positive real numbers with $\sum_1^N \alpha_k = 1$, $(m_i)_1^n$ is an increasing sequence in $\{1,2,\cdots m\}$, $\tilde{P}_k = \sum_1^n e_{m_i} \otimes e_i \in \mathcal{B}(\ell_\infty^m, \ell_\infty^n)$ $M_{\tau^k} \in \mathcal{B}(\ell_\infty^n, \ell_{r'}^n)$ with $\|\tau^k\|_{q'} \le 1$, $B_k \in \mathcal{B}(\ell_{r'}^n, E)$ with $\|B_k\| \le 1$ for $k = 1,\cdots N$.

We make use of Hölder's inequalty to obtain that

$$
\begin{aligned}
|\mathrm{tr}(B_k M_{\tau^k}\tilde{P}_k P_m M_\sigma RT)| &= |\sum_{j=1}^{n}\langle M_{\tau^k}\tilde{P}_k P_m M_\sigma RTB_k(e_j), e_j\rangle| \\
= |\sum_{j=1}^{n}\tau_j^k\langle e_{m_j}, P_m M_\sigma RTB_k(e_j)\rangle| &\le \sum_{j=1}^{n}|\tau_j^k\sigma_{m_j}|\ \|RTB_k(e_j)\| \\
&\le (\sum_{j=1}^{n}|\tau_j^k|^{q'})^{1/q'}(\sum_{j=1}^{n}(|\sigma_{m_j}|\ \|RTB_k(e_j)\|)^q)^{1/q} \\
\le \|\sum_{j=1}^{n}\|RTB_k(e_j)\|e_j\|_X\ \|\sigma\|_Y &\le \pi_{X,r}^n(RT)\,\|(B_k(e_j))_{j=1}^n\|_r^{\mathrm{weak}} \\
&\le \|R\|\,\pi_{X,r}^n(T)\ \|B_k\| \le \pi_{X,r}^n(T).
\end{aligned}
$$

Hence

$$
\mathrm{tr}(SP_m M_\sigma RT) \le \sum_{1}^{N}\alpha_k\ |\mathrm{tr}(B_k M_{\tau^k}\tilde{P}_k P_m M_\sigma RT)| \le \pi_{X,r}^n(T).
$$

This informs us the desired estimate. □

We now turn to the study of the p-convexity of a maximal symmetric Banach sequence space.

LEMMA 2. *Let $1 \le p < \infty$ and let X be a p-convex maximal symmetric Banach sequence space. Then there exists a maximal symmetric Banach sequence space Y such that $X = \mathcal{M}(Y, \ell_p)$.*

Proof. We set $Y = (X_{(p)}^+)_{(\frac{1}{p})}$, where $X_{(p)} = \{x : |x|^{1/p} \in X\}$. Note that $\|\lambda\|_Y = \||\lambda|^p\|_{X_{(p)}^+}^{1/p} = \sup\{\|\lambda^p\mu\|_1^{1/p} : \|\mu\|_{X_{(p)}} \le 1\} = \sup\{\|\lambda\nu\|_p : \|\nu\|_X \le 1\} = \|\lambda\|_{\mathcal{M}(X,\ell_p)}$. In other words, $Y = \mathcal{M}(X, \ell_p)$.

Since $\|\sigma\tau\|_p \le \|\sigma\|_X\ \|\tau\|_{\mathcal{M}(X,\ell_p)} = \|\sigma\|_X\ \|\tau\|_Y$, it follows that

$$
\|\sigma\|_{\mathcal{M}(Y,\ell_p)} \le \|\sigma\|_X.
$$

On the one hand,

$$
\begin{aligned}
\|\sigma\|_X = \|\sigma^p\|_{X_{(p)}}^{1/p} = \|\sigma^p\|_{\mathcal{M}(X_{(p)}^+,\ell_1)}^{1/p} \\
= \sup\{(\sum |\sigma_k|^p|\tau_k|)^{1/p} : \|\tau\|_{X_{(p)}^+} \le 1\} \\
\le \|\sigma\|_{\mathcal{M}(Y,\ell_p)} \sup\{\, \|\tau^{1/p}\|_Y : \|\tau\|_{X_{(p)}^+} \le 1\} \\
= \|\sigma\|_{\mathcal{M}(Y,\ell_p)} \sup\{\|\tau\|_{X_{(p)}^+}^{1/p} : \|\tau\|_{X_{(p)}^+} \le 1\} \le \|\sigma\|_{\mathcal{M}(Y,\ell_p)}.
\end{aligned}
$$

As a consequence $X = \mathcal{M}(Y, \ell_p)$.

□

A decisive step toward the proof of the theorem given below is provided by the following criterion.

Proposition. *Let $1 \le r < \infty$ and let X be a maximal symmetric Banach sequence space. An operator T from a Banach lattice L to a Banach space F is (X, r)-concave with $K_{X,r}(T) \le C$ if and only if for every positive operator $S : C(K) \to L$, the composition $TS : C(K) \to F$ is (X, r)-summing with $\pi_{X,r}(TS) \le C \cdot \|S\|$.*

Proof. Let $S : C(K) \to L$ be any positive operator, and pick $f_1, \cdots, f_n$ from $C(K)$. Then we have

$$
\begin{aligned}
(\sum_1^n |Sf_k|^r)^{1/r} = \sup\{\sum_1^n a_k\, Sf_k : \|a\|_{r'} \le 1\} \\
\le S(\sup\{\sum_1^n a_k\, f_k : \|a\|_{r'} \le 1\}) = S(\sum_1^n |f_k|^r)^{1/r}.
\end{aligned}
$$

Therefore if $T : L \to F$ is (X, r)-concave and $K_{X,r}(T) \le C$ then

$$
\begin{aligned}
\|\sum_1^n \|TSf_k\|\, e_k\|_X \le K_{X,r}(T)\, \|(\sum_1^n |Sf_k|^r)^{1/r}\| \\
\le K_{X,r}(T)\|S\|\ \|(\sum_1^n |f_k|^r)^{1/r}\| = K_{X,r}(T)\|S\|\ \|(f_k)\|_r^{\text{weak}}.
\end{aligned}
$$

This signifies that TS is (X,r)-summing with $\pi_{X,r}(TS) \leq K_{X,r}(T) \cdot \|S\|$.

For the converse, take vectors $x_1, \cdots, x_n$ in L. We can assume that $x = (\sum_1^n |x_k|^r)^{1/r}$ has norm one. Notice that $I(x) = \{y \in L : |y| \leq \lambda \cdot |x| \text{ for some } 0 < \lambda < \infty\}$ endowed with the norm $\|y\|_\infty = \inf\{\lambda > 0 : |y| \leq \frac{\lambda}{\|x\|} \cdot |x|\,\}, y \in I(x)$, can be identified with a space $C(K)$ for a suitably chosen compact Hausdorff space K. Let $J : I(x) = C(K) \to L$ be the canonical embedding and let $T : L \to F$. In view of our hypothesis, TJ is (X,r)-summing with $\pi_{X,r}(TJ) \leq C$. Accordingly

$$\begin{aligned}
\|\sum_1^n \|Tx_k\|\, e_k\|_X &= \|\sum_1^n \|TJx_k\| e_k\|_X \\
&\leq \pi_{X,r}(TJ)\|(\sum_1^n |x_k|^r)^{1/r}\|_{C(K)} \\
&= \pi_{X,r}(TJ)\|x\|_\infty = \pi_{X,r}(TJ)\|x\| \leq C.
\end{aligned}$$

This forces that T is (X,r)-concave with $K_{X,r}(T) \leq C$.

□

In the next two theorems we intend to generalize the result of B. Maurey [5], which is a description of Rademacher cotype q operators, to the setting of Banach sequence space.

THEOREM 3. *Let $2 < q < \infty$ and let X be a q-convex maximal symmetric Banach sequence space. The following are equivalent statements about an operator T from a Banach lattice L to a Banach space F.*

(i) *T is $(X,1)$-summing.*
(ii) *T is (X,r)-concave for all $1 \leq r < q$.*
(iii) *T is of Rademacher cotype X.*

Proof. (i) ⇒ (ii). By virtue of the hypothesis (i), for any vectors $x_1, \cdots, x_n$ from L, we have

$$\|\sum_1^n \|Tx_k\|\, e_k\|_X \le \pi^n_{X,1}(T)\ \|(x_k)_1^n\|_1^{\rm weak}$$

$$= \pi^n_{X,1}(T)\ \sup\{\|\sum_1^n a_k x_k\| : \|a\|_\infty \le 1\}$$

$$\le \pi^n_{X,1}(T)\|\sup\{\sum_1^n a_k x_k : \|a\|_\infty \le 1\}\| = \pi^n_{X,1}(T)\,\|\sum_1^n |x_k|\,\|.$$

This means that T is $(X,1)$-concave with $K^n_{X,1}(T) \le \pi^n_{X,1}(T)$. An appeal to proposition ensures that for every positive operator $S : C(K) \to L$, the composition $TS : C(K) \to F$ is $(X,1)$-summing. The q-convexity of X enables us to invoke lemma 2 to get that there exists a maximal symmetric Banach sequence space Y such that $X = \mathcal{M}(Y, \ell_q)$. Taking account of the fact that $(q,1)$-summing operators on $C(K)$ are always (q,r)-summing for $1 \le r < q$ and using lemma 1 we obtain that $TS : C(K) \to F$ is (X,r)-summing. It takes another appeal to proposition to see that T is (X,r)-concave for all $1 \le r < q$.

(ii) $\Rightarrow$ (iii). The hypothesis (ii) indicates that T is $(X,2)$-concave. We apply Khinchin's inequality to produce that for any vectors $x_1, \cdots, x_n$ from L,

$$\|\sum_1^n \|Tx_k\|\, e_k\|_X \le K^n_{X,2}(T)\|(\sum_1^n |x_k|^2)^{1/2}\|$$

$$\le C \cdot K^n_{X,2}(T) \cdot (\int_0^1 \|\sum_1^n r_k(t)x_k\|^2 dt)^{1/2}.$$

This gives that T is of Rademacher cotype X with $\mathrm{rc}^n_X(T) \le C \cdot K^n_{X,2}(T)$.

(iii) $\Rightarrow$ (i). Using Kahane's inequality, together with the hypothesis (iii), for any vectors $x_1, \cdots, x_n$ from L, we get

$$\begin{aligned}\|\sum_1^n \|Tx_k\|\, e_k\|_X &\le \mathrm{rc}_X^n(T)\big(\int_0^1 \|\sum_1^n r_k(t)x_k\|^2 dt\big)^{1/2}\\ &\le C\,\mathrm{rc}_X^n(T)\big(\int_0^1 \|\sum_1^n r_k(t)x_k\| dt\big)\\ &\le C\,\mathrm{rc}_X^n(T)\sup\{\|\sum \epsilon_i x_i\| : \epsilon_i = \pm 1\} = C\,\mathrm{rc}_X^n(T)\,\|(x_k)\|_1^{\mathrm{weak}}.\end{aligned}$$

This ensures that T is $(X,1)$-summing with $\pi_{X,1}^n(T) \le C \cdot \mathrm{rc}_X^n(T)$. □

THEOREM 4. *Let $2 < q < \infty$ and let X be a q-convex maximal symmetric Banach sequence space. An operator T from a Banach lattice L to a Banach space F is $(X,1)$-summing if and only if there is a constant C such that for all choices of finitely many disjoint vectors $x_1, \cdots, x_n$ from L, $\|\sum_1^n \|Tx_k\|\, e_k\|_X \le C \cdot \|\sum_1^n x_k\|$.*

Proof. Assume first that our condition holds. Let $S : C(K) \to L$ be a positive operator. We select disjointly supported functions $f_1, \cdots, f_n$ from $C(K)$. The very nature of S assures that $Sf_1, \cdots, Sf_n$ are disjoint vectors in L, so our hypothesis tells us that

$$\begin{aligned}\|\sum_1^n \|TSf_k\|\, e_k\|_X &\le C \cdot \|\sum_1^n Sf_k\|\\ &\le C \cdot \|S\|\, \|\sum_1^n f_k\| = C \cdot \|S\|\, \|(f_k)\|_1^{\mathrm{weak}}.\end{aligned}$$

Thus $TS : C(K) \to F$ is $(X,1)$-summing. It follows from the proof of theorem 3 that $TS : C(K) \to F$ is (X,r)-summing for $1 \le r < q$. Proposition permits us to have that T is (X,r)-concave for $1 \le r < q$. We use theorem 3 to find that T is $(X,1)$-summing.

To pass in the other direction, we suppose that $T : L \to F$ is $(X,1)$-summing. Let $x_1, \cdots, x_n$ be disjoint vectors in L. Observe that for any choice of $\epsilon_k = \pm 1$, we have

$$\|\sum_1^n \epsilon_k x_k\| = \| \, |\sum_1^n \epsilon_k x_k| \, \| = \| \sup_{k\le n} |\epsilon_k x_k| \, \|$$
$$= \| \sup_{k\le n} |x_k| \, \| = \| \, |\sum_1^n x_k| \, \| = \|\sum_1^n x_k\|.$$

We make double use of theorem 3 to ensure that $T : L \to F$ is of Rademacher cotype X, that is,

$$\|\sum_1^n \|Tx_k\| \, e_k\|_X \le \mathrm{rc}_X^n(T)\big(\int_0^1 \|\sum_1^n r_k(t)x_k\|^2 dt\big)^{1/2}.$$

Since $(\int_0^1 \|\sum_1^n r_k(t)x_k\|^2 dt)^{1/2}\| = \|\sum_1^n x_k\|$, we arrive at the conclusion. □

In the theorem stated below we establish a result which relates the Gaussian cotype X operators on $C(K)$ spaces to the Rademacher cotype X operators on $C(K)$ spaces. For this purpose, we need another quotient formula.

Lemma 3. *Let Y and Z be maximal symmetric Banach sequence space. If $X = \mathcal{M}(Y,Z)$ then for any operator $T \in \mathcal{B}(E,F)$ we have*

$$\pi_{X,1}^n(T) = \sup\{\pi_{Z,1}^n(TRM_\sigma) | R \in \mathcal{B}(\ell_\infty, E), M_\sigma \in \mathcal{B}(\ell_\infty, \ell_\infty),$$
$$\|R\| \le 1, \sigma \in B_Y\}.$$

Proof. Let us pick any vectors $x_1, \cdots, x_n$ in E such that

$$\|(x_k)\|_1^{\mathrm{weak}} \le 1.$$

A quick reference to the defition of $\|(\|Tx_k\|)_1^n\|_X$ assures us that for any $\epsilon > 0$ there exists a sequence $\sigma \in B_Y$ for which $\|\sum_1^n \|Tx_k\|e_k\|_X \le$

$(1+\epsilon)\|\sum_1^n \|Tx_k\|\sigma_k e_k\|_Z$. Define an operator $R \in \mathcal{B}(\ell_\infty, E)$ by $R = \sum_1^n e_k \otimes x_k$. It is clear that $\|R\| \le 1$ and we have

$$\begin{aligned}\frac{1}{1+\epsilon}\|\sum_1^n \|Tx_k\|e_k\|_X &\le \|\sum_1^n \|TRM_\sigma e_k\|_F e_k\|_Z \\ &\le \pi_{Z,1}^n(TRM_\sigma)\|(e_k)\|_1^{\text{weak}} \le \pi_{Z,1}^n(TRM_\sigma)\end{aligned}$$

Thus $\pi_{X,1}^n(T) \le (1+\epsilon)\,\pi_{Z,1}^n(TRM_\sigma)$. This leads to the upper estimate.

To obtain the lower estimate we select a sequence $\sigma \in B_Y$, $M_\sigma \in \mathcal{B}(\ell_\infty^m, \ell_\infty^m)$ and $R \in \mathcal{B}(\ell_\infty^m, E)$ with $\|R\| \le 1$. Let $S \in \mathcal{B}(\ell_\infty^n, \ell_\infty^m)$ be any operator with $\|S\| \le 1$. Here $m > n$. Then $\|(Se_k)\|_1^{\text{weak}} \le 1$. A result due to B. Maurey [1] guarantees that S has the form $S = \sum_1^n e_k \otimes g^k$, where $(g^k)_1^n \subset \ell_\infty^m$ with mutually disjoint supports and $0 < \|g^k\|_\infty \le 1$ for $k = 1, \cdots, n$. Define an operator $I \in \mathcal{B}(\ell_\infty^n, \ell_\infty^m)$ by $I = \sum_1^n e_k \otimes \frac{M_\sigma g^k}{\|M_\sigma g^k\|}$. Set $\tau = (\|M_\sigma g^k\|_\infty)_{k=1}^n$. Since there is an increasing sequence $(n_k)_{k=1}^n$ in $\{1, 2, \cdots, m\}$ such that $\|M_\sigma g^k\|_\infty = |\langle e_{n_k}, M_\sigma g^k\rangle|$, it follows that

$$\|\tau\|_Y = \|\,(|\langle g^k, \sigma_{n_k} e_{n_k}\rangle|)_{k=1}^n\,\|_Y \le \|\sum_1^n \sigma_{n_k} e_{n_k}\|_Y \le \|\sigma\|_Y \le 1.$$

Therefore we have

$$\begin{aligned}\|\sum_1^n \|TRM_\sigma Se_k\|e_k\,\|_Z &= \|\sum_1^n (\|TRIe_k\|\tau_k)e_k\|_Z \\ &\le \|\sum_1^n \|TRIe_k\|e_k\|_X\|\tau\|_Y \le \pi_{X,1}^n(T)\|(RIe_k)\|_1^{\text{weak}} \\ &= \pi_{X,1}^n(T)\|RI\| \le \pi_{X,1}^n(T),\end{aligned}$$

and so $\pi_{Z,1}^n(TRM_\sigma) \le \pi_{X,1}^n(T)$. This gives us the desired estimate. □

Next we are concerned with the estimation of Gaussian cotype X norms of an operator on $C(K)$ in terms of $(X, 2)$-summing norms of an operator.

Lemma 4. *Let X be a maximal symmetric Banach sequence space and let $1/2 \leq s < \infty$. Then for any operator $T \in \mathcal{B}(C(K), F)$ there is a constant C such that*

$$\frac{1}{C}\,\mathrm{gc}_X^n(T) \leq \sup\{\pi_{X,2}^n(TRM_\sigma) | R \in \mathcal{B}(c_0, C(K)), M_\sigma \in \mathcal{B}(c_0, c_0),$$
$$\|R\| \leq 1, \|\sigma\|_{\infty,\infty,s} \leq 1\} \leq C\,\mathrm{gc}_X^n(T).$$

Proof. Take $R \in \mathcal{B}(c_0, C(K))$ with $\|R\| \leq 1$. We set $\sigma_k = (\log(k+1))^{-s}$, $k = 1, 2, \cdots$, so that $\|\sigma\|_{\infty,\infty,s} = 1$. Since $\sup_k \sigma_k(\log(k+1))^{\frac{1}{2}} < \infty$, lemma of [10] steps in to ensure that M_σ, and hence RM_σ, is γ-summing. Then for any vectors $x_1, \cdots, x_n$ in c_0, we have

$$\|\sum_1^n \|TRM_\sigma x_k\|\, e_k\|_X \leq \mathrm{gc}_X^n(T)(\int_\Omega \|\sum_{k=1}^n g_k(\omega) RM_\sigma x_k\|^2 dP(\omega))^{1/2}$$
$$\leq \mathrm{gc}_X^n(T)\pi_\gamma(RM_\sigma)\|(x_k)_1^n\|_2^{\mathrm{weak}} \leq \mathrm{gc}_X^n(T)\pi_\gamma(M_\sigma)\|(x_k)_1^n\|_2^{\mathrm{weak}}.$$

Thus $\pi_{X,2}^n(TRM_\sigma) \leq \mathrm{gc}_X^n(T)\,\pi_\gamma(M_\sigma)$. This implies the right-hand inequality.

For the left-hand inequality we choose functions $x_1, \cdots, x_n$ in $C(K)$ with $(\int_\Omega \|\sum_{k=1}^n g_k(\omega)x_k\|^2 dP(\omega))^{1/2} \leq 1$. We invoke Talagrand's theorem [11] to infer that there exist operators $U \in \mathcal{B}(\ell_2^n, c_0)$ and $R \in \mathcal{B}(c_0, C(K))$ such that $\|U\| \leq C_1$, $\|R\| \leq 1$ and $RM_\sigma Ue_k = x_k$, where $\sigma_k = (\log(k+1))^{-s}$, for $k = 1, \cdots, n$. From this we find

$$\|\sum_1^n \|Tx_k\|\, e_k\|_X = \|\sum_1^n \|TRM_\sigma Ue_k\|\, e_k\|_X$$
$$\leq \pi_{X,2}^n(TRM_\sigma)\|(Ue_k)_1^n\|_2^{\mathrm{weak}} = \pi_{X,2}^n(TRM_\sigma)\|U\|.$$

As a result, $\mathrm{gc}_X^n(T) \leq C_1 \cdot \pi_{X,2}^n(TRM_\sigma)$. This gives us the left-hand inequality.

□

Having these preliminary results we draw the theorem given below.

Theorem 5. *Let $2 < q < \infty$ and let X be a q-convex maximal symmetric Banach sequence space. If $Y = \mathcal{M}(\ell_{\infty,\infty}(\log \ell)^s, X)$, $\frac{1}{2} \leq s < \infty$, then for any operator $T \in \mathcal{B}(C(K), F)$ there is a constant C such that $\frac{1}{C}\,\mathrm{rc}_Y^n(T) \leq \mathrm{gc}_X^n(T) \leq C\,\mathrm{rc}_Y^n(T)$.*

Proof. First note that q-convexity of X implies the q-convexity of $Y = \mathcal{M}(\ell_{\infty,\infty}(\log \ell)^s, X)$. For any choice of vectors $y_1, \cdots, y_n$ in Y, we have

$$\begin{aligned}
&\|(\sum_{j=1}^n |y_j|^q)^{1/q}\|_Y \\
&\qquad = \sup\{\|\sum_k (\sum_{j=1}^n |y_j(k)|^q)^{1/q} \sigma_k e_k\|_X : \|\sigma\|_{\infty,\infty,s} \leq 1\} \\
&\qquad \leq \sup\{\|(\sum_{j=1}^n |\sum_k y_j(k)\sigma_k e_k|^q)^{1/q}\|_X : \|\sigma\|_{\infty,\infty,s} \leq 1\} \\
&\qquad \leq K^q(X) \sup\{(\sum_{j=1}^n \|\sum_k y_j(k)\sigma_k e_k\|_X^q)^{1/q} : \|\sigma\|_{\infty,\infty,s} \leq 1\} \\
&\qquad\qquad \leq K^q(X)(\sum_{j=1}^n \|y_j\|_Y^q)^{1/q}.
\end{aligned}$$

An appeal to lemma 4 in combination with theorem 3 and lemma 3 reveals that $\mathrm{gc}_X^n(T)$ and $\pi_{Y,1}^n(T)$ are equivalent. It takes another appeal to theorem 3 to see that $\pi_{Y,1}^n(T)$ and $\mathrm{rc}_Y^n(T)$ are equivalent. This completes the proof. □

Theorem 5 permits us to find a necessary condition which implies that an operator with domain a $C(K)$ space is of Gaussian cotype q.

Corollary. *Let $2 < q < \infty$ and $\frac{1}{2} \leq s < \infty$. If an operator $T \in \mathcal{B}(C(K), F)$ is of Gaussian cotype q then $(\sum_k (x_k(T)(\log(k+1))^{-s})^q)^{1/q} < \infty$.*

Proof. Using first theorem 5 and then theorem 3 we get that T is $(\ell_{q,q}(\log \ell)^{-s}, 2)$-summing because $\ell_{q,q}(\log \ell)^{-s} = \mathcal{M}(\ell_{\infty,\infty}(\log \ell)^s, \ell_q)$.

For simplicity write $Y = \ell_{q,q}(\log \ell)^{-s}$. Choose any operator U in $\mathcal{B}(\ell_2, C(K))$. Lemma 2.7.1. of [8] tells us that for every $\epsilon > 0$, there exists an orthonormal family $\{o_1, \cdots, o_n\}$ in ℓ_2 such that $a_k(TU) \leq (1+\epsilon)\|TUo_k\|$ for $k = 1, \cdots, n$, where $(\|TUo_k\|)$ is a decreasing sequence. Thus we have

$$\Big(\sum_1^n (a_k(TU)\,(\log(k+1))^{-s})^q\Big)^{1/q} \leq (1+\epsilon)\,\|(\|TUo_k\|)\|_{q,q,-s}$$
$$\leq (1+\epsilon)\,\pi_{Y,2}(TU)\,\|(o_k)\|_2^{\text{weak}} \leq (1+\epsilon)\,\pi_{Y,2}(T)\,\|U\|.$$

So we end up with $(\sum_k (x_k(T)\,(\log(k+1))^{-s})^q)^{1/q} < \infty$.

□

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