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SOME PROPERTIES OF PRODUCT FUZZY GROUPS, IDEALS, AND SUBRINGS

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Abstract. We define a product fuzzy group, which is weaker than the standard fuzzy group defined by Rosenfeld, and characterize some properties of product fuzzy groups, product fuzzy ideals, and product fuzzy subrings.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh ([6]). Rosenfeld ([2]) used this concept to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts based on the Rosenfeld's fuzzy groups were developed. Anthony and Sherwood ([1]) redefined fuzzy groups in terms of t-norm which replaced the minimum operation of Rosenfeld's definition. Some properties of these redefined fuzzy groups, which we call t-fuzzy groups, have been developed by Sherwood ([4]), Sessa ([3]), Sidky and Mishref ([5]). However the definition of the t-fuzzy groups seems to be too general. We define a product fuzzy group as a special case of the t-fuzzy groups, which is weaker than the fuzzy group defined by Rosenfeld ([2]), and develop some properties of product fuzzy groups, product fuzzy ideals, and product fuzzy subrings.

2. p-fuzzy groups, p-fuzzy ideals, and p-fuzzy subrings

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Definition 2.1. A function B from a set X to the closed unit interval $[0, 1]$ in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B. The set $\{x \in X : B(x) > 0\}$ is called the *support* of B and is denoted by $\text{supp}(B)$. A fuzzy set in X is called a *fuzzy point* iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α, where the point x is called its *support*. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$.

For fuzzy sets U, V in a set X, $U \circ V$ has been defined in most articles by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

We weaken this definition as follows.

Definition 2.2. Let X be a set and let U, V be two fuzzy sets in X. $U \circ V$ is defined by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} U(a)V(b) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

Proposition 2.3. *Let A, B be fuzzy sets in a groupoid X and let x_p, y_q be fuzzy points in X. Then $x_p \circ y_q = (xy)_{pq}$ and $A \circ B = \bigcup\limits_{x_p \in A, y_q \in B} x_p \circ y_q$, where $(x_p \circ y_q)(z) = \sup\limits_{cd=z} x_p(c)y_q(d)$.*

Proof. $(x_p \circ y_q)(xy) = \sup\limits_{ab=xy} x_p(a)y_q(b) = pq$. Thus $(x_p \circ y_q) = (xy)_{pq}$. If $x_p \in A$ and $y_q \in B$, then $A(s) \geq x_p(s)$ and $B(t) \geq y_q(t)$. Thus $(A \circ B)(z) = \sup\limits_{st=z} A(s)B(t) \geq \sup\limits_{st=z} \sup\limits_{x_p \in A, y_q \in B} x_p(s)y_q(t) = \sup\limits_{x_p \in A, y_q \in B} \sup\limits_{st=z} x_p(s)y_q(t) = \sup\limits_{x_p \in A, y_q \in B} (x_p \circ y_q)(z) = (\cup\, x_p \circ y_q)(z)$. Since $s_{A(s)} \in A$ and $t_{B(t)} \in B$, $(\bigcup\limits_{x_p \in A, y_q \in B} x_p \circ y_q)(z) = \sup\limits_{x_p \in A, y_q \in B} \sup\limits_{st=z} x_p(s)y_q(t) \geq \sup\limits_{st=z} s_{A(s)}(s)t_{B(t)}(t) = \sup\limits_{st=z} A(s)B(t) = (A \circ B)(z)$. $\square$

Definition 2.4. Let X be a group. We define U^{-1} by $U^{-1}(x) = U(x^{-1})$ for $x \in X$.

Proposition 2.5. *Let X be a set. Then*

(1) *If X is associative, commutative, respectively, then so is $\circ$.*
(2) *If X has a unit e, then $A \circ e_p = e_p \circ A$ for a fuzzy set A in X.*

Proof. Straightforward.

□

The standard definition of a fuzzy group by Rosenfeld ([2]) is that a fuzzy set B in a group X is a fuzzy group iff $B(xy) \geq \min\,(B(x),\ B(y))$ and $B(x^{-1}) = B(x)$ for all $x, y \in X$. We weaken this definition as follows.

Definition 2.6. Let S be a groupoid. A function $B : S \to [0,1]$ is a *product fuzzy groupoid* in S iff for every x, y in S, $B(xy) \geq B(x)B(y)$. We denote a product fuzzy groupoid by a p-fuzzy groupoid. If X is a group, a p-fuzzy groupoid B in X is a *p-fuzzy group* in X iff for each $x \in X$, $B(x^{-1}) = B(x)$. We denote a product fuzzy group by a p-fuzzy group.

Since $\min(p, q) \geq pq$, our definition of a p-fuzzy group is weaker than the standard definition by Rosenfeld ([2]). It is easy to see that if G is a fuzzy group in a group X and e is the identity of X, $G(e) \geq G(x)$ for all $x \in X$. If G is a p-fuzzy group in a group X, $G(e) = G(xx^{-1}) \geq G(x)G(x^{-1}) = [G(x)]^2$ for all $x \in X$.

Proposition 2.7. *Let G be a fuzzy subset in a group X such that $G(e) = 1$, where e is the identity of X. Then G is a p-fuzzy group iff $G(xy^{-1}) \geq G(x)G(y)$ for all $x, y \in X$.*

Proof. Suppose G is a p-fuzzy group. Then

$$G(xy^{-1}) \geq G(x)G(y^{-1}) = G(x)G(y).$$

Suppose $G(xy^{-1}) \geq G(x)G(y)$. Then

$$\begin{aligned} G(x^{-1}) &= G(ex^{-1}) \geq G(e)G(x) = G(x) = G(ex) \\ &\geq G(e)G(x^{-1}) = G(x^{-1}). \end{aligned}$$

That is, $G(x) = G(x^{-1})$. $G(xy) = G(x(y^{-1})^{-1}) \geq G(x)G(y^{-1}) = G(x)G(y)$. □

PROPOSITION 2.8. *Let G be a p-fuzzy groupoid in a group X such that $G(a) = G(a^{-1}) = 1$. Let $r_a : X \to X$ be a right translation defined by $r_a(x) = xa$ and let $l_a : X \to X$ be a left translation defined by $l_a(x) = ax$. Then $r_a(G) = l_a(G) = G$.*

Proof.

$$\begin{aligned} r_a(G)(x) &= \sup_{z \in r_a^{-1}(x)} G(z) = G(xa^{-1}) \\ &\geq G(x)G(a^{-1}) = G(x)G(a) = G(x) = G(xa^{-1}a) \\ &\geq G(xa^{-1})G(a) = G(xa^{-1}) = r_a(G)(x). \end{aligned}$$

Thus $r_a(G)(x) \geq G(x) \geq r_a(G)(x)$. That is, $r_a(G) = G$. Similarly we may show $l_a(G) = G$. □

DEFINITION 2.9. Let B be a fuzzy set in a groupoid X. B is a *p-fuzzy left (or right) ideal* of X iff $B(xy) \geq B(y)$ (or $B(xy) \geq B(x)$) for all $x, y \in X$. B is a *p-fuzzy ideal* of X iff B is p-fuzzy left and right ideal of X.

PROPOSITION 2.10. *Let B be a fuzzy subset in a semigroup S. Then $S \circ B$ (or $B \circ S$) is a p-fuzzy left (or right) ideal of S.*

Proof. From Proposition 2.5, $(S \circ S) \circ B = S \circ (S \circ B)$ and $B \circ (S \circ S) = (B \circ S) \circ S$. Since $S(x) = 1$ for all $x \in S$, $S \circ S \subseteq S$ from Definition 2.2. Since $S(x) = 1$, $(S \circ B)(xy) \geq ((S \circ S) \circ B)(xy) = (S \circ (S \circ B))(xy) \geq S(x)(S \circ B)(y) = (S \circ B)(y)$. Thus $S \circ B$ is a p-fuzzy left ideal. Since $S(y) = 1$, $(B \circ S)(xy) \geq (B \circ (S \circ S))(xy) = ((B \circ S) \circ S)(xy) \geq (B \circ S)(x)S(y) = (B \circ S)(x)$. Thus $B \circ S$ is a p-fuzzy right ideal. □

PROPOSITION 2.11. *Let B be a fuzzy set in a semigroup S. Then $S \circ B \circ S$ is a p-fuzzy ideal of S.*

Proof. From Proposition 2.5, $(S \circ S) \circ B \circ S = S \circ (S \circ B \circ S)$ and $S \circ B \circ (S \circ S) = (S \circ B \circ S) \circ S$. Since $S(x) = 1$ for all $x \in S$, $S \circ S \subseteq S$ from Definition 2.2. Thus $(S \circ B \circ S)(xy) \geq ((S \circ S) \circ B \circ S)(xy) = (S \circ (S \circ B \circ S))(xy) \geq S(x)(S \circ B \circ S)(y) = (S \circ B \circ S)(y)$. That is, $S \circ B \circ S$ is a p-fuzzy left ideal. Since $(S \circ B \circ S)(xy) \geq (S \circ B \circ (S \circ S))(xy) = ((S \circ B \circ S) \circ S)(xy) \geq (S \circ B \circ S)(x)S(y) = (S \circ B \circ S)(x)$, $S \circ B \circ S$ is a p-fuzzy right ideal. Hence $S \circ B \circ S$ is a p-fuzzy ideal. □

Definition 2.12. Let X be a ring with respect to two binary operations $+$ and $\cdot$. Let B be a fuzzy set in X. B is called a *p-fuzzy subring* of X if B is a p-fuzzy group for the operation $+$ and A is a p-fuzzy groupid for the operation $\cdot$ in X.

Theorem 2.13. *Let A and B be p-fuzzy subrings of a commutative ring X. Then $A \circ B$ is a p-fuzzy subring of X.*

Proof. Since X is associative and commutative with respect to the operation $+$, $\circ$ is associative and commutative with respect to $+$ by Proposition 2.5. Thus $(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B)$. Let $x_p, y_q \in A$. Then $A(x) \geq p$ and $A(y) \geq q$. Since A is a p-fuzzy group with respect to $+$, $(x_p \circ y_q)(z) = pq \leq A(x)A(y) \leq A(x+y) = A(z)$ for $z = x + y$ and $(x_p \circ y_q)(z) = 0 \leq A(z)$ for $z \neq x + y$. That is, $x_p \circ y_q \in A$. By Proposition 2.3, $(A \circ A)(z) = [\bigcup_{x_p \in A, y_q \in A} x_p \circ y_q](z) = \sup_{x_p \in A, y_q \in A} (x_p \circ y_q)(z)$. Since $x_p \circ y_q \in A$ for $x_p, y_q \in A$, $\sup_{x_p \in A, y_q \in A} (x_p \circ y_q)(z) \leq A(z)$. Thus $A \circ A \subseteq A$ with respect to $+$. Similarly we may show $B \circ B \subseteq B$ with respect to $+$. Hence $(A \circ B) \circ (A \circ B) = (A \circ A) \circ (B \circ B) \subseteq A \circ B$. Thus $(A \circ B)(x + y) \geq [(A \circ B) \circ (A \circ B)](x + y) = \sup_{a+b=x+y} [(A \circ B)(a)(A \circ B)(b)] \geq (A \circ B)(x)(A \circ B)(y)$. That is, $A \circ B$ is a p-fuzzy groupoid with respect to $+$. $(A \circ B)(-x) = \sup_{y+z=-x} A(y)B(z) = \sup_{(-z)+(-y)=x} B(-z)A(-y) = (B \circ A)(x)$. Since $\circ$ is commutative, $(A \circ B)(-x) = (B \circ A)(x) = (A \circ B)(x)$. Hence $A \circ B$ is a p-fuzzy group with respect to $+$. Since X is associative and commutative with respect to the operation $\cdot$, $\circ$ is associative and commutative with respect to $\cdot$ by Proposition 2.5. Thus $(A \circ B) \circ (A \circ B) = (A \circ A) \circ (B \circ B)$. Let $x_p, y_q \in A$. Then

$A(x) \geq p$ and $A(y) \geq q$. Since A is a p-fuzzy group with respect to $\cdot$, $(x_p \circ y_q)(z) = pq \leq A(x)A(y) \leq A(x \cdot y) = A(z)$ for $z = x \cdot y$ and $(x_p \circ y_q)(z) = 0 \leq A(z)$ for $z \neq x \cdot y$. That is, $x_p \circ y_q \in A$. By the same way as shown for $+$, we may show $A \circ A \subseteq A$ and $B \circ B \subseteq B$ with respect to $\cdot$. Hence $(A \circ B) \circ (A \circ B) = (A \circ A) \circ (B \circ B) \subseteq A \circ B$. Thus $(A \circ B)(x \cdot y) \geq [(A \circ B) \circ (A \circ B)](x \cdot y) = \sup_{a \cdot b = x \cdot y} (A \circ B)(a)(A \circ B)(b) \geq (A \circ B)(x)(A \circ B)(y)$. That is, $A \circ B$ is a p-fuzzy groupoid with respect to $\cdot$. Hence $A \circ B$ is a p-fuzzy subring of X. $\square$

THEOREM 2.14. *Let A and B be p-fuzzy groups in an abelian group X. Then $A \circ B$ is a p-fuzzy group.*

Proof. The proof is similar to that of Theorem 2.13. $\square$

COROLLARY 2.15. *Let A and B be p-fuzzy groupoids in a commutative semigroup X. Then Then $A \circ B$ is a p-fuzzy groupoid.*

Proof. Immediate from Theorem 2.14. $\square$

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