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CONGRUENCE-FREE SIMPLE SEMIGROUP

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Abstract. If a semigroup S has no nontrivial congruences then S is either simple or 0-simple.([2]) By contrast with ring theory, not every congruence on a semigroup is associated with an ideal, hence some simple(or 0-simple) semigroup may have a nontrivial congruence. Thus it is a short note for the characterization of a simple(or 0-simple) semigroup that is congruence-free. A semigroup that has no nontrivial congruences is said to be congruence-free.

1. Introduction

If H is a subgroup of a group G then the relation ρ on G defined by $a\rho b$ if and only if $ab^{-1} \in H$ is a right congruence on G, and the equivalence classes of ρ are the right coset Ha of H in G. Thus it is clear that there is a one-to-one correspondence between subgroups(normal subgroups resp.) and right congruences(congruences resp.) on G.
The study of the structure of semigroups has been approached by means of right congruences. Such an approach seems appropriate since a right congruence on a semigroup is one of the possible analogues of both the right ideal of a ring and the subgroup of a group. However, even though there is a one-to-one correspondence between (right, left) ideals of a ring and (right, left) congruences of that ring, such a correspondence does not prevail for the semigroup.

For a semigroup S we denote the set of all right congruences on S by $L_r(S)$. If α and β are in $L_r(S)$ then the right congruence $\alpha \vee \beta$, called the join of α and β, is the smallest right congruence that contains both α and β. Also the right congruence $\alpha \wedge \beta$, called the meet of α and β, is the largest right congruence contained in both

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α and β. Each of these is well defined. The unique largest element of $L_r(S)$ is the universal congruence $\upsilon = \{(s,t) \mid \forall s,t \in S\}$. The unique smallest element of $L_r(S)$ is the identity congruence $\iota = \{(t,t) \mid \forall t \in S\}$. Thus $L_r(S)$ becomes a complete lattice.

If S is the finite cyclic group of prime order then it clearly has no nontrivial (right)congruences. If S is a ring which has proper (right) ideals then S also has no nontrivial (right)congruences. However, since it is not at all clear which semigroup has no nontrivial (right)congruences, we now characterize the semigroups having this property.
In this paper we only consider semigroups having more than one element and then the semigroup which has no nontrivial (right)congruences is said to be (right)congruence-free.

2. Congruence-free simple(or 0-simple) semigroup

DEFINITION 1. *A* **right congruence** *on a semigroup S is a right compatible equivalence relation.* **Right compatible** *means that for a, b, s in S, $a\rho b$ implies $as\rho bs$.*
An Equivalence relation on a semigroup is called **a congruence** *if it is both left and right compatible.*

DEFINITION 2. *For a semigroup S, an element x is called* **a zero** *of S if $xs = x = sx$ for all s in S.*

DEFINITION 3. *A semigroup S is called* **simple** *if it does not properly contain any two-sided ideal. Similarly a semigroup S with a zero element 0 is called* **0-simple** *if $S^2 \neq 0$ and 0 is the only proper two-sided ideal of S.*

THEOREM 1. *Let S be a finite semigroup having more than one element. Then S is right congruence-free if and only if there is either $S = \{1, 0\}$ or S is isomorphic to Z_p for some prime p.*

Proof. If S is either $S = \{1, 0\}$ where 1 is the identity and 0 is the zero element of S or isomorphic to Z_p for some prime p then it is clear that S has no nontrivial right congruences, hence we now assume that S is right congruence-free and let $I = \{a \in S \mid S \neq aS^1\}$ where S^1 is the semigroup adjoined S with the identity 1. If I is empty then S is a right simple semigroup and hence there is an element u in S^1 such that

$a = au$. Define a right congruence μ on S by $s\mu t$ if and only if $as = at$. Since the set S/μ of equivalence classes of μ on S is isomorphic to aS^1, μ should be the identity congruence ι so that $us = s$ for all s in S. If we show that u is the only idempotent of S then it is clear that S is a group.

If there were two distinct idempotents u and v in S let λ be the principal right congruence on S generated by (u, v). Since it is not the identity congruence it must be the universal congruence so that all elements of S are related by λ. Hence if x, y are any elements of S then there are elements $s_1, s_2, \cdots, s_n$ in S such that $x \in \{us_1, vs_1\}, y \in \{us_n, vs_n\}$, and $\{us_j, vs_j\} \cap \{us_{j+1}, vs_{j+1}\} \neq \emptyset$ for all $j, 1 \leq j < n$. Since every idempotent of a right simple semigroup is a left identity of S, we have that $x = s_1 = s_2 = \cdots = s_n = y$, hence S is the singleton. Thus if S has more that one element, then S has exactly one idempotent and the right simple semigroup S having exactly one idempotent is clearly a group. Thus if S is right congruence-free then S is also a simple group. In particular S is isomorphic to Z_p for some prime p if S is finite.

Next we assume that the set I is nonempty. Then it is easily seen that I is the largest right ideal contained in S properly. Let σ be Rees right congruence on S modulo I. Then the set S/σ of equivalence classes of σ on S is $\{I, \{x\}\}$ where x is an element not in I. If σ is the universal congruence then the set S/σ is clearly the singleton which contradicts to the choice of an element x. Thus σ should be the identity congruence and S/σ is isomorphic to S so that $\mid I \mid= 1$. It means that for every element a of S except one, say z, $S = aS^1$. Moreover such z is a zero element of S and if we denote it by 0 then S is a right 0-simple semigroup. Let $T = S \setminus \{0\}$ and $a, b \in T$. Then ab is either in T or not. If ab is not in T, then $ab = 0$ so that $\{0\} = a(bS^1) = aS \subset aS^1 = S$. Since $a \in aS$ implies that $S = aS = \{0\}$, a should be not in aS so that $S = aS^1 = \{a\} \cup aS = \{a, 0\}$ is the null semigroup of order 2. But since the fact that S is right congruence-free implies that S is the singleton, ab must be in T so that T is a subsemigroup of S. To show that T is a right simple semigroup, let J be any right ideal of T. Then $J \cup \{0\}$ is a right ideal of S and then $J \cup \{0\} = S$ so that T is equal to J.

For any nonzero a in S let ρ be a right congruence on S defined by $s\rho t$ if and only if $as = at$. Since ρ is clearly not the universal congruence, it must be the identity congruence so that T is also a group by a similar argument mentioned before. Thus S is a group adjoined with 0.

We denote the identity element of a group S by 1 and consider the principal right congruence μ on S generated by $(a, 1)$ for nonzero element a of S. If μ were the universal congruence then the zero element 0 should be related to 1 by μ. Hence there are elements $s_1, s_2, \cdots, s_n$ in S such that $0 \in \{1s_1, as_1\}, 1 \in \{1s_n, as_n\}$ and $\{1s_j, as_j\} \cap \{1s_{j+1}, as_{j+1}\} \neq \emptyset$ for all j, $1 \leq j < n$. Since it implies that $0 = s_1 = s_2 = \cdots = s_n = 1$, this case does not happen. Hence μ is the identity congruence and then 1 is the only nonzero element so that $S = \{1, 0\}$.

Lemma 1. ([2]) *Let S be a semigroup having more than one element. If S has no nontrivial congruences, then S is either simple or 0-simple.*

Remark. By contrast with ring theory, not every congruence on a semigroup is associated with an ideal, hence a simple(or 0-simple) semigroup may have a nontrivial congruence.
Consider the semigroup $S = \{a, b, 0\}$ defined by $aa = b = bb, ab = a = ba$ where 0 is the zero of S. Clearly S is a 0-simple semigroup. If ρ is a congruence on S containing (a, b) then we have that $\rho = \{(a, b), (b, a)\} \cup \iota$ is a nontrivial congruence on S.

Definition 4. *A semigroup S is* **left(right) reductive** *if for $a, b \in S$, whenever $xa = xb(ax = bx)$ for all x in S then $a = b$.*

Lemma 2. *([4]) Any congruence-free semigroup with $\mid S \mid > 2$ is left and right reductive.*

Note. For semigroups S and S', a mapping ϕ of S into S' is called a **homomorphism** if $(ab)\phi = (a\phi)(b\phi)$ for all a, b in S. Similarly ϕ is called a **anti-homomorphism** if $(ab)\phi = (b\phi)(a\phi)$ for all a, b in S.
For a semigroup S let T_S be the full transformation semigroup on S. An (anti-)homomorphism ψ of S into T_S is called an (anti-)representation of S by transformations of S. With each element a of S we associate a transformation $\rho_a(\lambda_a)$ of S defined by $x\rho_a = xa(x\lambda_a = ax)$ for all x in S. We call $\rho_a(\lambda_a)$ the inner right(left) translation of S corresponding to a of S. Obviously the transformations ρ_a and λ_a are in T_S. Since S is associative, it is clearly that the mapping $a \to \rho_a$ is an representation of S by transformations of S and the mapping $a \to \lambda_a$ is an anti-representation of S.

Theorem 2. *If a semigroup S is 0-simple then S is congruence-free if and only if $\mid S \mid = 2$.*

Proof. Let S be a 0-simple semigroup. If $\mid S \mid= 2$ then clearly it is congruence-free, hence we just show that the converse is true. For this we suppose not, that is, we assume that S is congruence-free with $\mid S \mid> 2$. Then S is both left and right reductive by lemma 2. We now consider the (anti- resp.)representation $\phi(\psi$ resp.) of S into T_S given by $a\phi = \rho_a(a\psi = \lambda_a$ resp.) where $\rho_a(\lambda_a$ resp.) is the map of S into S given by $x\rho_a = xa(x\lambda_a = ax$ resp.). If $a\phi = b\phi$ for a, b in S then $\rho_a = \rho_b$. Hence for every x in S, $xa = xb$ and the fact that S is left reductive implies $a = b$. It is enough to say that ϕ is one-to-one and the fact that S is also right reductive shows that ψ is also one-to-one. If both ϕ and ψ are one-to-one then S is clearly both left and right cancellative. We next claim that $S \setminus \{0\}$ is a simple sub-semigroup of S. For this we assume that $ab = 0$ for nonzero a and b in S. Then it implies that either a or b is a zero element since they are both left and right cancellable. Thus ab must be not equal to 0 and it shows that $S \setminus \{0\}$ is a sub-semigroup of S. Moreover it is clearly simple by the same argument in the proof of Theorem 2. Therefore S is a simple semigroup adjoined with 0 and then we can construct a congruence on S having two equivalence classes $S \setminus \{0\}$ and $\{0\}$. But if $\mid S \mid> 2$, then the equivalence class $S \setminus \{0\}$ contains more than one element so that this congruence should not be the identity. Thus $\mid S \mid= 2$ if 0-simple semigroup S is congruence-free.

Remark. Two elements of a semigroup S are said to be L-equivalent if they generate the same principal left ideal of S. R-equivalence is defined dually. The join of the equivalence relations L and R is denoted by D and their intersection by H. These fundamental equivalence relations for a semigroup were first introduced and studied by Green.([1])
For a, b in S we define aLb by $S^1a = S^1b$. Clearly L is an equivalence relation such that if aLb then $asLbs$ for all s in S, that is, L is a right congruence. Dually we define aRb by $aS^1 = bS^1$ and note that R is a left congruence on S.

Theorem 3. *Let S be a simple semigroup without zeros. If S is congruence-free, then either (i) $\mid S \mid= 2$, (ii) S is a simple group or (iii) S has no idempotents and every H-class of S contains a single element.*

Proof. Assume that S is a congruence-free simple semigroup. Since a semigroup under our consideration has more than one element, there are two cases; $\mid S \mid= 2$ or $\mid S \mid> 2$.
We assume that $\mid S \mid> 2$ and consider the equivalence relation H. Since the equivalence relation H is a congruence on S, H is either ι or υ. If

the equivalence relation H is the universal congruence v then for every a and b in S, they are H-equivalent. Since they are also both L- and R- equivalent, there are u, v, x, y in S^1 such that $a = bu, b = av$ and $a = xb, b = ya$. Hence $a = xb = xya, b = av = buv$ and then $xy(uv$ resp.) is the right(left resp.) identity of S since a, b are both left and right cancellable. If S contains a both left and right identity then they are equal and then it is an identity of S. Thus S is a group that is congruence-free, that is, a simple group. If the equivalence relation H is the identity congruence ι then every H-class of S contains a single element and clearly S has no idempotents.

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