Kangweon-Kyungki Math. Jour. **13** (2005), No. 2, pp. 249–254

SEVERAL TYPES FUZZY HALF-COMPACTNESS ON AN INTUITIONISTIC FUZZY TOPOLOGICAL SPACE

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Abstract. In this paper, we introduce the concepts of intuitionistic fuzzy half-compactness, nearly intuitionistic fuzzy half-compactness and almost intuitionistic fuzzy half-compactness defined by intuitionistic gradations of openness, and obtain some characterizations.

1. Introduction

In 1992 [8], Chattopadyay et al. introduced the concept of fuzzy topology redefined by a gradation of openness and investigated some fundamental properties. In particular, Gayyar, Kerre, Ramadan [7] and Demirci [5, 6] introduced the concepts of fuzzy closure and fuzzy interior of a fuzzy set, and obtained some properties of them. Atanassov [1] introduced the concept of intuitionistic fuzzy set which is a generalization of fuzzy set in Zadeh's sense [12]. Çoker introduced the concept of intuitionistic fuzzy topological spaces [4], which it is an extended concept of fuzzy topological spaces [2] in Chang's sense. In 2002, Mondal and Samanta introduced and investigated the concept of intuitionistic gradation of openness [9] which is a generalization of the concept of gradation of openness defined by Chattopadyay. In [10] we introduced the concepts of half-interior, half-closure, half-gp-map and half-gp-open map and also obtained some characterizations.

In this paper, we introduce the concepts of intuitionistic fuzzy half-compactness, nearly intuitionistic fuzzy half-compactness and almost

Received September 7, 2005.

2000 Mathematics Subject Classification: 54A40.

Key words and phrases: intuitionistic gradation of openness, half-interior, half-closure, half-gp-map, half-gp-open map, intuitionistic fuzzy half-compact, nearly intuitionistic fuzzy half-compactness, almost intuitionistic fuzzy half-compactness.

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intuitionistic fuzzy half-compactness in intuitionistic fuzzy topological spaces and investigate some properties of them.

2. Preliminaries

Let X be a set and $I = [0,1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

DEFINITION 2.1 ([3, 8, 11]). Let X be a non-empty set and $\tau : I^X \to I$ be a mapping satisfying the following conditions:
(O1) $\tau(0_X) = \tau(1_X) = 1$;
(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(O3) For every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$.
Then the mapping $\tau : I^X \to I$ is called a *fuzzy topology* (or *gradation of openness* [10]) on X. We call the ordered pair (X, τ) a *fuzzy topological space*. The value $\tau(A)$ is called the *degree of openness* of A.

DEFINITION 2.2 ([1]). An *intuitionistic fuzzy set* A in a set X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

DEFINITION 2.3 ([9]). An *intuitionistic gradation of openness* (briefly *IGO*) of fuzzy subsets of a set X is an ordered pair (τ, τ^*) of functions $\tau, \tau^* : I^X \to I$ such that
(IGO1) $\tau(A) + \tau^*(A) \leq 1$, for all $A \in I^X$;
(IGO2) $\tau(0_X) = \tau(1_X) = 1, \tau^*(0_X) = \tau^*(1_X) = 0$;
(IGO3) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ and $\tau^*(A \cap B) \leq \tau^*(A) \vee \tau^*(B)$;
(IGO4) For every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$ and $\tau^*(\cup_{i\in J} A_i) \leq \vee_{i\in J} \tau^*(A_i)$.

Then the triplet (X, τ, τ^*) is called an *intuitionistic fuzzy topological space* (briefly $IFTS$) on X. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.4 ([9]). Let X be a nonempty set and two functions $\mathcal{F}, \mathcal{F}^* : I^X \to I$ be satisfying

(IGC1) $\mathcal{F}(A) + \mathcal{F}^*(A) \leq 1$, for all $A \in I^X$;

(IGC2) $\mathcal{F}(0_X) = \mathcal{F}(1_X) = 1, \mathcal{F}^*(0_X) = \mathcal{F}^*(1_X) = 0$;

(IGC3) $\forall A, B \in I^X$, $\mathcal{F}(A \cup B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$ and $\mathcal{F}^*(A \cup B) \leq \mathcal{F}^*(A) \vee \mathcal{F}^*(B)$;

(IGC4) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\mathcal{F}(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \mathcal{F}(A_i)$ and $\mathcal{F}^*(\cap_{i\in J} A_i) \leq \vee_{i\in J} \mathcal{F}^*(A_i)$.

Then the ordered pair $(\mathcal{F}, \mathcal{F}^*)$ is called an *intuitionistic gradation of closedness* [9] (briefly *IGC*) on X. $\mathcal{F}$ and $\mathcal{F}^*$ may be interpreted as gradation of closedness and gradation of nonclosedness, respectively.

Theorem 2.5 ([9]). *Let X be a nonempty set. If (τ, τ^*) is an IGO on X, then the pair $(\mathcal{F}, \mathcal{F}^*)$, defined by $\mathcal{F}_\tau(A) = \tau(A^c)$, ${\mathcal{F}^*}_{\tau^*}(A) = \tau^*(A^c)$ where A^c denotes the complement of A, is an IGC on X. And if $(\mathcal{F}, \mathcal{F}^*)$ is an IGC on X, then the pair $(\tau_\mathcal{F}, {\tau^*}_{\mathcal{F}^*})$, defined by $\tau_\mathcal{F}(A) = \mathcal{F}(A^c)$, ${\tau^*}_{\mathcal{F}^*}(A) = \mathcal{F}^*(A^c)$ is an IGO on X.*

Definition 2.6 ([9]). Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *gp-map* if $\tau(f^{-1}(A)) \geq \sigma(A)$ and $\tau^*(f^{-1}(A)) \leq \sigma^*(A)$ for every $A \in I^Y$.

Definition 2.7 ([10]). Let (X, τ, τ^*) be an IFTS and $A \in I^X$. Then the *half-closure* (resp., *half-interior*) of A, denoted by A_- (resp., A_o), is defined by $A_- = \cap\{K \in I^X : \mathcal{F}_\tau(A) > 0 \text{ and } \mathcal{F}^*_{\tau^*}(A) \leq \frac{1}{2}, A \subseteq K\}$ (resp., $A_o = \cup\{K \in I^X : \tau(K) > 0 \text{ and } \tau^*(A) \leq \frac{1}{2}, K \subseteq A\}$).

Definition 2.8 ([10]). Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is a *half-gp-map* iff for every $A \in I^Y$ such that $\sigma(A) > 0$ and $\sigma^*(A) \leq \frac{1}{2}$, $\tau(f^{-1}(A)) > 0$ and $\tau^*(f^{-1}(A)) \leq \frac{1}{2}$.

Definition 2.9 ([10]). Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs. A mapping $f : X \to Y$ is called a *half-gp-open* map iff for every $A \in I^X$ such that $\tau(A) > 0$ and $\tau^*(A) \leq \frac{1}{2}$, $\sigma(f(A)) > 0$ and $\sigma^*(f(A)) \leq \frac{1}{2}$.

3. Several types compactness in intuitionistic fuzzy topological spaces

In this section, we introduce the concepts of intuitionistic fuzzy half-compactness, nearly intuitionistic fuzzy half-compactness and almost intuitionistic fuzzy half-compactness in intuitionistic fuzzy topological spaces and investigate some properties of them.

DEFINITION 3.1. An IFTS (X, τ, τ^*) is called *intuitionistic fuzzy half-compact* iff for every family $\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq \frac{1}{2}, i \in J\}$ covering X, there exists a finite subset J_o of J such that $\cup_{i \in J_o} A_i = 1_X$.

THEOREM 3.2. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs and $f : X \to Y$ a surjective half-gp-map. If (X, τ, τ^*) is intuitionistic fuzzy half-compact, then so is (Y, σ, σ^*).*

Proof. Let a family $\{A_i \in I^Y : \sigma(A_i) > 0 \text{ and } \sigma^*(A_i) \leq \frac{1}{2}, i \in J\}$ be a cover of Y; then by Definition 2.8, the family $\{f^{-1}(A_i) \in I^X : \tau(f^{-1}(A_i)) > 0 \text{ and } \tau^*(f^{-1}(A_i)) \leq \frac{1}{2}, i \in J\}$ covers X. From the surjectivity of f and intuitionistic fuzzy half-compactness, it follows that Y also is intuitionistic fuzzy half-compact. □

DEFINITION 3.3. An IFTS (X, τ, τ^*) is called *nearly intuitionistic fuzzy half-compact* iff for every family $\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq \frac{1}{2}, i \in J\}$ covering X, there exists a finite subset J_o of J such that $\cup_{i \in J_o}((A_i)_-)_o = 1_X$.

THEOREM 3.4. *An intuitionistic fuzzy half-compact space (X, τ, τ^*) is nearly intuitionistic half-compact.*

Proof. Let $\{A_i \in I^X : \tau(A_i) > 0 \text{ and } \tau^*(A_i) \leq \frac{1}{2}, i \in J\}$ be a cover of X; then there exists a finite subset J_o of J such that $\cup_{i \in J_o} A_i = 1_X$. Since $\tau(A_i) > 0$ for all $i \in J$, we have $A_i = (A_i)_o \subseteq (A_{i-})_o$. Consequently the IFTS (X, τ, τ^*) is nearly intuitionistic fuzzy half-compact. □

REMARK 3.5. In Theorem 3.4, the converse of implication may not be true. For if (X, τ, τ^*) is an IFTS and $\tau^*(\mu) = 0$ for all $\mu \in I^X$, then the (X, τ, τ^*) is a fuzzy topological space in Sostak's sense, that is, a fuzzy topological space is a special case in IFTSs. And in general, a nearly fuzzy compact space is not fuzzy compact, so we can say an nearly intuitionistic fuzzy half-compact space (X, τ, τ^*) is not always intuitionistic fuzzy half-compact.

Definition 3.6. An IFTS (X,τ,τ^*) is called *almost intuitionistic fuzzy half-compact* iff for every family $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \frac{1}{2}, i \in J\}$ covering X, there exists a finite subset J_o of J such that $\cup_{i\in J_o} A_{i-} = 1_X$.

Theorem 3.7. *A nearly intuitionistic fuzzy half-compact space (X,τ,τ^*) is almost intuitionistic fuzzy half-compact.*

Proof. Let $\{A_i \in I^X : \tau(A_i) > 0$ and $\tau^*(A_i) \leq \frac{1}{2}, i \in J\}$ be a cover of X; then there exists a finite subset J_o of J such that $\cup_{i\in J_o}(A_{i-})_o = 1_X$. Since $(A_{i-})_o \subseteq A_{i-}$ for each $i \in J$, we can say (X,τ,τ^*) is almost intuitionistic fuzzy half-compact. □

As Remark 3.5, we can show that the almost intuitionistic fuzzy half-compactness is not always the nearly intuitionistic fuzzy half-compactness.

Theorem 3.8. *Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs and $f : X \to Y$ a surjective half-gp-map. If X is almost intuitionistic fuzzy half-compact, then so is Y.*

Proof. Let $\{A_i \in I^Y : \sigma(A_i) > 0$ and $\sigma^*(A_i) \leq \frac{1}{2}, i \in J\}$ be a cover of Y. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is a half-gp-map, $\{f^{-1}(A_i) \in I^X : \tau(f^{-1}(A_i)) > 0$ and $\tau^*(f^{-1}(A_i)) \leq \frac{1}{2}, i \in J\}$ is a cover of X.

Since X is almost intuitionistic fuzzy half-compact, there exists a finite subset J_o of J such that $\cup_{i\in J_o} f^{-1}(A_i)_- = 1_X$. From the surjectivity of f, (Y,σ,σ^*) is almost intuitionistic fuzzy half-compact.

□

Corollary 3.9. Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs and $f : X \to Y$ a surjective half-gp-map. If X is nearly intuitionistic fuzzy half-compact, then Y is almost intuitionistic fuzzy half-compact.

Theorem 3.10. *Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs and $f : X \to Y$ a surjective, half-gp-map and half-gp-open map. If X is intuitionistic nearly half-compact, then so is Y.*

Proof. Let a family $\{A_i \in I^Y : \sigma(A) > 0$ and $\sigma^*(A_i) \leq \frac{1}{2}, i \in J\}$ be a cover of Y. Since X is nearly intuitionistic fuzzy half-compact, there exists a finite subset J_o of J such that $\cup_{i\in J_o}((f^{-1}(A_i))_-)_o = 1_X$. From the surjectivity of f, we have

$$\begin{aligned} 1_Y &= \cup_{i\in J_o} f(((f^{-1}(A_i))_-)_o) \\ &\subseteq \cup_{i\in J_o} (f(f^{-1}(A_i))_-)_o \\ &\subseteq \cup_{i\in J_o} (f(f^{-1}((A_i)_-)_o \\ &= \cup_{i\in J_o} ((A_i)_-)_o. \end{aligned}$$

Hence $\cup_{i\in J_o}((A_i)_-)_o = 1_Y$. Thus (Y, σ) is nearly intuitionistic fuzzy half-compact.

□

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